

Handbook of Mathematical Psychology

Volume III, Chapters 15-21

Handbook of

Volume III, Chapters 15-21

WITH CONTRIBUTIONS BY

Jozef Zwislocki	Edwin Martin
Leo M. Hurvich	R. Duncan Luce
Dorothea Jameson	Patrick Suppes
David H. Krantz	J. Laurie Snell
Robert R. Bush	Richard Bellman

New York • London • Sydney

Mathematical Psychology

EDITED BY

R. Duncan Luce, *University of Pennsylvania*

Robert R. Bush, *University of Pennsylvania*

Eugene Galanter, *University of Washington*

John Wiley and Sons, Inc.

Copyright © 1965 by John Wiley & Sons, Inc

All Rights Reserved

This book or any part thereof
must not be reproduced in any form
without the written permission of the publisher.

Library of Congress Catalog Card Number. 63-9428
Printed in the United States of America

Preface

A general statement about the background, purpose, assumptions, and scope of the *Handbook of Mathematical Psychology* can be found in the Preface to Volume I

Of the seven chapters of this volume, the first five are concerned with mathematical applications to more-or-less traditional psychological problems, and so they do not require any special comment. The last two, on stochastic processes and functional equations, exposit purely mathematical matters. Although several books on stochastic processes exist, we are inclined to believe that a discussion which lies somewhere in difficulty between an elementary text and an advanced treatise, such as Doob's,¹ will be useful for many of those applying stochastic models to psychological problems. Even fewer expositions can be found in the area of functional equations. The only book of which we are aware, Aczel's,² is in German (fortunately, an English revision will soon be available) and is, perhaps, not within easy mathematical reach of the majority of those working in mathematical psychology. There seemed, therefore, reason to include a brief outline of some of the basic results about certain classes of functional equations that have played a role in contemporary mathematical psychology.

Our editorial efforts have been greatly eased by Mrs. Sally Kraska, who handled the numerous details of coordinating the authors, publisher, and editors, and by Mrs. Kay Estes, who has skillfully prepared the index for this as well as the two preceding volumes. We thank them both.

Finally, we are indebted to the Universities of Pennsylvania and Washington, to the National Science Foundation, and to the Office of Naval Research for the support—intellectual and personal, as well as financial—that they have provided us throughout this project.

R. DUNCAN LUCE
ROBERT R. BUSH
EUGENE GALANTER

Philadelphia, Pennsylvania
September 1964

¹ Doob, J. L. *Stochastic processes*. New York: Wiley, 1953.

² Aczél, J. *Vorlesungen über Functionalgleichungen und ihre Anwendungen*. Basel and Stuttgart: Birkhäuser, 1961.

Basic References in Mathematical Psychology (Continued)

In Volume I of this *Handbook* we listed the then published books that we felt constitute a basic library in mathematical psychology. Here we add thirteen items that have appeared in the intervening eighteen months. Mention should also be made here of the existence of the *Journal of mathematical psychology*, the first volume beginning in 1964, which, together with *Psychometrika*, should prove to be a primary source of articles in this area.

- Atkinson, R. C. *Studies in mathematical psychology*. Stanford: Stanford University Press, 1964.
- Berger, J., Cohen, B. P., Snell, J. L., & Zelditch, M., Jr. *Types of formalizations in small group research*. Boston: Houghton Mifflin, 1962.
- Cohen, B. P. *Conflict and conformity: a probability model and its application*. Cambridge: M.I.T. Press, 1963.
- Coombs, C. H. *A theory of data*. New York: Wiley, 1964.
- Feigenbaum, E. A., & Feldman, J. (Eds.) *Computers and thought*. New York: McGraw-Hill, 1963.
- Flament, C. *Applications of graph theory to group structure*. Englewood Cliffs, N.J.: Prentice-Hall, 1963.
- Fredricksen, N., & Gulliksen, H. *Contributions to mathematical psychology*. New York: Holt, Rinehart & Winston, 1964.
- Green, B. F., Jr. *Digital computers in research*. New York: McGraw Hill, 1963.
- Kemeny, J. G., & Snell, J. L. *Mathematical models in the social sciences*. Boston: Ginn, 1962.
- Kyburg, H. E., Jr., & Smokler, H. E. *Studies in subjective probability*. New York: Wiley, 1964.
- Messick, S., & Brayfield, A. H. (Eds.) *Decision and choice: Contributions of Sidney Siegel*. New York: McGraw Hill, 1964.
- Miller, G. A. *Mathematics and psychology*. New York: Wiley, 1964.
- Swets, J. A. *Signal detection and recognition by human observers*. New York: Wiley, 1964.

Contents

15	ANALYSIS OF SOME AUDITORY CHARACTERISTICS	1
	by Jozef Zwislocki, <i>Syracuse University</i>	
16	THEORETICAL TREATMENTS OF SELECTED VISUAL PROBLEMS	99
	by Leo M. Hurvich, <i>University of Pennsylvania</i> Dorothea Jameson, <i>University of Pennsylvania</i> and David H. Krantz, <i>University of Pennsylvania</i>	
17.	IDENTIFICATION LEARNING	161
	by Robert R. Bush, <i>University of Pennsylvania</i>	
18	CONCEPT UTILIZATION	205
	by Edwin Martin, <i>University of Michigan</i>	
19	PREFERENCE, UTILITY, AND SUBJECTIVE PROBABILITY	249
	by R. Duncan Luce, <i>University of Pennsylvania</i> and Patrick Suppes, <i>Stanford University</i>	
20	STOCHASTIC PROCESSES	411
	by J. Laurie Snell, <i>Dartmouth College</i>	
21.	FUNCTIONAL EQUATIONS	487
	by Richard Bellman, <i>the RAND Corporation</i>	
	INDEX	515

Analysis of Some Auditory Characteristics^{1 2 3}

Jozef Zwislocki
Syracuse University

1. This work was supported in part by Contract No. B-3950 between the National Institute of Neurological Diseases and Blindness, Public Health Service, and Syracuse University.

2. There are several persons I would like to thank for their help in preparing the chapter. They have no responsibility for its shortcomings. My secretary, Mrs. Carole Selkirk, typed the first draft, combining speed with care, infinite patience, and good disposition. Mr. Erdogan Ozkaratay copied all the mathematical formulas, and Mr. Allen Ayres and Mr. Charles Reiner did a superb job on the ink drawing and photocopying of the many figures. My very special gratitude goes to my wife who edited parts of the chapter, proofread the whole, and gave me the moral support necessary for writing it despite the extreme dearth of time.

I would also like to thank Dr. David M. Green for his careful review of the chapter and valuable suggestions.

3. In a not very successful effort to meet a deadline, the chapter has certain obvious omissions in its content as well as form. The most important deletion that had to be made concerns interaural effects. Readers of the Handbook are referred to an excellent article on the subject by J. L. Hall and C. M. Pyle in the Readings, Vol II.

Contents

1	Mechanical Vibration and Sound Propagation	5
1 1	Mathematical nomenclature,	6
1 2	Forced oscillation,	8
1 3	Concept of impedance,	10
1 4	Theory of electromechanical and electroacoustic analogies,	11
1 5	Mechanical wave propagation and reflection,	14
2	Sound Transmission in the Ear	21
2 1	Specification of the auditory stimulus,	21
2 2	Sound transmission in the cochlea,	23
2 3	Sound transmission in the middle ear,	38
2 4	Transmission characteristics of the ear,	44
3	Pitch Function	46
4	Threshold of Audibility	56
4 1	Threshold of audibility as a function of temporal stimulus pattern,	56
4 2	Threshold of audibility as a function of sound frequency,	66
5	Masking, Critical Band, and Contrast Phenomena	68
6	The Loudness Function	80
	References	93

Analysis of Some Auditory Characteristics

In the second half of the nineteenth, and even in the first quarter of the twentieth century, this chapter might have been called "Theory of Hearing." Past the halfway mark of the twentieth century, the accumulated empirical knowledge inspires more prudence. We know too much to delude ourselves that we understand the phenomenon of hearing as a whole. A physiological or pseudo-physiological explanation of one or another auditory attribute should not be misconstrued as a theory of the total process. Hypotheses about such a process are possible, but they should await a confirmation by a sufficient body of empirical data before being called theories.

This chapter is traditional in that it attempts to account for some of the most prominent auditory characteristics in terms of physiological mechanisms. The first characteristic so analyzed is pitch as a function of sound frequency—the classical subject of the "theories of hearing." The following section is devoted to the effects of sound frequency and temporal stimulus patterns on the threshold of audibility. The threshold is defined in this context as the stimulus intensity that produces a 50% chance of signal detection under constant experimental conditions. Further sections deal with masking, contrast, and critical band phenomena, and with loudness. All these sections are preceded by an elementary theory of the sound stimulus and by a theory of sound transmission in the ear.

Readers with a background in physical acoustics should be able to omit the first section. The second section which is purely physical and physiological may appear too long for a handbook of mathematical psychology. It is included, nevertheless, because it leads to the specification of the auditory stimulus at the level of sensory cells and constitutes a foundation for further psychophysiological analysis. Several auditory characteristics depend to a large extent on the stimulus transformation between the entrance to the auditory canal and the sensory cells in the cochlea, as may be seen from the following examples. The interrelationship between pitch and sound frequency appears to be determined primarily by the location of the vibration maximum in the cochlea. Frequency discrimination seems to be controlled in part by the shift of this location with frequency and by the sharpness of the vibration maximum. The critical bands, which appear to result from a self-adapting filter mechanism, may be shown to be correlated with the distribution of

the vibration amplitude along the cochlear partition. The mechanical transmission characteristic of the ear has a marked effect on the auditory sensitivity as a function of frequency.

There is another reason for emphasis on the mechanical transmission characteristic. Although some may regard it as almost trivial, in fact, it is very complex and not completely known. Its elimination from the total auditory process by the ability of specifying mathematically the stimulus at the sensory cells considerably simplifies the analysis of the neural coding of auditory information and leads to worthwhile psychophysiological conclusions.

In the sections dealing with psychological characteristics of hearing, an effort has been made to avoid purely hypothetical postulates and to remain as close as possible to empirical evidence. The derived theorems have been validated by means of numerical comparison with experimental data.

In general, the mathematical theory has been used for interpolation rather than extrapolation. At present, it appears more important to integrate the already observed wealth of phenomena into a coherent system rather than to predict new ones. For the limited scope of this chapter, such a system seems to have been achieved—no mutually inconsistent postulates have been accepted, and no theorems have been found to be in conflict with each other. Admittedly, absence of inconsistency does not necessarily mean consistency, it could mean mutual independence. This situation does not arise, however, and a meaningful overlap does exist.

All postulates necessary for the calculation of the pitch function from the physical and anatomical constants of the ear enter into the calculation of the absolute threshold as a function of frequency. With one exception, all added factors have been derived in a straightforward manner from purely acoustic measurements. The exception is a theorem which follows from a theory of temporal auditory summation. This theory has been derived from one psychoacoustic experiment dealing with the threshold of audibility for pairs of short pulses. It has accurately predicted the threshold for all temporal stimulus patterns used so far, it also applies to other sense modalities and to motor activity. In the last section of the chapter, it is shown that the same theory holds for temporal loudness summation. The loudness function is combined with the theory of temporal summation with the help of three postulates: (1) that the absolute threshold of audibility is controlled by the intrinsic noise of the auditory system, (2) that the rate of neural firing decays with stimulus duration, and (3) that the loudness function is independent of stimulus duration except for a multiplicative constant. The first two postulates are widely accepted, and there is good physiological evidence for them. It is

shown that the first agrees numerically with the masking function near the absolute threshold, and that both are numerically consistent with physiological measurements on the peripheral auditory system of monkeys and cats. The empirical evidence for the third has begun to accumulate. The steady-state loudness function itself is derived from S. S. Stevens's power law, the postulate of intrinsic noise, and the phenomenon of additivity of stimulus power within a critical band. A further postulate of central loudness additivity has been at least approximately verified by a substantial number of experiments. The loudness function appears to agree with the measured firing rate in the auditory nerve.

Somewhat more independent of other psychoacoustic phenomena is the demonstration that the critical band may be accounted for by v. Békésy's neural unit developed for the eye and the skin. Nevertheless, even this demonstration relies heavily on the calculated distribution of the vibration amplitude in the cochlea.

Because of their overlap and mutual consistency, the various derivations of the chapter may be considered as parts of one theory.

1 MECHANICAL VIBRATION AND SOUND PROPAGATION

In psychoacoustics, human or animal responses are correlated with physical stimulus parameters. As a consequence, it is not sufficient to describe the behavior of experimental subjects—a thorough knowledge of the stimulus is also necessary. In quantitative experiments, the stimulus must be described in mathematical terms.

The auditory stimulus consists of mechanical vibrations propagated in the form of waves. The vibration may be produced mechanically by striking an elastic object capable of vibration, or electromechanically by activating such an object by means of an electrostatic or an electromagnetic field. Strings, bars, membranes, and thin plates are used for the purpose. Modern psychoacoustics almost exclusively uses electromechanical instruments, called electroacoustic transducers. Depending on whether they are coupled to the ear by means of a small volume of air or radiate sound into larger enclosures, like rooms or auditoriums, they are called earphones or loudspeakers. One advantage of electroacoustic transducers is that their acoustic output can be easily controlled within a considerable range of sound frequencies and intensities.

Sound waves can be measured by microphones which are electroacoustic transducers converting mechanical energy into electrical energy. They may be regarded as earphones or loudspeakers working in reverse.

The function of the peripheral portion of the auditory system is similar to that of microphones and some of the mechanically active parts of the ear resemble known microphone structures. Because of this similarity, the mathematical theory that applies to electroacoustic transducers is also useful in analyzing the sound transmitting system of the ear.

1.1 Mathematical Nomenclature

We now shall develop some of the theoretical concepts of acoustics, especially those of forced vibration, impedance, electroacoustic and electromechanical analogies, and of wave propagation. The development is limited to the simple harmonic functions, i.e., sine and cosine functions. For many purposes, it is more convenient, however, to express such functions in complex nomenclature and to regard the sine and cosine functions as projections of a special vector, called a *phasor*, on two orthogonal axes. The convention is illustrated in Fig. 1. The real numbers are plotted along the horizontal axis, the imaginary numbers along the vertical one. The phasor of magnitude r is at a phase angle θ from the real axis. Then the horizontal projection has the magnitude

$$x = |r| \cos \theta \quad (1)$$

and the vertical projection
$$y = |r| \sin \theta, \quad (2)$$

with $x_{\max} = y_{\max} = |r|$ called the *amplitude*.

The phasor r may be expressed as a vector sum of the orthogonal components

$$r = |r| (\cos \theta + j \sin \theta) \quad (3)$$

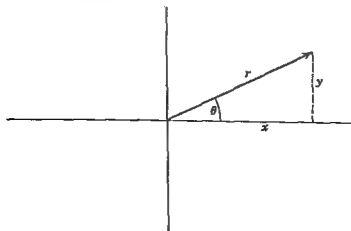


Fig. 1 Phasor in a complex plane

It can be shown that such a sum is equal to $|r| e^{j\theta}$, and this symbol is used to describe the phasor. One has to remember, however, that the actual physical situation, that is, the position of a vibrating body, is described by the projection of the phasor on one of the orthogonal axes rather than by the phasor itself. We take either the real or the imaginary part of the phasor and write

$$x = \operatorname{Re}(|r| e^{j\theta}) \quad (4)$$

$$y = \operatorname{Im}(|r| e^{j\theta})$$

By making the angle θ proportional to time, which means a rotation of the phasor r with a constant velocity, we obtain

$$r = |r| e^{j\omega t} = |r| (\cos \omega t + j \sin \omega t), \quad (5)$$

where $\cos \omega t$ and $\sin \omega t$ are simple harmonic functions of time and ω is the angular velocity. Denoting one full rotation by 2π radians and the number of rotations per second (frequency) by f , we have $\omega = 2\pi f$. The time needed for a complete rotation, $T = 1/f$, is called the *period*. There is no need to draw a sinusoidal curve here; instead, we can take another look at Fig. 1. Let us consider θ to be the phase angle at the time $t = 0$, then

$$r = |r| e^{j(\omega t + \theta)} = |r| [\cos(\omega t + \theta) + j \sin(\omega t + \theta)] \quad (6)$$

When a particle of matter vibrates around its equilibrium position so that at the time $t = 0$ its excursion is zero, which is usually identified with $\theta = 0$, its position at any time is defined by

$$y = |r| \sin \omega t$$

If the initial position of the particle is at $y = |r|$, and therefore $\theta = \pi/2$, its motion follows the function

$$y = |r| \sin \left(\omega t + \frac{\pi}{2} \right) = |r| \cos \omega t$$

Restriction of the developments of this section to simple harmonic functions does not mean loss of generality, provided that we deal with linear systems, that is, provided that the intervening forces remain proportional to kinematic parameters. This condition is usually fulfilled in acoustics, which deals with small oscillations. According to the Fourier theorem, any periodic function can be represented by a sum of simple harmonic components,

$$f(t) = \sum_{n=1}^{\infty} a_n e^{2\pi n f t + \theta_n}, \quad (7)$$

and any aperiodic function can be represented within a finite interval of time by a continuous frequency spectrum, such that

$$f(t) = \int_0^{\infty} g(\omega) d\omega \quad (8)$$

As a consequence, any linear system may be fully described in terms of simple harmonic functions. More detailed information on Fourier analysis as well as frequency spectra of various functions can be found, for instance, in *The Fourier Integral and its Applications* (Papoulis, 1962).

1.2 Forced Oscillation

Let us now consider a simple mechanical system consisting of a rigid plate suspended elastically in a heavy frame (Fig. 2a), and a mechanical, electrostatic, or electromagnetic force F acting on the plate perpendicularly to its surface. If the force is periodic, the plate will move up and down like a piston. Such a system in a somewhat more complex form is found in many electroacoustic transducers whose diaphragm vibrates in almost the same way as a rigid piston at sufficiently low frequencies. The system may be represented symbolically by a mass, a spring, and a frictional element, as shown in Fig. 2b. When the mass is pulled away from its equilibrium position to a position x , the spring is stretched and produces an opposing force $-sx$ that tends to bring the mass back to its equilibrium

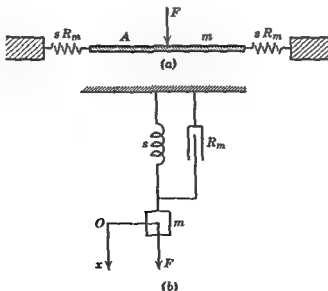


Fig. 2 A simple mechanical system. A rigid plate suspended elastically in a heavy frame (a) The mechanical structure, (b) its symbolic representation

position. If the motion occurs with a velocity $u = dx/dt$, friction produces an opposing force $-r(dx/dt)$. Finally, the acceleration of the mass leads to an inertial force $-m(d^2x/dt^2)$. In general, all forces act at the same time, and a dynamic equilibrium is established when

$$m \frac{d^2x}{dt^2} + r \frac{dx}{dt} + sx = F \quad (9)$$

The expression is an inhomogeneous linear differential equation whose solution consists of the general solution of the homogeneous part and of a particular solution of the whole equation. The solution of the homogeneous equation,

$$m \frac{d^2x}{dt^2} + r \frac{dx}{dt} + sx = 0, \quad (10)$$

is

$$x = e^{-\alpha t}(A_1 e^{j\omega_d t} + A_2 e^{-j\omega_d t}), \quad (11)$$

where

$$\alpha = \frac{r}{2m}, \quad \text{and} \quad \omega_d = \left(\frac{s}{m} - \alpha^2\right)^{1/2}$$

By selecting the constants A_1 and A_2 so that the right-hand expression of Eq. 11 is real, the equation can be transformed into a trigonometric function

$$x = Ae^{-\alpha t} \cos(\omega_d t + \phi) \quad (12)$$

It should be noted that Eq. 12 describes an oscillation with an amplitude decreasing according to the function $e^{-\alpha t}$. The amplitude decay becomes more rapid with increasing α . The rapidity of amplitude decay is often expressed as a natural logarithm of the ratio of two consecutive excursions on the same side of equilibrium. Since such excursions are spaced in time by one period, we obtain

$$D = \ln \frac{Ae^{-\alpha t}}{Ae^{-\alpha(t+T)}} = \alpha T \quad (13)$$

The expression is called a *logarithmic decrement*.

For the particular solution of the inhomogeneous equation, we assume that the external force is a simple harmonic function of time, $F = \hat{F}e^{j\omega t}$. The particular solution then becomes

$$x = \frac{\hat{F}e^{j\omega t}}{j\omega r + (s - \omega^2 m)}, \quad (14)$$

and the total solution can be written in the form

$$x = Ae^{-\alpha t}e^{j(\omega_d t + \phi)} + \frac{\hat{F}e^{j\omega t}}{j\omega r + (s - \omega^2 m)} \quad (15)$$

The first term decays after a while and constitutes the so called transient, well known to anybody who has worked in psychoacoustics. Transient responses of earphones or loudspeakers are often heard as clicks and may interfere with the measurement of auditory responses. Because of them, most auditory stimuli have to be turned on gradually, so that it is difficult to measure short duration effects. The second term of Eq. 15 describes a sustained oscillation whose amplitude and phase with respect to the amplitude and phase of the driving force depend on frequency and on the parameters m , s , and r .

1.3 Concept of Impedance

The steady state part of Eq. 15 can be expressed in terms of velocity of motion rather than of displacement, and we obtain

$$u = \frac{dx}{dt} = \frac{\hat{F}e^{j\omega t}}{r + j(\omega m - s/\omega)}, \quad (16)$$

or

$$\frac{F}{u} = r + j\left(\omega m - \frac{s}{\omega}\right) \quad (17)$$

The ratio F/u is usually denoted by Z_m and is called *mechanical impedance*.

The concept of mechanical impedance does not apply exclusively to the structure of Fig. 2 but it may be generalized to any mechanical structure driven at a certain point. As soon as the impedance is known it is possible to find the velocity produced by a given force without having recourse to a differential equation

$$u = \frac{F}{Z_m} \quad (18)$$

Since displacement is equal to the time integral of velocity, we can simply write

$$x = \int u \, dt = \frac{1}{Z_m} \int \hat{F}e^{j\omega t} \, dt \quad (19)$$

or

$$x = \frac{F}{j\omega Z_m}$$

Similarly, the acceleration follows as

$$a = \frac{du}{dt} = \frac{j\omega F}{Z_m} \quad (20)$$

Note that the impedance term does not enter into the integration and differentiation operations

These few examples should suffice as a demonstration of the operational simplification introduced by the concept of impedance. A further advantage is that the impedance of any linear structure consisting of lumped elements can be calculated by means of algebraic operations. For instance, the impedance of the structure in Fig. 2 consists of three parts: the impedance of the mass m , the impedance of the spring s , and the impedance of the frictional element r . The expressions for these partial impedances are $Z_1 = j\omega m$, $Z_2 = s/j\omega$, and $Z_3 = r$. The total impedance is equal to the sum of the components

$$Z_m = Z_1 + Z_2 + Z_3 = j\left(\omega m - \frac{s}{\omega}\right) + r \quad (21)$$

The mechanical impedance generally consists of a real part R , called resistance, and of an imaginary part X , called reactance. A purely frictional element has an impedance consisting of a resistance, a mass element, of a positive reactance, a stiffness element (spring), of a negative reactance.

(See also Beranek, 1949)

1.4 Theory of Electromechanical and Electroacoustic Analogies

The concept of impedance was borrowed from electrical network analysis where it has proved to be extremely useful. This brings us to the theory of electromechanical analogies. Let us consider an electrical series circuit consisting of an inductance L , a resistance R , and a capacitance C . The circuit can be described by the differential equation

$$L \frac{d^2 q}{dt^2} + R \frac{dq}{dt} + \frac{q}{C} = \epsilon, \quad (22)$$

where q means electric charge and ϵ the applied voltage. The steady-state solution of this equation is

$$q = \frac{E e^{j\omega t}}{j\omega R + (1/C - \omega^2 L)}, \quad (23)$$

and since electric current is a time derivative of electric charge,

$$i = \frac{\epsilon}{R + j(\omega L - 1/\omega C)} \quad (24)$$

The first term decays after a while and constitutes the so-called transient, well known to anybody who has worked in psychoacoustics. Transient responses of earphones or loudspeakers are often heard as clicks and may interfere with the measurement of auditory responses. Because of them, most auditory stimuli have to be turned on gradually, so that it is difficult to measure short-duration effects. The second term of Eq. 15 describes a sustained oscillation whose amplitude and phase with respect to the amplitude and phase of the driving force depend on frequency and on the parameters m , s , and r .

1.3 Concept of Impedance

The steady-state part of Eq. 15 can be expressed in terms of velocity of motion rather than of displacement, and we obtain

$$u = \frac{dx}{dt} = \frac{\hat{F}e^{j\omega t}}{r + j(\omega m - s/\omega)}, \quad (16)$$

or

$$\frac{F}{u} = r + j\left(\omega m - \frac{s}{\omega}\right) \quad (17)$$

The ratio F/u is usually denoted by Z_m and is called *mechanical impedance*.

The concept of mechanical impedance does not apply exclusively to the structure of Fig. 2 but it may be generalized to any mechanical structure driven at a certain point. As soon as the impedance is known, it is possible to find the velocity produced by a given force without having recourse to a differential equation,

$$u = \frac{F}{Z_m} \quad (18)$$

Since displacement is equal to the time integral of velocity, we can simply write

$$x = \int u \, dt = \frac{1}{Z_m} \int \hat{F}e^{j\omega t} \, dt, \quad (19)$$

or

$$x = \frac{F}{j\omega Z_m}$$

Similarly, the acceleration follows as

$$a = \frac{du}{dt} = \frac{j\omega F}{Z_m} \quad (20)$$

Note that the impedance term does not enter into the integration and differentiation operations

These few examples should suffice as a demonstration of the operational simplification introduced by the concept of impedance. A further advantage is that the impedance of any linear structure consisting of lumped elements can be calculated by means of algebraic operations. For instance, the impedance of the structure in Fig. 2 consists of three parts: the impedance of the mass m , the impedance of the spring s , and the impedance of the frictional element r . The expressions for these partial impedances are $Z_1 = j\omega m$, $Z_2 = s/j\omega$, and $Z_3 = r$. The total impedance is equal to the sum of the components

$$Z_m = Z_1 + Z_2 + Z_3 = j\left(\omega m - \frac{s}{\omega}\right) + r \quad (21)$$

The mechanical impedance generally consists of a real part R , called resistance, and of an imaginary part X , called reactance. A purely frictional element has an impedance consisting of a resistance, a mass element, of a positive reactance, a stiffness element (spring), of a negative reactance.

(See also Beranek, 1949)

1.4 Theory of Electromechanical and Electroacoustic Analogies

The concept of impedance was borrowed from electrical network analysis where it has proved to be extremely useful. This brings us to the theory of electromechanical analogies. Let us consider an electrical series circuit consisting of an inductance L , a resistance R , and a capacitance C . The circuit can be described by the differential equation

$$L \frac{d^2 q}{dt^2} + R \frac{dq}{dt} + \frac{q}{C} = \epsilon, \quad (22)$$

where q means electric charge and ϵ the applied voltage. The steady-state solution of this equation is

$$q = \frac{E e^{j\omega t}}{j\omega R + (1/C - \omega^2 L)}, \quad (23)$$

and since electric current is a time derivative of electric charge,

$$i = \frac{\epsilon}{R + j(\omega L - 1/\omega C)} \quad (24)$$

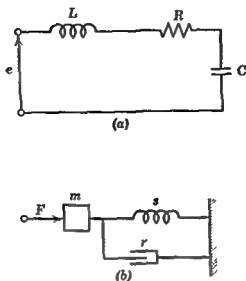


Fig 3 An electric series circuit (a), and its mechanical analog (b)

The electric impedance of the circuit follows as

$$Z_e = \frac{e}{i} = R + j\left(\omega L - \frac{1}{\omega C}\right) \quad (25)$$

It is easy to see that the equations describing the electrical series circuit are analogous to those for the mechanical structure of Fig 2, and that the latter can be obtained from the former by simply replacing the voltage E by the force F , the electric charge q by the displacement x , etc. We obtain the following analog pairs

$$\begin{aligned} E &- F, & q &- x, & i &- u, \\ R &- r, & L &- m, & C &- \frac{1}{s} \end{aligned} \quad (26)$$

The electric circuit of Fig 3a, is an analog of the mechanical system of Fig 3b. We can find analog circuits for other systems. For instance, the parallel circuit of Fig 4a is an analog of the mechanical system of Fig 4b.

The concept of impedance can also be applied to acoustic systems. For instance, a constriction in a tube acts like a mass combined with a resistance, a dilation like a stiffness or its reciprocal, the compliance. Thus, the acoustic system of Fig 5 is an analog of the mechanical system of Fig 2.

In acoustics, the mechanical impedance is often replaced by the specific impedance or the acoustic impedance. The specific impedance is defined as the ratio of sound pressure to the velocity of motion. If in Fig 2a we denote the surface area of the moving plate by A and assume that the

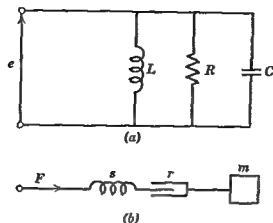


Fig 4 An electric parallel circuit (a), and its mechanical analog (b)

force F is evenly distributed over that area, the sound pressure follows from

$$p = \frac{F}{A} \quad (27)$$

The specific impedance is then

$$Z_{sp} = \frac{p}{u} = \frac{F/A}{u} \quad (28)$$

The acoustic impedance is defined as the ratio of sound pressure to volume velocity. Again, in Fig. 2a, the volume velocity of the plate amounts to $v = uA$, so that the acoustic impedance is

$$Z_a = \frac{p}{v} = \frac{p}{uA} \quad (29)$$

Summarizing,

$$Z_{sp} = \frac{Z_m}{A}, \quad \text{and} \quad Z_a = \frac{Z_m}{A^2} \quad (30)$$

The concept of acoustic impedance can be very helpful in the control of the auditory stimulus and also in the analysis of the acoustic function of the ear. When an earphone is placed over the pinna, the sound pressure it generates at the entrance to the ear canal is equal to the volume velocity

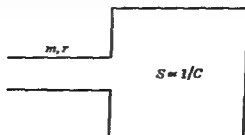


Fig 5 An acoustic resonator. It is an analog of the mechanical system of Fig 2 and of the electrical circuit of Fig 3

of the diaphragm multiplied by the acoustic impedance of the ear as measured at the level of the diaphragm,

$$p_e = Z_{ae} i \quad (31)$$

The volume displacement depends, in turn, on the electromagnetic force acting on the diaphragm and on the sum of the acoustic impedances of the diaphragm and of the ear,

$$v = \left(\frac{F}{A}\right) \left(\frac{1}{Z_{ae} + Z_{ad}}\right) \quad (32)$$

Consequently, the sound pressure at the ear amounts to

$$p_e = \left(\frac{F}{A}\right) \left(\frac{Z_{ae}}{Z_{ae} + Z_{ad}}\right) \quad (33)$$

If the earphone is calibrated in a standard way on a 6 cc coupler, the sound pressure generated in the coupler is

$$p_c = \left(\frac{F}{A}\right) \left(\frac{Z_{ac}}{Z_{ac} + Z_{ad}}\right), \quad (34)$$

and the ratio of sound pressures for a constant F , i.e., a constant voltage across the earphone terminals, is

$$\frac{p_e}{p_c} = \frac{Z_{ae}(Z_{ac} + Z_{ad})}{Z_{ac}(Z_{ae} + Z_{ad})} \quad (35)$$

This relation makes it possible to determine fairly accurately the sound pressure in the ear canal once the earphone has been calibrated on the standard coupler. Data for the average impedance of the ear are available and most of the time it is possible to assume that $Z_{ad} \gg Z_{ae} \cong Z_{ac}$, so that

$$\frac{p_e}{p_c} \cong \frac{Z_{ae}}{Z_{ac}} \quad (36)$$

In the analysis of sound transmission in the ear, the impedance at the eardrum determines how much sound is absorbed by the ear and how much is reflected back. It helps in calculating the frequency responses of the middle and inner ear, permitting us to specify the stimulus at the level of the sensory cells. By these means many psychoacoustic characteristics may be explained. See Olson (1958).

1.5 Mechanical Wave Propagation and Reflection

The vibration of a mechanical instrument is ordinarily transmitted to the ear by waves propagated through the ambient air. In the event that a

vibrator is applied directly to the head, the transmission can occur through the skull bones. The second mode is used in some clinical tests but is of limited value in psychoacoustics so that we shall focus our attention on the first.

Vibration of a plate or a similar object produces compressions and dilations of air directly in front of it. These changes of pressure, which are accompanied by motion of air particles, spread through the medium as a result of elastic restoring forces acting among adjacent particles. The particles themselves oscillate around their equilibrium positions and do not change their average positions. In contradistinction to electromagnetic waves for example light waves, the mechanical waves cannot be propagated in vacuum—they require a material medium. They can be generated in fluids as well as in solids provided that a deformation or compression of the medium evokes a restoring force. Except for gravity waves, such a restoring force is elastic in nature and can usually be described by Hooke's law for solids and by the bulk modulus of elasticity for fluids. Hooke's law simply states that the restoring force is proportional to deformation. Thus

$$F = -YS \frac{d\xi}{dx}, \quad (37)$$

where F is the restoring force, Y the proportionality constant called Young's modulus, S the cross sectional area perpendicular to the force, and $d\xi$ the elongation of the elemental length dx .

The bulk modulus of elasticity is defined by

$$B = - \frac{dP}{dV/V}, \quad (38)$$

where dP is the change of pressure and dV is the associated change of volume V . The bulk modulus can be derived from the general gas law and is equal to

$$B = \gamma P, \quad (39)$$

where γ is the ratio of specific heats and P is the static pressure.

The change of volume or compression of fluid is described by the continuity equation which states that the divergence of the flow of fluid through a space element is equal to the change of density within the space element

$$\text{div } u = \frac{1}{\rho_0} \frac{\partial \rho}{\partial t} \quad (40)$$

is one form of the continuity equation, where

$$\text{div } u = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z},$$

u is the velocity vector with the orthogonal components u_x , u_y , and u_z . ρ_0 denotes the resting density, and $\partial\rho/\partial t$ the time derivative of density.

Propagation of mechanical or acoustic waves requires a continuous exchange between potential and kinetic energy. In order to describe the process in simple terms, let us consider longitudinal waves in a bar of moderate cross-sectional area S . The restoring force produces an acceleration of particles such that

$$\frac{\partial F}{\partial x} dx = -a\rho_0 S dx, \quad (41)$$

with $\partial F/\partial x$ the change of restoring force with x , $a = \partial^2 \xi / \partial t^2$ is the acceleration, ρ_0 the density, and dx the element of length. Substituting Eq. 37 for F , we obtain

$$SY \frac{\partial^2 \xi}{\partial x^2} = S\rho_0 \frac{\partial^2 \xi}{\partial t^2}, \quad (42)$$

or

$$\frac{\partial^2 \xi}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 \xi}{\partial t^2},$$

the differential equation of motion for longitudinal plane waves. The quantity

$$c = \sqrt{Y/\rho_0} \quad (43)$$

denotes the *velocity of wave propagation*, which is also called phase velocity.

It can be shown that any function of the argument $ct \pm x$ is a solution of the differential Eq. 42. Thus,

$$\xi = f_1(ct - x) + f_2(ct + x) \quad (44)$$

is the most general solution, where f_1 and f_2 are arbitrary well-behaved functions. It indicates that events that take place at the location x_1 occur at x_2 with a time delay $\Delta t = (x_2 - x_1)/c$, and are propagated, therefore, with the velocity $c = (x_2 - x_1)/\Delta t$. Since the functions f_1 and f_2 do not change with x or t , a disturbance produced at x_1 is transmitted through the medium without distortion—a phenomenon of fundamental importance in communication. This is strictly true only within the validity of Hooke's law and when there are no dissipative losses. Hooke's law almost always holds for small disturbances. Dissipation of energy is always present to a greater or lesser extent, it limits the useful range of information transmission.

Equation 44 is called a *wave equation*. The function f_1 indicates wave propagation in the positive x direction, f_2 in the negative x direction. One

of the most useful forms of the wave equation is the *simple harmonic function* which can be written either in terms of complex numbers, that is,

$$x = Ae^{jk(ct-x)} + Be^{jk(ct+x)}, \quad (45)$$

or in terms of sine or cosine functions,

$$x = A \sin k(ct - x) + B \sin k(ct + x) \quad (46)$$

In these equations, $k = 2\pi f/c$ with f meaning frequency of oscillation. The constant k is called the wave constant because the wavelength $\lambda = c/f$ and, consequently, $k = 2\pi/\lambda$. The wavelength is the distance covered by the disturbance in one period $T = 1/f$. For simple harmonic functions, Eq. 42 may be simplified to

$$\frac{d^2\xi}{dx^2} = -\frac{4\pi^2 f^2}{c^2} \xi, \quad (47)$$

since

$$\frac{\partial^2}{\partial t^2} e^{i2\pi ft} = -4\pi^2 f^2 e^{i2\pi ft}$$

Wave equations for fluids are similar to those derived for longitudinal plane waves in solids. Their exact form depends on the geometry of the system, however. In order to derive a general three-dimensional differential equation, we introduce the velocity potential ϕ such that

$$u_x = -\frac{\partial\phi}{\partial x}, \quad u_y = -\frac{\partial\phi}{\partial y}, \quad \text{and} \quad u_z = -\frac{\partial\phi}{\partial z}$$

Equation 40 can now be written in the form

$$\nabla^2\phi = \frac{1}{\rho_0} \frac{\partial\rho}{\partial t} \quad (48)$$

Let us denote by p the excess pressure over the static pressure P and consider the forces acting on an element $dx\,dy\,dz$. The restoring force in the x direction is equal to $-(\partial p/\partial x)\,dx\,dy\,dz$, and the inertia force for small disturbances to $\rho(\partial u_x/\partial t)\,dx\,dy\,dz$. Applying Newton's second law, we obtain

$$\frac{\partial p}{\partial x} dx + \rho \frac{\partial u_x}{\partial t} dx = 0$$

Similarly,

$$\frac{\partial p}{\partial y} dy + \rho \frac{\partial u_y}{\partial t} dy = 0, \quad (49)$$

and

$$\frac{\partial p}{\partial z} dz + \rho \frac{\partial u_z}{\partial t} dz = 0$$

Addition of the three expressions yields

$$\frac{\partial p}{\partial x} dx + \frac{\partial p}{\partial y} dy + \frac{\partial p}{\partial z} dz + \rho \frac{\partial}{\partial t} (u_x dx + u_y dy + u_z dz) = 0,$$

which is equivalent to

$$dp - \rho \frac{\partial}{\partial t} (d\phi) = 0 \quad (50)$$

since

$$u_x = -\frac{\partial \phi}{\partial x}, \text{ etc}$$

Integration of Eq 50 leads to

$$p - \rho \frac{\partial \phi}{\partial t} = K, \quad (51)$$

where K is a constant of integration. When no disturbance is present in the fluid, both ϕ and p vanish, so that $K = 0$.

In a further step, we use the definition of density

$$\rho = \frac{M}{V}, \quad (52)$$

where M is the mass of the volume V . Differentiating, we obtain the expression

$$d\rho = M \frac{-dV}{V^2},$$

which can be rewritten in the form

$$\frac{d\rho}{\rho_0} = -\frac{dV}{V} \quad (53)$$

Replacement of $-dV/V$ by $d\rho/\rho_0$ in Eq 38 produces

$$B = \frac{dP}{d\rho/\rho_0} \quad (54)$$

Introducing $P = p_0 + p$ and expressing both P and ρ as a function of time, we obtain

$$\frac{\partial p}{\partial t} = \frac{B}{\rho_0} \frac{\partial \rho}{\partial t} \quad (55)$$

By combining Eqs 51 and 55 it is possible to eliminate p ,

$$\frac{1}{\rho_0} \frac{\partial \rho}{\partial t} = \frac{\rho_0}{B} \frac{\partial^2 \phi}{\partial t^2}, \quad (56)$$

and substituting $(\rho_0/B)(\partial^2\phi/\partial t^2)$ for $(1/\rho_0)(\partial\rho/\partial t)$ in Eq 48, we finally obtain

$$\nabla^2\phi = \frac{1}{c^2} \frac{\partial^2\phi}{\partial t^2}, \quad (57)$$

with $c = \sqrt{B/\rho}$ determining the velocity of propagation

Acoustics often deals with small sound sources and with spherical waves emitted by them. Under such conditions Eq 57 is expressed in polar coordinates. Since it is well beyond the scope of this chapter to discuss these matters, however, we shall limit ourselves to the consideration of plane waves. They are approximated by spherical waves at a sufficiently large distance from the source. Waves of this kind are also propagated in tubes whose diameter is small compared to the wavelength. To a sufficient approximation, the ear canal may be regarded as such a tube. In plane waves, $\partial\phi/\partial y$ and $\partial\phi/\partial z$ vanish, and Eq 57 becomes

$$\frac{\partial^2\phi}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2\phi}{\partial t^2}, \quad (58)$$

which is analogous to Eq 42. As a consequence,

$$\phi = \phi_+ + \phi_- = A e^{jk(\alpha - x)} + B e^{jk(\alpha + x)} \quad (59)$$

is a solution of Eq 58 for simple harmonic time functions. The variable pressure p , called sound pressure for audible frequencies of vibration, follows from Eq 51

$$p = \rho \frac{\partial\phi}{\partial t} = j\rho kc(\phi_+ + \phi_-), \quad (60)$$

and the particle velocity from the definition of the potential

$$u = -\frac{\partial\phi}{\partial x} = jk(\phi_+ - \phi_-) \quad (61)$$

In Eqs 59, 60, and 61, ϕ_+ denotes wave propagation in the positive x direction, ϕ_- in the opposite direction. The second wave is usually due to reflection of a primary wave at some boundary, so that in the absence of reflection ϕ_- vanishes. In this situation,

$$Z_{sp} \approx \frac{p}{u} = \rho c \quad (62)$$

defines the specific impedance of the medium for progressive plane waves. In air, under normal atmospheric conditions, its numerical value amounts to approximately $41.5 \text{ g/cm}^2 \text{ sec}$. It should be noted that the impedance is real, so that sound pressure and particle velocity are in phase.

If waves are propagated in a tube of cross-sectional area S , the acoustic impedance for progressive waves amounts to

$$Z_a = \frac{Z_{ap}}{S} = \frac{\rho c}{S} \quad (63)$$

For instance, an average ear canal has a cross-sectional area of about 0.38 cm^2 , so that its acoustic impedance would amount to approximately $110 \text{ g/cm}^4 \text{ sec}$ if there were no reflection at the eardrum

When the tube in which acoustic waves are propagated is not of infinite length, but is terminated by an acoustic impedance different from its own impedance, a partial wave reflection takes place. At the place of reflection the sound-pressure ratio between the reflected and the incident wave amounts to

$$\frac{p_r}{p_i} = \frac{Z_2 - Z_1}{Z_2 + Z_1}, \quad (64)$$

where Z_1 is the impedance of the tube and Z_2 the terminating impedance. When the real and imaginary parts of the impedances are separated, the equation becomes

$$\frac{p_r}{p_i} = \frac{R_2 - R_1 + j(x_2 - x_1)}{R_2 + R_1 + j(x_2 + x_1)}, \quad (65)$$

or

$$\frac{p_r}{p_i} = \beta e^{j\theta}, \quad (66)$$

with

$$\beta = \sqrt{\frac{(R_2 - R_1)^2 + (x_2 - x_1)^2}{(R_2 + R_1)^2 + (x_2 + x_1)^2}}, \quad (67)$$

and

$$\theta = \arctg \frac{x_2 - x_1}{R_2 - R_1} - \arctg \frac{x_2 + x_1}{R_2 + R_1} \quad (68)$$

At a distance l from the place of reflection the ratio of pressures becomes

$$\frac{p_{rl}}{p_{il}} = \beta e^{j(\theta - 2kl)}, \quad (69)$$

so that the total pressure amounts to

$$p = p_{il}(1 + \beta e^{j(\theta - 2kl)}), \quad (70)$$

with its amplitude varying as a function of frequency between

$$p_{il}(1 + \beta) \quad \text{and} \quad p_{il}(1 - \beta)$$

The minimum occurs when $\theta - 2kl = -\pi$, or $l = \lambda[(\theta + \pi)/4\pi]$. For a very large terminating impedance, $\theta \rightarrow 0$ and $l = \lambda/4$. For a very small terminating impedance, $\theta \rightarrow \pi$ and $l = \lambda/2$. In the first instance, the sound pressure at the end of the tube ($x = 0$) may be considerably higher than at $x = l$. This phenomenon affects quite considerably the frequency response of the ear. See Kinsler & Frey (1962).

2 SOUND TRANSMISSION IN THE EAR

2.1 Specification of the Auditory Stimulus

In auditory experiments the sound stimulus is usually specified in terms of sound pressure in dynes/cm², or in terms of sound pressure level (SPL) in decibels above 0.0002 dyne/cm². As long as we deal with plane sound waves or with sound energy generated in a small cavity, like that enclosed under an earphone cushion, such a specification is sufficient. In plane waves, the particle velocity is in phase with the sound pressure and its magnitude is determined by the specific impedance

$$u = \frac{p}{\rho c} \quad (71)$$

In a small enclosure the ratio between sound pressure and particle velocity is equal to the acoustic input impedance. As a consequence, all sound parameters, including sound intensity, are uniquely determined by the sound pressure. The sound intensity of a plane wave follows from the equation

$$I = \frac{p\hat{u}}{2} = \frac{p^2}{2\rho c}, \quad (72)$$

and it is expressed in ergs/cm² sec in the cgs system. One watt sec is equal to 10⁷ ergs. The sound pressure and particle velocity are expressed in peak values. If, however, a small sound source is placed close to the listener's head, the waves are spherical rather than plane, and the relations 62 and 71 do not hold. A measurement of sound pressure does not specify the acoustic event completely. Since in laboratory experiments such conditions are usually avoided, we will not discuss them any further.

Even if we restrict ourselves to consideration of plane waves and small enclosures, measurement of sound pressure does not completely specify the stimulus. It is necessary to specify the conditions of measurement. For instance, it is possible to determine the sound pressure at the entrance to the ear canal or, with a probe microphone, in the vicinity of the eardrum.

Two procedures have become standard. In one of them, sound is generated by a loudspeaker placed in an anechoic chamber, that is, a room with acoustically treated walls. The listener is placed 6 ft from the loudspeaker, facing it. The sound pressure is measured at the location of the listener's head before introducing the listener into the sound field. The measured sound pressure is different from that at the entrance to the ear canal because of sound diffraction at the listener's head. In the second method, the test earphone is calibrated on a standard 6 cc cavity serving as a coupling device between the earphone and the measuring microphone. Since the acoustic impedance of the standard cavity differs from that of the ear, the sound pressure produced at the microphone is not the same as that produced in the ear. The difference depends on sound frequency and the mechanical impedance of the earphone. Both methods lead to an arbitrary specification of the stimulus. This is of no serious disadvantage in a purely empirical approach to auditory problems. However, it does constitute a handicap in an analytical treatment of auditory characteristics. From the latter point of view, it is helpful to define the stimulus at the eardrum, or even further, at the level of sensory cells. In so doing, we knowingly eliminate a part of the sensory process. No generality is lost, however, if we are able to specify the stimulus transformation that takes place in the sound transmitting part of the ear. The transformed stimulus may be reintroduced into the stimulus-response relationship as a new variable. We obtain

$$\psi = \phi[f(s)], \quad (73)$$

where ψ means the response function, s is the acoustic stimulus, $f(s)$ is the stimulus at the level of sensory cells, and ϕ the response function in terms of $f(s)$. The main advantage of the operation is a substantial simplification. Since the acoustic transfer function $f(s)$ is quite complex, the stimulus-response function $\phi(f)$ is simpler than $\psi(s)$. The operation also achieves an almost complete separation of mechanical and neural variables. A complete separation is not possible because sound transmission in the middle ear is controlled to a certain extent by two muscles that are an integral part of a neuromechanical feedback loop. The effect is of secondary importance, however, since the feedback becomes effective only at high sound intensities.

We shall now attempt to derive the transfer function $f(s)$ and, more specifically, to express the vibration amplitude at the level of sensory cells as a function of sound pressure measured in a free sound field at the location of the center of the listener's head. The anatomical drawing of Fig. 6 may be helpful in explaining the situation. The sound waves enter the opening of the auditory meatus and are propagated up to the eardrum, where part of their energy is reflected and part is transformed into vibration

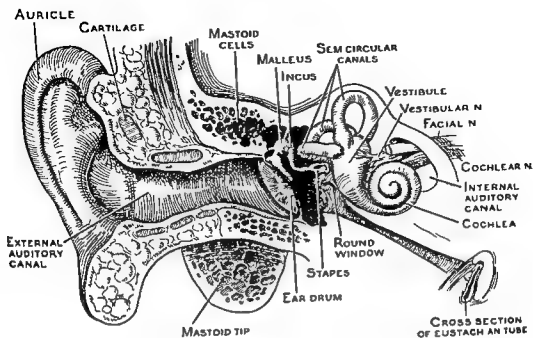


Fig 6 A semidiagrammatic drawing of a longitudinal section through the outer and the middle ear, which also shows the inner ear and the cochlear nerve Reproduced with permission from Davis (1959 p 566)

of the middle ear system. This vibration is communicated through the stapes and the oval window to the inner ear and, in particular, to the cochlea. The cochlear spiral is the last station of the sound-transmitting channel of the ear. Its mechanical properties have a considerable effect on the action of the middle ear which, in turn, controls the wave reflection at the eardrum. For this reason, the analysis has to begin with the cochlea and proceed backwards.

2.2 Sound Transmission in the Cochlea

The cochlea is a spiral canal of two and one-half turns which is filled with liquid and surrounded by hard, bony walls. The canal is divided lengthwise into the scala vestibuli and the scala tympani. The scala vestibuli is connected to the vestibular apparatus of the ear and may be considered to begin at the oval window. The scala tympani begins at the round window and ends at the apex of the cochlea in a small opening providing a passage to the scala vestibuli (Fig 7). The oval window is filled almost completely by the footplate of the stapes, the last ossicle in the ossicular chain of the middle ear. The footplate is attached to the perimeter of the oval window by a flexible ligament. The round window

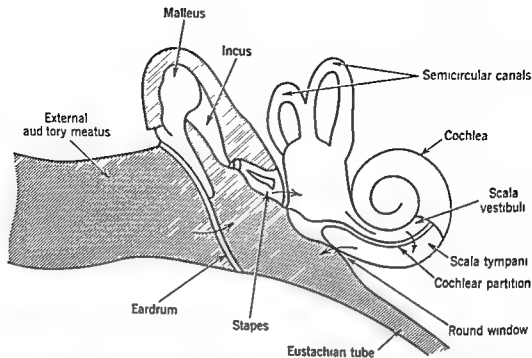


Fig 7 A schematic drawing of sound transmitting structures of the middle and inner ear Adapted with permission from v Békésy (1960a p 11)

opens into the tympanic cavity and is separated from it by a thin membrane. The cochlear partition itself consists in part of a bony plate, the lamina spiralis, and in part, of an almost triangular canal, the scala media (Fig 8). The canal is filled with endolymph and surrounded by two flexible walls, the Reissner's membrane and the basilar membrane, and by the rigid outer wall of the cochlea. The basilar membrane supports the organ of Corti containing the sensory hair cells. According to v Békésy, its mechanical properties control the sound transmission in the cochlea to a considerable extent. The Reissner's membrane is very thin and its mechanical effect appears to be negligible.

For a mathematical treatment of sound transmission in the cochlea, we unroll the spiral and place the x axis along the cochlear canals in the plane of the basilar membrane (Fig 9). With the help of v Békésy's observations, we can assume that the waves that are propagated along the cochlear canals are long compared to the linear cross-sectional dimensions of the canals (v Békésy, 1947, 1960). Under these conditions the problem may be reduced to a linear one, and there is no need for the remaining two orthogonal space coordinates (Zwislocki, 1946, 1948, 1950, 1953).

When, because of sound pressure at the eardrum, the stapes is pressed into the cochlea, it compresses the perilymph in the scala vestibuli and in the scala tympani. The latter occurs through the deformation of the cochlear partition. This compression is equalized almost completely by

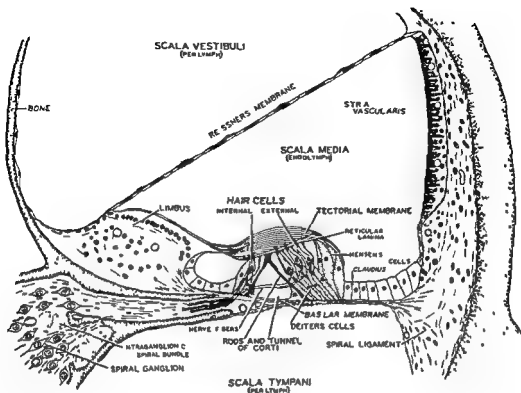


Fig 8 Cross section of the cochlear partition of the guinea pig Reproduced with permission from Davis (1959, p 567)

the outward motion of the round window membrane. Fletcher (1951, 1953) has shown that the remaining compression can be neglected and the cochlear fluids regarded as incompressible. Let us now take a volume element of length dx and of a cross-sectional area of the scala vestibuli. Applying the continuity equation for incompressible fluids ($dp/dt = 0$) (Lamb, 1945, Lindsay, 1960), we see that

$$\frac{\partial(\bar{u}_0 S_0)}{\partial x} dx = v dx, \quad (74)$$

where v means the volume velocity of the cochlear partition per unit

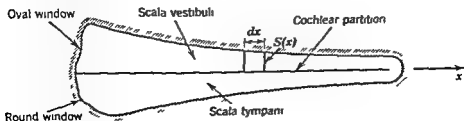


Fig 9 A schematic drawing of an unrolled cochlea in a longitudinal section

length, and \bar{u}_v is the average particle velocity in x direction, i.e.,

$$\bar{u}_v = \frac{1}{S_v} \int u_v dS$$

For small \bar{u} and small $\partial S_v / \partial x$, and neglecting second-order small terms, Eq. 74 can be rewritten as follows

$$\frac{\partial(\bar{u}_v S_v)}{\partial x} = \frac{\partial \bar{u}_v}{\partial x} S_v + \frac{\partial S_v}{\partial x} \bar{u}_v \approx \frac{\partial \bar{u}_v}{\partial x} S_v \quad (75)$$

Similarly, we have for the scala tympani

$$S_t \frac{\partial \bar{u}_t}{\partial x} = -v \quad (76)$$

The sign in front of v in Eq. 76 is reversed because the fluid in the scala tympani moves in the opposite direction from that in the scala vestibuli.

Assuming a sound pressure p_v in the scala vestibuli and p_t in the scala tympani, the pressure difference amounts to

$$p = p_v - p_t \quad (77)$$

This pressure difference must be equal to the acoustic impedance of the partition per unit length multiplied by the volume velocity v

$$p = Z_p v \quad (78)$$

The force acting on an element of unit length in the x direction amounts to

$$-S_v \frac{\partial p_v}{\partial x} = \rho S_v \frac{\partial u_v}{\partial t} + R_{fv} S_v \bar{u}_v \quad (79)$$

in the scala vestibuli, and to

$$-S_t \frac{\partial p_t}{\partial x} = \rho S_t \frac{\partial \bar{u}_t}{\partial t} + R_{ft} S_t \bar{u}_t \quad (80)$$

in the scala tympani. In both equations, ρ is the density of the perilymph and R_f is the resistance per unit length of the canal. Differentiating Eqs. 79 and 80 with respect to x and neglecting small terms of the second order, we obtain

$$\begin{aligned} S_v \frac{\partial^2 p_v}{\partial x^2} &= -\rho S_v \frac{\partial^2 \bar{u}_v}{\partial x \partial t} - R_{fv} S_v \frac{\partial u_v}{\partial x}, \\ S_t \frac{\partial^2 p_t}{\partial x^2} &= -\rho S_t \frac{\partial^2 \bar{u}_t}{\partial x \partial t} - R_{ft} S_t \frac{\partial \bar{u}_t}{\partial x}, \end{aligned} \quad (81)$$

and substituting $+v$ for $S_v(\partial \bar{u}_v/\partial x)$ and $-v$ for $S_t(\partial \bar{u}_t/\partial x)$,

$$\begin{aligned} S_v \frac{\partial^2 p_v}{\partial x^2} &= -\rho \frac{\partial v}{\partial t} - R_{fv}v, \\ S_t \frac{\partial^2 p_t}{\partial x^2} &= +\rho \frac{\partial v}{\partial t} + R_{ft}v \end{aligned} \quad (82)$$

Because of the relation given in Eq 77, it is possible to sum Eqs 82

$$\frac{\partial^2 p}{\partial x^2} = \frac{\partial^2 p_v}{\partial x^2} - \frac{\partial^2 p_t}{\partial x^2} = -\left(\frac{1}{S_v} + \frac{1}{S_t}\right)\rho \frac{\partial v}{\partial t} - \left(\frac{R_{fv}}{S_v} + \frac{R_{ft}}{S_t}\right)v \quad (83)$$

Substitution for v from Eq 78 finally produces

$$\frac{\partial^2 p}{\partial x^2} = -\left(\frac{1}{S_v} + \frac{1}{S_t}\right)\frac{\rho}{Z_p} \frac{\partial p}{\partial t} - \left(\frac{R_{fv}}{S_v} + \frac{R_{ft}}{S_t}\right)\frac{1}{Z_p} p, \quad (84)$$

the desired differential equation in terms of p (Zwislocki, 1946, 1948)

Equation 84 can be abbreviated by introducing $S = S_v S_t / (S_v + S_t)$, and $R_f/S = (R_{fv} S_t + R_{ft} S_v) / S_v S_t$, and simplified by restricting it to simple harmonic functions of time

$$\frac{\partial^2 p}{\partial x^2} = -\frac{R_f + j\omega\rho}{SZ_p} p \quad (85)$$

In order to determine the vibration pattern of the cochlear partition and have an insight into the stimulus conditions at the sensory cells, it is necessary to solve the differential equation, Eq 85. Because of the dependence of S , R_f , and Z_p on x , no general solution is available, and so, before proceeding any further, these parameters must be expressed as functions of x .

Only meager data exist with respect to the cross sectional area of the cochlea. Figure 10 shows this area, that is, the area of the scala vestibuli plus that of scala tympani, as derived from graphs of three cochleas published by E. G. Wever (1949) and of one cochlea measured by the author (Zwislocki, 1948). The cross sectional area is relatively large near the oval window and decreases to about one-fifth of that value toward the helicotrema. The functional relationship can be described by the equation

$$S_v + S_t = 5 \times 10^{-2} e^{-0.5x}, \quad (86)$$

which produces the straight line in Fig 10. The approximation is not very accurate, but the events in the cochlea depend only on the square root of the cross-sectional area, so that the effective error remains small. The area

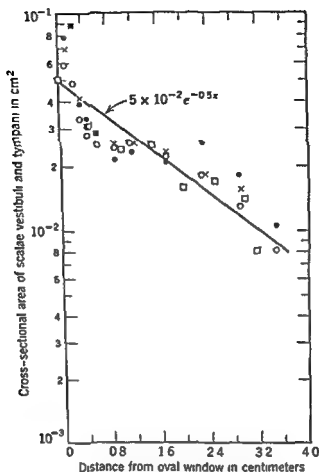


Fig 10 Cross sectional area of the scalae vestibuli and tympani as a function of the distance from the oval window. The points indicate two series of measurements, the straight line is a mathematical approximation

S of Eq 85 follows from $S = S_v S_t / (S_v + S_t)$ or, accepting that to the first order of approximation $S_v \cong S_t$, $S = (S_v + S_t) / 4$. With the help of Eq 86 we obtain

$$S = 1.25 \times 10^{-2} e^{-0.52x} = S_0 e^{-ax} \quad (87)$$

The resistance R_v , which is due to the viscosity of the fluid, can be derived from equations of fluid motion in circular pipes. Unfortunately, the complete derivation is rather long, and we shall have to content ourselves with the final results (Kinsler & Frey, 1950, Olson, 1957). For conditions encountered in the cochlea, that is, for a tube of radius $a > 10\sqrt{\mu/\rho\omega}$, where μ is the coefficient of viscosity, the resistance per unit length amounts to

$$R = \frac{\sqrt{2\mu\rho\omega}}{a}, \quad (88)$$

which can be expressed in terms of cross-sectional area by introducing $S = \pi a^2$.

$$R = \sqrt{\frac{2\pi\mu\rho\omega}{S}} \quad (89)$$

Maintaining the approximation $S_v \cong S_t$, which leads to $R_{fv} \cong R_{ft}$, we obtain from the definition of R_f in Eq. 85

$$R_f = S \frac{R_{fv}S_t + R_{ft}S_v}{S_vS_t} \cong \frac{S_t}{2} \frac{2R_{ft}S_t}{S_t^2} = R_{ft} \cong R_{fv} \quad (90)$$

Introducing $S^* = (S_v + S_t)/2$ as the best approximation of the cross-sectional area, we obtain from Eq. 89

$$R_f = \sqrt{\frac{2\pi\mu\rho\omega}{(S_v + S_t)/2}} = r\omega^{1/2}e^{ax/2} \quad (91)$$

The numerical value for the coefficient of viscosity was found by v. Békésy (1942, 1960) to amount approximately to $2 \cdot 10^{-2}$ cgs units, the density of the perilymph is very nearly that of water, i.e., $\rho = 1$ gram/cm³. With these values, and in agreement with Eq. 86, $r = 2.24$ cgs units and $\alpha = 0.5$ cm⁻¹.

The remaining parameter to be determined is the acoustic impedance of the cochlear partition Z_p . It can be expressed in general terms by the formula

$$Z_p = R_p + j\left(\omega M + \frac{1}{\omega C}\right), \quad (92)$$

where R_p , M , and C mean respectively the acoustic resistance, mass, and compliance per unit length. It is possible to show that the mass M is negligible in those portions of the cochlea where the vibration amplitude is substantial (Ranke, 1942, Zwislocki, 1948). The compliance C can be derived from a static measurement performed by v. Békésy (1941). He loaded the cochlear partition over its entire length with a pressure equivalent to a column of 1 cm of water and measured its displacement. The empirical results can be closely approximated by the function

$$C = C_0 e^{\beta x}, \quad (93)$$

where $C_0 = 4 \times 10^{-10}$ cm⁴/dyne and $\beta = 1.5$ cm⁻¹.

The agreement between the empirical and theoretical values is shown in Fig. 11. v. Békésy (1947, 1960) later performed another series of experiments in which he measured the compliance of the basilar membrane and of other structures by pressing small hairs against their surface. Since in these experiments the distribution of forces was quite different

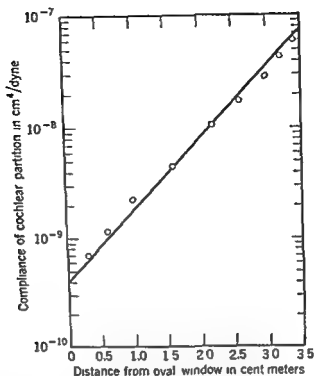


Fig 11 Acoustic compliance per unit length of the cochlear partition as a function of the distance from the oval window. The points indicate v Békésy's measurements, and the curve is a mathematical approximation.

from that produced by the inner ear fluid, their results are inadequate for calculating acoustic events in the cochlea.

The compliance of the cochlear partition, as measured by means of fluid pressure, is low near the oval window and increases by a factor of over 100 toward the helicotrema. It leads to the characteristic wave pattern observed by v Békésy (1928, 1942, 1943, 1947, 1960).

There are no direct data for the acoustic resistance R_p . Consequently, its numerical value must be adjusted so that the theoretical computations agree with v Békésy's dynamic measurements. We shall assume that the value of R_p is independent of x , in agreement with v Békésy's (1943, 1960) finding that the logarithmic decrement of transients in the cochlea is the same over the whole length of the partition, with the exception of the apical portion, that is, near the helicotrema.

After introducing the expressions derived for the parameters R_p , S , and Z_p , Eq. 85 becomes

$$\frac{\partial^2 p}{\partial x^2} = - \frac{r\omega^{1/2}e^{(a/2)x} - j\omega\rho}{S_0 e^{-ax}(R_p - j(1/\omega C_0)e^{-\beta x})} p \quad (94)$$

Before Eq. 94 can be solved analytically, it has to be simplified. It is possible to deduce from v Békésy's experiments that $R_p \ll 1/\omega C$ in those

portions of the cochlea where the amplitude of vibration is substantial. Consequently, we can approximate the expression $1/(R_p + 1/j\omega C)$ by $j\omega C + \omega^2 C^2 R_p$ and rewrite Eq. 94 in the form

$$\frac{\partial^2 p}{\partial x^2} = - \frac{(R_f + j\omega\rho)(\omega^2 C^2 R_p + j\omega C)}{S} p,$$

or regrouping the terms

$$\frac{\partial^2 p}{\partial x^2} = \left[(1 - CR_p R_f) - j \left(\omega CR_p + \frac{R_f}{\omega\rho} \right) \right] \frac{\omega^2 \rho C p}{S} \quad (95)$$

In this equation, $CR_p R_f \ll 1$ for all values of ω and x that are of interest, so that we can simplify it further to

$$\frac{\partial^2 p}{\partial x^2} = \left[1 - j \left(\omega CR_p + \frac{R_f}{\omega\rho} \right) \right] \frac{\omega^2 \rho C p}{S} \quad (96)$$

The last equation can be solved rather easily if we are allowed to consider the expression in the bracket as independent of x (Zwislocki, 1946, 1948). This is an acceptable approximation in view of the fact that the variable expression $(\omega CR_p + R_f/\omega\rho) \ll 1$ for all combinations of values of ω and x that produce a substantial vibration amplitude. We can introduce, therefore,

$$K = 1 - j \left(\omega CR_p + \frac{R_f}{\omega\rho} \right), \quad (97)$$

and attempt to solve the simplified equation

$$\frac{\partial^2 p}{\partial x^2} = \frac{K\omega^2 \rho C_0 e^{(\beta+\alpha)x}}{S_0} p \quad (98)$$

After some formal transformations, it is found that the solution can be written as a Hankel function of the zero order (Jahnke & Emde, 1945). Hankel functions belong to the class of Bessel functions, and they form complex conjugate pairs, $H^{(1)}$ and $H^{(2)}$. The function $H^{(1)}$ converges to zero for infinite complex arguments when the imaginary part is positive, the function $H^{(2)}$ does the same for negative imaginary terms. Since there seems to be no noticeable reflection at the apex of the cochlea and the wave amplitude decays to zero for large x , $H_0^{(2)}$ is the Hankel function that satisfies the boundary conditions. Consequently, the appropriate solution of Eq. 98 is

$$p = A' H_0^{(2)} \left[\frac{2K_0^{1/4} \omega \rho^{1/4} C_0^{1/4}}{(\alpha + \beta) S_0^{1/4}} e^{(\alpha+\beta)x/2} \right] \quad (99)$$

In order to evaluate the argument of the Hankel function, we first investigate the complex term $K^{1/4}$. Using the definition given by Eq 97 and remembering that

$$\left(\omega CR_p + \frac{R_l}{\omega \rho}\right)^2 \ll 1$$

for the combinations of values of ω and x that are of interest, we can write to a close approximation

$$K^{1/4} \cong 1 - j \frac{1}{2} \left(\omega R_p C_0 e^{\beta x} + \frac{r}{\omega^{1/4} \rho} e^{\alpha x/2} \right) \quad (100)$$

Equation 99 can now be written in the form

$$p = A H^{(2)} \left\{ Q \omega \left[1 - j \frac{1}{2} \left(\omega R_p C_0 e^{\beta x} + \frac{r}{\omega^{1/4} \rho} e^{\alpha x/2} \right) \right] e^{(x+\beta)x/2} \right\}, \quad (101)$$

with

$$Q = \frac{2\rho^{1/4} C_0^{1/4}}{(\alpha + \beta) S_0^{1/4}}$$

It describes pressure waves that are propagated from the oval window toward the helicotrema with a monotonically decreasing amplitude and wavelength (Zwislocki, 1948). This becomes more evident when Eq 101 is approximated by an exponential function (Jahnke & Emde, 1945)

$$p = A \exp \left[-\frac{(\alpha + \beta)x}{4} - \frac{1}{2} Q \omega^2 R_p C_0 e^{(\alpha+\beta)x/2} - \frac{1}{2} Q \frac{r}{\rho} \omega^{1/4} e^{(2\alpha+\beta)x/2} \right] \exp [-j Q \omega e^{(\alpha+\beta)x/2}] \quad (102)$$

The approximation holds for values of the argument greater than one, which, in fact, prevail in the cochlea.

The vibration pattern of the scala media is found by dividing Eq 102 by $j\omega Z_p$ (Zwislocki, 1948). We obtain

$$\eta = \frac{p}{j\omega Z_p} \cong \eta_0 \exp \left[\frac{(3\beta - \alpha)x}{4} - \frac{1}{2} Q \omega^2 R_p C_0 e^{(\alpha+\beta)x/2} - \frac{1}{2} Q \frac{r}{\rho} \omega^{1/4} e^{(2\alpha+\beta)x/2} \right] \exp [-j Q \omega e^{(\alpha+\beta)x/2}] \quad (103)$$

The last equation indicates that the displacement of the partition is approximately in phase with the pressure difference between the scala vestibuli and scala tympani. Its amplitude does not decay monotonically toward the helicotrema, however, but reaches a maximum at a location along the scala media which depends on the frequency. This result agrees with v. Békésy's (1943, 1947, 1960) visual observations. The

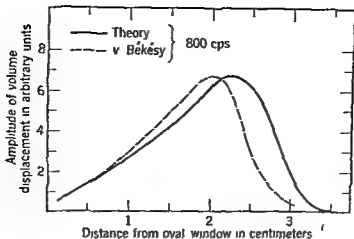


Fig 12 Amplitude distribution along the cochlear partition for an 800 cps tone

theoretical amplitude distribution for an 800 cps tone is compared to that recorded by v Békésy (1943, 1960) on one inner-ear specimen in Fig 12. The agreement between the two curves with respect to both the location of the maximum and the shape lies within the range of individual variability that can be inferred from v Békésy's data.

The vibration maximum occurs where $dn/dx = 0$. Its locus is plotted in Fig 13 together with three series of v Békésy's (1942, 1943, 1947, 1960) measurements. A further validation of the theory as well as of v Békésy's measurements on ear preparations is provided by the correlation between the frequency of maximum hearing loss and the locus of maximum damage.

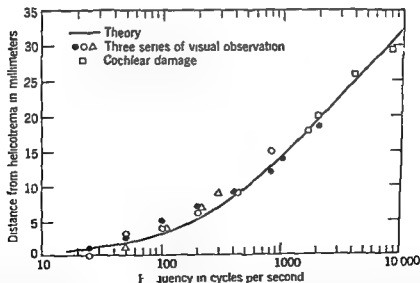


Fig 13 Location of the vibration maximum along the cochlear partition as a function of sound frequency. The points indicate two kinds of experimental data, the curve is theoretical.

to the cochlear structures. Evidence to this effect was obtained by Crowe, Guild, and Polvogt (cit Wever, 1949). The position of the open rectangles in Fig 13 was inferred from their data. The agreement between the two series of empirical results and the theory appears quite satisfactory for the present state of the art.

When calculating the locus of maximum vibration at low frequencies, a small modification has to be made in the resistance R_r in order to obtain results that agree with those of v. Bekésy. It seems that the resistance R_r does not vary with the square root of frequency, as indicated by Eqs 89 and 91, but is somewhat less. Since it plays a secondary role in the mechanical events taking place in the cochlea, it was simply assumed to be independent of frequency and equal to

$$R_r = 124e^{2x/2} \quad (104)$$

An intermediate formulation would have given a still better approximation to the empirical data.

There are three additional parameters of wave motion in the cochlea that are relevant to the analysis of auditory characteristics: the phase of displacement, the phase (wave) velocity, and the input impedance of the cochlea.

The phase follows from the last exponential term of Eq 103,

$$\theta = Q\omega e^{(\alpha+\beta)x/2} \quad (105)$$

Phase relationships at the vibration maximum have been considered as a possible factor in the frequency analysis performed by the ear. The phase at the locus of vibration maximum can be calculated by substituting for ω and x in Eq 105 pairs of values that satisfy the condition $d\eta/dx = 0$. The phase angle obtained in this way refers to the vibration of the cochlear partition at $x = 0$. In order to compare the theoretical results with v. Bekésy's (1947, 1960) observations, it is necessary to use the phase of stapes displacement as a reference. As will be explained later, the displacement of the stapes lags behind the cochlear partition by $\pi/2$ at $x = 0$.

The phase angle at the locus of maximum vibration relative to the phase angle of the stapes motion is plotted in Fig 14. Closed circles indicate v. Bekésy's observations. In view of experimental difficulties and the dependence of the phase on the numerical values of several parameters, the agreement must be considered satisfactory. It is particularly noteworthy that the theory as well as the experiments indicate a slightly increasing phase with frequency. Although such an increase could easily be avoided by a moderate alteration of certain parameters, this fact in itself indicates that no particular analytical function should be assigned to the phase at the vibration maximum.

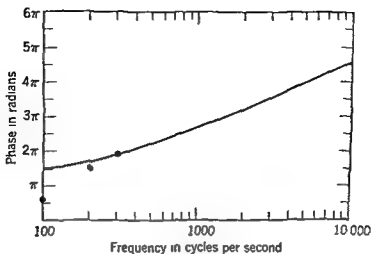


Fig 14 Phase angle at the locus of maximum vibration along the cochlear partition as a function of sound frequency. The phase angle is plotted relative to the phase angle of the stapes. Points indicate values obtained by v. Békésy, the curve is theoretical.

The phase velocity is of relevance to localization experiments. It can be calculated from the imaginary exponent in Eq 103 by introducing the time variable according to the definition $p = Pe^{j\omega t}$. It follows that

$$\theta_t = \omega t - \omega Q e^{(\alpha+\beta)x/2}, \quad (106)$$

and, after differentiation with respect to t while keeping the phase constant,

$$0 = \omega - \frac{\alpha + \beta}{2} \omega Q e^{(\alpha+\beta)x/2} \frac{dx}{dt} \quad (107)$$

Since, by definition, dx/dt is the phase velocity, we obtain by rewriting Eq 107,

$$c = \frac{dx}{dt} = \frac{2}{(\alpha + \beta)Q} e^{-(\alpha+\beta)x/2} \quad (108)$$

Note that, according to Eq 108, the phase velocity is independent of frequency, which agrees with model experiments of Tonndorf (1957). A more exact calculation would reveal some frequency dependence, but the effect is very small except for the lowest frequencies. The calculated numerical values are shown by the curve of Fig 15. They are in excellent agreement with the values derived from v. Békésy's (1947, 1960) measurements, indicated by solid circles. Physiological measurements on guinea pigs (Tasaki, Davis, & Legoux, 1952) are also in support of the theoretical curve.

The specific input impedance of the cochlea is by definition

$$Z_c = \frac{p}{u}. \quad (109)$$

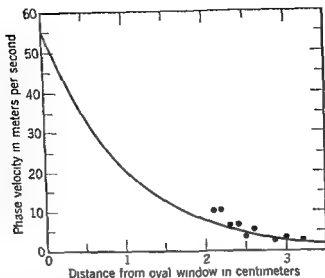


Fig 15 Phase velocity of the transversal waves in the cochlea, plotted as a function of the distance from the oval window. The points indicate values derived from v. Békésy's experiments, the curve is theoretical.

In this formula, the pressure p follows from Eq. 102. The velocity u along the cochlear partition has to be determined by using Eqs. 79 and 80. Neglecting the small resistive term and adding both equations so as to obtain the total pressure gradient, we obtain

$$\frac{\partial p}{\partial x} = -2\rho \frac{\partial \bar{u}}{\partial t} \quad (110)$$

on the assumption that $S_t \cong S_v$ or, with $p = Pe^{j\omega t}$ and $\bar{u} = \bar{U}e^{j\omega t}$,

$$\frac{dP}{dx} = -j2\rho\omega\bar{U} \quad (111)$$

The pressure gradient follows from Eq. 99

$$\frac{dP}{dx} = -\omega \frac{K^{1/2} \rho^{1/2} C_0^{1/2}}{S_0^{1/2}} e^{(a+\beta)x/2} A' H_1^{(2)}, \quad (112)$$

or, since $H_1^{(2)} \cong jH_0^{(2)}$ for argument values greater than one,

$$\frac{dP}{dx} \cong -j\omega \frac{K^{1/2} \rho^{1/2} C_0^{1/2}}{S_0^{1/2}} e^{(a+\beta)x/2} P \quad (113)$$

With Eq. 113, the expression for the velocity becomes

$$U = \frac{K^{1/2} C_0^{1/2}}{2\rho^{1/2} S_0^{1/2}} e^{(a+\beta)x/2} P, \quad (114)$$

and that for the input impedance ($x = 0$),

$$Z_e = \frac{2\rho^{\frac{1}{2}}S_0^{\frac{1}{2}}}{K^{\frac{1}{2}}C_0^{\frac{1}{2}}} \quad (115)$$

Except for very low and very high frequencies, $K^{\frac{1}{2}} \cong 1$, so that

$$Z_e \cong \frac{\rho^{\frac{1}{2}}S_0^{\frac{1}{2}}}{C_0^{\frac{1}{2}}} \cong 11,000 \text{ cgs units} \quad (116)$$

Thus, for most practical purposes, the specific input impedance of the cochlea is independent of frequency and is real. It has a rather high numerical value.

The nature of the cochlear input impedance has important consequences for sound transmission in the ear. First, it shows that sound pressure generated in the vicinity of the oval window is in phase with the velocity of the stapedial motion, and it precedes the stapedial displacement by 90° . We have accordingly

$$\chi = \frac{\bar{U}}{j\omega}, \quad (117)$$

so that

$$P = \frac{\bar{U}}{Z_e} = j\omega \frac{\chi}{Z_e} \quad (118)$$

We note that the sound pressure across the cochlear partition increases in direct proportion to frequency when the amplitude of the stapes displacement is kept constant.

From the point of view of the transmission characteristic of the ear, it is of interest to determine the maximum amplitude of vibration of the cochlear partition as a function of frequency. This is particularly so when we assume that the vibration maximum controls the neural excitation in the vicinity of the threshold of audibility. The desired relation may be found from Eq. 103 by substituting for x the frequency expression that satisfies the condition $d\eta/dx = 0$. By neglecting the insignificant contribution of R_p , this expression can be written in the form

$$x = \frac{2}{\alpha + 3\beta} \{ \ln(3\beta - \alpha) - \ln[(\alpha + 3\beta)QR_pC_0] - 2 \ln \omega \} \quad (119)$$

Substituted in Eq. 103, it produces after elementary simplifications

$$\hat{\eta}_{\max} = B\beta f^{-0.4}, \quad (120)$$

where B includes all constant terms except the pressure amplitude. Now, applying Eq. 118, we obtain

$$\hat{\eta}_{\max} = \frac{B}{|Z_e|} \hat{\chi} f^{0.6}, \quad (121)$$

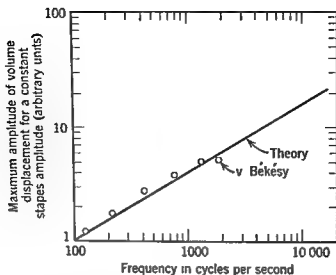


Fig 16 Magnitude of the maximum vibration of the cochlear partition for a constant amplitude of the stapes, plotted as a function of sound frequency. The points are experimental, the curve is theoretical.

the maximum amplitude of the volume displacement of the cochlear partition for a constant displacement of the stapes. According to Eq. 121, this amplitude increases with the 0.6 power of frequency. Figure 16 demonstrates that v. Békésy (1943, 1960) obtained a similar result by direct observation.

The preceding analysis shows that we are able to deal reasonably well with the mechanical events in the cochlea. Given a specified input, for instance, the displacement amplitude of the stapes, it is possible to calculate the vibration of the cochlear partition, which probably constitutes the direct mechanical stimulus for the sensory cells. We now shall attempt to calculate the motion of the stapes for a given sound pressure at the eardrum.

2.3 Sound Transmission in the Middle Ear

The main acoustic function of the middle ear appears to be to adapt the high input impedance of the cochlea to the low specific impedance of air. In this way the transfer of acoustic energy is improved (Helmholtz, *cit.* Wever & Lawrence, 1954; v. Békésy, 1941).

The average acoustic energy per unit volume of a plane wave amounts to

$$E = \frac{\bar{p} \bar{U}}{2c}, \quad (122)$$

where \bar{p} is the pressure amplitude and \bar{U} the amplitude of particle velocity. In most acoustic problems, the quantity introduced is the average rate of

energy flow through a unit area normal to the direction of wave propagation. Such energy flow is called *intensity* and is defined by the formula

$$I = \frac{\bar{p}^2 \bar{U}^2}{2}, \quad (123)$$

which, because of Eq. 62, can also be written in the form

$$I = \frac{\bar{p}^2}{2\rho c} \quad (124)$$

When a plane wave encounters a discontinuity where the impedance changes from Z_1 to Z_2 , part of its energy is reflected and the ratio of sound pressure of the reflected and incident waves follows from Eq. 64. Squaring this equation, we obtain the ratio of the reflected and incident intensities

$$\frac{I_r}{I_i} = \left(\frac{Z_2 - Z_1}{Z_2 + Z_1} \right)^2 \quad (125)$$

The transmitted intensity is

$$I_t = I_i - I_r = I_i \frac{4Z_2/Z_1}{(1 + Z_2/Z_1)^2} \quad (126)$$

It is equal to the incident energy when $Z_2 = Z_1$ and decreases as the impedance ratio either increases or decreases. Ideally, the specific impedance measured at the eardrum should be equal to the specific impedance of air for plane waves.

The impedance transformation in the middle ear is achieved by a high ratio between the surface areas of the eardrum and of the oval window and by a lever ratio of the ossicular chain. The situation is schematized in Fig. 17 where P_d and A_d indicate the sound pressure and the area of the eardrum, respectively, P_o and A_o , the corresponding parameters of the oval window, l_m , the lever of the malleus, and l_i , the lever of the incus. The system is in equilibrium when the acting moments are in balance, that is, when

$$P_d A_d l_m = P_o A_o l_i \quad (127)$$

The average velocity of motion of the eardrum is related to the average velocity of the stapes by the equation

$$\frac{\bar{U}_d}{l_m} = \frac{\bar{U}_o}{l_i} \quad (128)$$

As a consequence, the ratio of impedances amounts to

$$\frac{Z_d}{Z_o} = \frac{P_d}{P_o} \frac{\bar{U}_o}{\bar{U}_d} = \frac{A_o}{A_d} \frac{l_i^2}{l_m^2}. \quad (129)$$

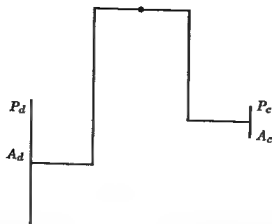


Fig 17 Schematic representation of the mechanical transformer of the middle ear. The symbols P_d and A_d indicate the sound pressure at the eardrum and the effective area of the eardrum respectively. P_c and A_c indicate the corresponding parameters at the entrance to the cochlea.

According to v. Bekésy and Rosenblith (1951), the effective area of the eardrum amounts to 0.55 cm^2 , the effective area of the oval window, which is equal to the surface area of the stapes footplate, 0.032 cm^2 , and the lever ratio $l_m/l_s = 1.3$. With these numerical values

$$\frac{Z_d}{Z_c} = 0.0345,$$

and the input impedance of the inner ear⁴ is reduced from the theoretical value of 5600 cgs units to 193 cgs units. The specific impedance of air is still $4\frac{1}{2}$ times smaller, so that even in this idealized situation 40% of incident acoustic intensity would be reflected from the eardrum. This figure, by the way, is very small for an acoustic system.

In reality, the situation is much more complex. The middle ear adds its own impedance to that of the cochlea and introduces a transmission loss through imperfect coupling.

The acoustically important parts of the middle ear are the eardrum, the ossicular chain, and the middle ear cavities. When the eardrum is displaced, the air in the cavities is compressed or rarefied so that the acoustic impedance of the cavities affects the eardrum motion. This impedance was measured by Onchi (1961) on one specimen of temporal bone, and was later corrected to fit average dimensions (Zwislocki, 1962). The motion of the eardrum is also imparted to the malleus of the ossicular chain, but only incompletely, since part of it is absorbed by the flexibility of the eardrum. Thus, there is a slight energy leak between the eardrum and the malleus. The incus appears to be rigidly attached to the malleus, so that no energy loss should occur at the junction between the two.

⁴ The ratio between the surface areas of the cross section at the entrance to the cochlea and the oval window decreases the cochlear impedance by a factor of two.

ossicles. However, some energy becomes absorbed by the ligaments and muscles holding the ossicular chain in place. A further energy leak occurs at the incudo-stapedial joint which appears to provide a considerable freedom of relative motion between the incus and the stapes (Onchi, 1949, 1961, Zwislocki, 1957, 1962).

On the basis of the anatomy (Fig 6) and of the consideration of energy flow, the middle ear system may be represented by the block diagram of Fig 18 (Zwislocki, 1962). With the help of electroacoustic and electro-mechanical analogies, the blocks may be considered as representing electrical networks with the impedances Z_1, Z_2 , etc. Then the input impedance of the system, which is equivalent to the acoustic impedance at the eardrum, follows from the equation

$$Z = \frac{Z_1 Z_2 Z_4 + Z_1 Z_2 Z_5 + Z_1 Z_3 Z_4 + Z_1 Z_3 Z_5 + Z_1 Z_4 Z_5 + Z_2 Z_3 Z_4 + Z_2 Z_3 Z_5 + Z_2 Z_4 Z_5}{Z_2 Z_4 + Z_2 Z_5 + Z_3 Z_4 + Z_3 Z_5 + Z_4 Z_5} \quad (130)$$

It is not difficult to see that an analysis of the middle ear function by means of Eq 130 is rather tedious, especially since each of the Z terms is a complex expression when written in terms of the circuit elements. It is easier to build an analog network and to match its characteristics to those of the middle ear. Figure 19 shows the result of such an analysis (Zwislocki, 1962). By comparing the groups of elements in Fig 19 to the position of the blocks in Fig 18, it is possible to see which parts of the middle ear they represent. The numerical values of the elements have been determined on the basis of anatomical measurements and of impedance measurements on normal and pathological ears. It may be of interest to note that some pathologies, like otosclerosis or interrupted

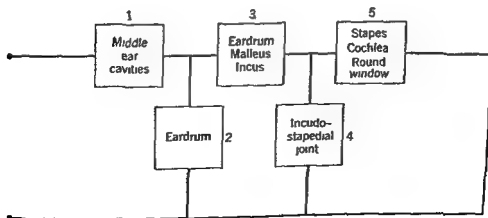


Fig 18 Block diagram of the middle ear system. Adapted with permission from Zwislocki (1962, p 1515)

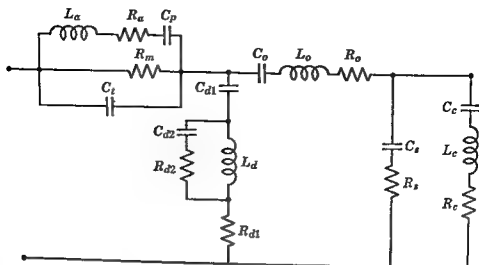


Fig 19 A network analog of the middle ear Adapted with permission from Zwislöcki (1962 p 1520)

incudostapedial joint, lead to a substantial simplification of the network (Zwislöcki, 1957, 1962)

Figures 20a and b compare the input impedance of the analog of Fig 19 to that measured at the eardrum of normal ears in several experimental series. The analog curves fall within the range of the experimental data. Empirical observations indicate that the slightly increased resistance at low frequencies is probably due to an artifact in measurements. The relative minima and maxima found in the analog characteristics in the vicinity of 2000 cps reflect resonances and antiresonances of the system. A similar pattern resulted from Onchi's (1961) measurements on ear preparations and is apparent in individual impedance curves determined on normal ears (Møller, 1961). In the average data shown in Figs 20a and b, the maxima and minima, whose location varies somewhat from person to person, have been smoothed out. The impedance values above 2000 cps are not known exactly. They have been measured in only one experimental series (Morton & Jones, 1956), and the difficulties of measurement increase rapidly above 1000 cps. Consequently, it will not be possible to test the analog at high frequencies until more empirical data are gathered. The analog agrees with the middle ear structurally as well as with respect to the numerical values of all its elements that could be checked out. It is highly probable, therefore, that the analog transmission characteristic closely approximates that of an average middle ear. This characteristic, expressed by the volume displacement of the stapes for a constant sound pressure at the eardrum, is plotted in Fig 21.

It has not been possible to measure the transmission characteristic on live human beings, so that the curve of Fig 21 is probably the best available

estimate Both v Bekesy (1942) and Onchi (1961) determined the transmission characteristic on ear preparations. However, more recent comparative impedance measurements have revealed that the mechanical properties of the middle ear change considerably after death. The post-mortem impedance at the eardrum is several times larger than in a living subject (Zwislocki & Feldman, 1963). Accordingly, v Bekesy's and Onchi's data indicate a transmission loss that is one order of magnitude

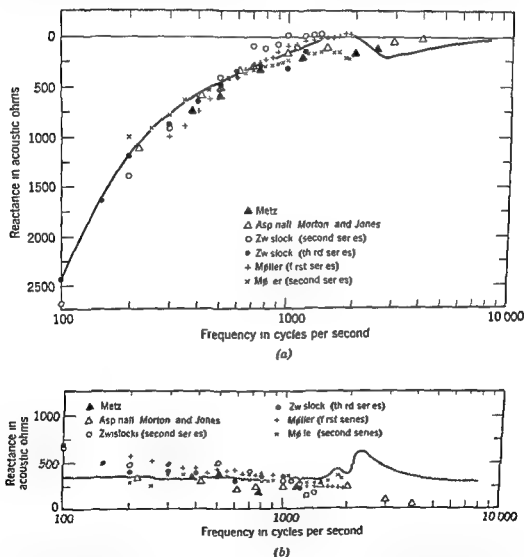


Fig 20(a) Acoustic reactance at the eardrum. The points indicate the average values obtained in several experimental series, the curve was obtained on the network analog. Adapted with permission from Zwislocki (1962, p. 1520). (b) Acoustic resistance at the eardrum. The points indicate the average values obtained in several experimental series, the curve was obtained on the network analog. Adapted with permission from Zwislocki (1962, p. 1520).

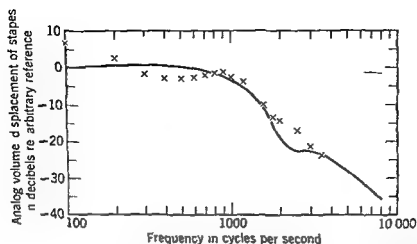


Fig 21 Transmission characteristic of the middle ear, that is the volume displacement of the stapes for a constant sound pressure at the eardrum plotted as a function of sound frequency. The crosses indicate data obtained on ear preparations, the curve was obtained on the middle ear analog.

larger than in the analog. Nevertheless, the shape of the transmission characteristic seems to remain reasonably well preserved, as indicated by crosses in Fig 21. Their position has been derived from Onchi's measurements.

In order to be able to compute the transmission characteristic of the whole ear, two more steps are necessary.

2.4 Transmission Characteristic of the Ear

The complete transmission characteristic of the ear can be obtained by adding the pressure transformation in the ear canal and the pressure change produced by sound diffraction at the head to the already calculated transmission characteristic of the cochlea and the middle ear.

The ear canal can be regarded as acoustically equivalent to a rigid tube of approximately 2.5 cm length and 0.7 cm diameter. Because of wave reflection at the eardrum, the total pressure at any distance x from the eardrum follows from Eq. 69

$$p_x = p_{ix}(1 + \beta e^{j(\theta - 2kx)}) \quad (131)$$

Here p_{ix} denotes the sound pressure of the incident wave and

$$\beta e^{j\theta} = \frac{Z_2 - Z_1}{Z_2 + Z_1}, \quad (132)$$

where Z_1 is the acoustic impedance of the ear canal in the absence of

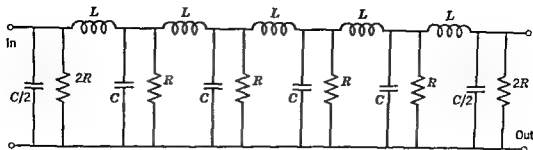


Fig 22 Network analog of the ear canal

reflections and Z_2 is the acoustic impedance at the eardrum. At the eardrum, the sound pressure amounts to

$$p_0 = p_{i0}(1 + \beta e^{i\theta}), \quad (133)$$

so that the pressure ratio between the eardrum and the entrance to the ear canal is

$$\frac{p_0}{p_i} = \frac{p_{i0}}{p_{i1}} \frac{(1 + \beta e^{i\theta})}{(1 + \beta e^{i(\theta - 2kl)})}, \quad (134)$$

or, since p_{i0} lags p_{i1} by kl ,

$$\frac{p_0}{p_i} = \frac{1 + \beta e^{i\theta}}{1 + \beta e^{i(\theta - 2kl)}} e^{-ikl} \quad (135)$$

The pressure transformation can either be calculated from Eq 135 or determined on an analog network. For the latter purpose the ear canal may be considered as a transmission line with a structure like that in Fig 22. The pressure transformation in the ear canal has been measured by Wiener and Ross (1946) using a probe microphone. Since the probe might have altered somewhat the acoustic conditions in the ear canal and since the network model is only an approximation of the real conditions, it is comforting to find that the analog results agree reasonably well with Wiener's and Ross's data, as shown in Fig 23.

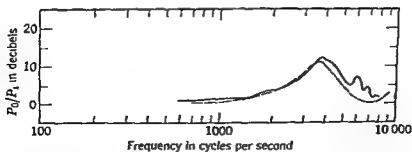


Fig 23 Sound pressure at the eardrum for a constant sound pressure at the entrance to the ear canal. The solid curve indicates average experimental results, the intermittent curve was obtained by means of the ear analogs.

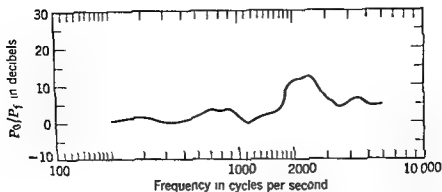


Fig 24 Sound pressure at the entrance to the ear canal for a constant pressure measured in a free field at the location of the center of a listener's head. Experimental data was obtained by Wiener and Ross. Adapted with permission from Wiener & Ross (1946, p 407)

We shall not concern ourselves here with a calculation of sound diffraction at the listener's head. The resulting change in sound pressure can be measured without great difficulty, and Fig 24 shows the corresponding data obtained by Wiener and Ross (1946).

The total transmission characteristic of the ear is a simple decibel sum of the four component characteristics of Figs 16, 21, 23, and 24 (see also Zwislocki, 1958). It is shown in a slightly modified form in Sec 4. With its help it is possible to transfer approximately the stimulus specification from the free field to the level of sensory cells. This transformation together with certain additional characteristics of the cochlea will help us to interpret some of the known psychoacoustic functions.

3 PITCH FUNCTION

Pitch is a psychological attribute of sound. It depends mainly on sound frequency, but it also varies with sound intensity and stimulus duration. The pitch produced by complex sounds does not necessarily correspond to a spectral frequency component, but it may depend on the periodicity of the envelope (Schouton, 1940). For such conditions, the term "periodicity pitch" has been proposed. In this section, attention is focused on the classical concept of pitch, that is, pitch that is associated with pure tones.

In everyday life, we order pitch on an ordinal scale from low to high. Stevens, Volkman, and Newman (1937) succeeded in constructing a numerical interval scale by means of a method of fractionation. They presented alternately tones of fixed and variable frequency, and asked the observers to adjust the variable stimulus until its pitch appeared

to be half the pitch of the fixed reference stimulus. They called the unit of pitch a "mel" and defined it by assigning a pitch of 1000 mels to a tone of 1000 cps at a loudness level of 60 db. Subsequently, Stevens and Volkman (1940) revised the original mel scale on the basis of fractionation and equisection experiments. The revised mel scale is plotted in Fig 25.

More recently, Stevens and Galanter (1957) have shown that, under certain precautions, methods of magnitude estimation and category judgments yield results that are consistent with the mel scale. Their magnitude estimation data are indicated by open circles in Fig 25. A similar conclusion was reached by Beck and Shaw (1962), who found that, depending on the reference standard, magnitude estimation can agree either with the original or with the revised mel scale.

Thus, within the range determined by the data of Fig 25, the pitch scale appears to be fairly well established. Its validity is strengthened by the simple relationship it bears to frequency jnd's, critical bands, and other auditory parameters (Licklider, 1951), as well as to anatomical and physiological findings (Stevens & Davis, 1938, Wever, 1949).

Out of a considerable number of jnd studies, those of Shower and Biddulph (1931) and of Zwicker (1952) are probably the most extensive. In both, pitch variation was produced by means of frequency modulation

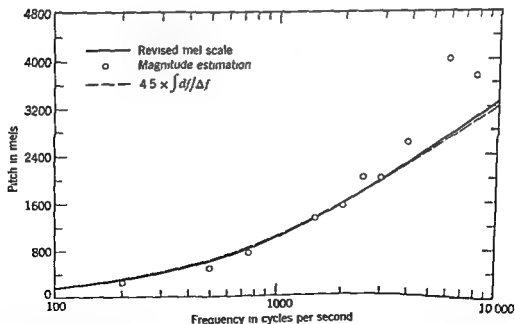


Fig 25 Pitch as a function of sound frequency for a loudness level of 60 db. The solid line shows results of fractionation and equisection, the circles are those of magnitude estimation, and the intermittent line resulted from an integration of frequency jnd's.

The simplest frequency modulated tone may be described by the equation

$$p = P \cos \left(2\pi f_0 t - \frac{\delta f}{f_m} \cos 2\pi f_m t \right), \quad (136)$$

where p is the sound pressure, P its amplitude, f_0 the center frequency, f_m frequency of modulation, and $2\delta f$ the frequency range. At slow modulation rates, a periodically varying pitch is perceived, at higher modulation rates, a steady, complex sound. The latter effect is produced as a result of spectrum analysis in the auditory system. When a frequency modulated signal is transmitted through a bank of narrow band filters, several fixed frequencies instead of a varying frequency can be detected at the output. For moderate δf and sufficiently high f_m , such that $\delta f/f_m \ll 1$, three predominating spectral components arise: f_0 , the center component, and two sidebands located at $(f_0 + f_m)$ and $(f_0 - f_m)$, respectively.

When the difference between the spectral frequencies, that is, $2f_m$, is sufficient, the auditory system behaves in some ways like a bank of filters, and it resolves the frequency modulated signal into its steady-state components. Under such conditions a just noticeable frequency difference can still be determined, although its magnitude is expressed as the amplitude of the sidebands. This amplitude is equal to $\delta f/2f_m$.

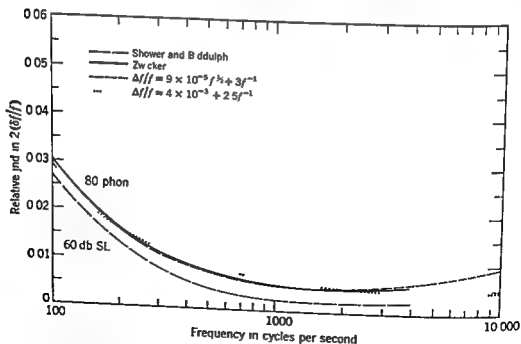


Fig. 26 Relative frequency jnd as a function of sound frequency. Two series of experimental data and two mathematical approximations.

Since, at low modulation rates, $2\delta f$ defines a frequency range and, at high modulation rates, $2f_m$ does the same, both can be used as different measures of the resolving power of the auditory system. First, Fig. 26 shows two series of jnd values plotted in terms of $2\delta f/f$ against center frequency. The Shower and Biddulph data were obtained at a sensation level of 60 db with a modulation rate $f_m = 2$ cps. At this modulation rate the jnd reaches a minimum. Zwicker's data were obtained at a sound pressure level of 80 db and a modulation rate of 4 cps. Their higher numerical values are probably due to the higher modulation rate. Otherwise, they appear to be consistent with those of Shower and Biddulph, and they show approximately the same functional relationship to the center frequency. At low frequencies, the absolute jnd appears to be approximately constant, at high frequencies, it is roughly proportional to frequency, and more exactly to its $3/2$ power. Zwicker's results can be described quite accurately by the equation

$$2\delta f = af^{3/2} + b, \quad (137)$$

or

$$2 \frac{\delta f}{f} = af^{1/2} + bf^{-1}, \quad (138)$$

where $a = 9 \cdot 10^{-5}$ cps $^{3/2}$ and $b = 3$ cps.

Equation 137 is characteristic not only for frequency but also for differential sensitivity in general. Its constant term indicates an absolute sensitivity maximum that persists up to a certain stimulus level. At higher levels, the sensitivity decreases, indicating a self-adapting system, that is, a system whose characteristics are controlled to a certain extent by the input signal. A stimulus-dependent sensitivity is essential in handling large dynamic ranges so typical of sensory organs.

The independent determination of the pitch scale and of the frequency jnd's provides an opportunity to test Fechner's hypothesis of the equal psychological magnitude of just noticeable increments. It is only necessary to compare the size of the jnd's to that of the mel steps. For this purpose, either the pitch function of Fig. 25 can be differentiated with respect to sound frequency or the jnd function of Fig. 26 can be integrated. Let us follow the classical procedure and integrate the jnd's (Stevens & Davis, 1938)

If $\Delta f = 2\delta f$ denotes the physical size of the jnd, then $n = 1/\Delta f$ is the number of jnd steps per cycle. There are

$$N = \int_{f_1}^{f_2} n \, df \quad (139)$$

steps between the frequency limits f_1, f_2 . Using Eq 137, we obtain

$$N = \int_{f_1}^{f_2} \frac{df}{af^{3/2} + b}, \quad (140)$$

The integration can be performed graphically, or analytically by approximation. In the first instance, it is convenient to plot $f/\Delta f$ against $\ln f$ and measure the area bounded by f_1, f_2 and the $f/\Delta f$ curve. That this is equivalent to the integral, Eq 139, results from the following consideration

$$d(\ln f) = \frac{1}{f} df,$$

consequently,

$$nf d(\ln f) = n df$$

In order to perform an analytical integration, we note that, for numerical values of Fig 26, the equation

$$\Delta f = kf + l, \quad (141)$$

with $k = 4 \cdot 10^{-3}$ and $l = 2.5$ cps, describes the jnd curve almost as well as Eq 137. Thus

$$N = \int_{f_1}^{f_2} \frac{df}{kf + l} = \frac{1}{k} \ln \frac{kf_2 + l}{kf_1 + l} \quad (142)$$

When the integration begins at $f = 0$, the number of jnd's as a function of frequency is

$$N = \frac{1}{k} \ln \frac{kf_2 + l}{l} = 2.5 \times 10^3 \ln (1.6 \times 10^{-3}f + 1) \quad (143)$$

The numerical values of N increased by a factor of 4.5 are plotted in Fig 25. The resulting curve coincides almost exactly with the mel scale, leading to the conclusion that the frequency jnd's have equal psychological magnitudes. More specifically,

$$1 \text{ jnd} \cong 4.5 \text{ mels}$$

at approximately 80 db SPL. The functional relationship between the jnd's and the mel scale holds approximately for all sensation levels, although the absolute size of jnd's changes (Stevens & Davis, 1938).

Although the psychologically equal size of frequency jnd's is of considerable theoretical interest, its importance should not be overestimated since the rule does not hold for differential sensitivity along other continua.

We next consider frequency modulation at such a high rate that pitch changes can no longer be perceived and the sensation becomes that of a

steady complex sound. For this purpose, Eq 136 can be expressed in terms of spectral components

$$p = P \cos 2\pi f_0 t + \frac{P}{2} \frac{\delta f}{f_m} \sin 2\pi(f_0 + f_m)t + \frac{P}{2} \frac{\delta f}{f_m} \sin 2\pi(f_0 - f_m)t \quad (144)$$

Zwicker (1952) measured the just noticeable amplitude ratio between the sidebands and the carrier for various sensation levels and carrier frequencies. Figure 27 shows his results for $f_0 = 1000$ cps. He also measured the just noticeable amplitude ratio for an amplitude-modulated tone as a function of modulation frequency and found that, at low frequencies, the auditory system is much more sensitive to amplitude modulation than to frequency modulation. At high modulation frequencies, the difference vanished, however (Fig 27). This is of considerable theoretical interest since, for small modulation amplitudes, the spectrum of an amplitude-modulated oscillation is the same as that of a frequency modulated one, except for the phase difference between the sidebands and the carrier. In a frequency modulated oscillation, the phase difference amounts to $\pi/2$, in an amplitude modulated oscillation, it is zero. The spectrum of the latter can be expressed as follows

$$p = P \cos 2\pi f_0 t + \frac{P}{2} m \cos 2\pi(f_0 + f_m)t + \frac{P}{2} m \cos 2\pi(f_0 - f_m)t \quad (145)$$

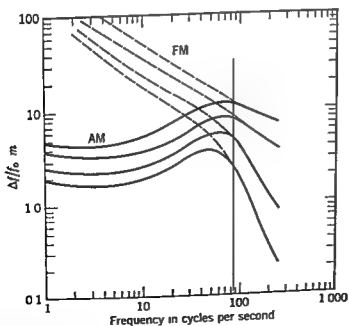


Fig 27 Just noticeable amplitude and frequency modulation as a function of modulation frequency for a 1000 cps carrier frequency. Adapted with permission from Zwicker (1952, p 130)

Since the spectral phase relation is the only differentiating parameter between the two kinds of oscillation, Zwicker concluded that the ear is phase sensitive for low modulation frequencies, that is, when the frequency difference between the sidebands is small. Beyond a certain critical spectrum width, the phase sensitivity disappears. A further investigation revealed that the critical spectrum width depends on the carrier frequency in the same way as do the frequency jnd's at low modulation rates and the size of mel steps. This means that the critical spectrum width is equal to a constant number of mels. It is possible to deduce from Zwicker's data that, for 1000 cps tones, this number is approximately 115 mels, independent of sensation level.

The critical spectrum width manifests itself in several kinds of auditory experiments and may be regarded as a fundamental concept in auditory theory (Feldkeller & Zwicker, 1956, Zwicker, Flottorp, & Stevens, 1957). The concept was introduced previously by Fletcher (1940) and called *critical band*. The absolute size of Fletcher's critical band resulted from postulated rather than from empirical data, although a direct measurement was attempted. For this reason, Zwicker, Flottorp, and Stevens (1957) suggested that it be called the *critical ratio* and that the name *critical band* be transformed to the more recently modified concept. Fletcher determined the size of the critical ratio as a function of sound frequency, and he found a similar relationship to that for the critical band. As a consequence, the quotient of the critical band and the critical ratio is practically independent of frequency, it amounts to approximately 2.5.

The similarity between pitch, frequency jnd, and critical band functions suggests the hypothesis that all three have a common physiological substrate. Helmholtz (1863) postulated that pitch is determined by the maximally excited nerve fibers and that this excitation is controlled by the locus of maximum vibration amplitude in the cochlea. The mechanism he invoked to explain the occurrence of such a maximum may have been inaccurate, but v. Bekésy (1942, 1943) has demonstrated that the maximum occurs and that its locus changes with frequency in the way Helmholtz predicted. For low frequencies, it is situated near the helicotrema, for high frequencies, in the vicinity of the oval window. Theories linking pitch to location along the cochlear canals are now called *place theories*.

Some simple postulates that may be proposed with respect to a place theory are (1) pitch is directly proportional to the distance between the vibration maximum and the helicotrema and (2) the size of mel steps, jnd's, and critical bands is inversely proportional to the derivative of the distance with respect to sound frequency.

In Sec. 1, the functional relationship between the locus of maximum vibration and sound frequency was derived theoretically on the basis of the

anatomy and the physical constants of the cochlea. It was shown that the obtained relation is consistent with empirical measurements. An expression was found that determines the amplitude of vibration η of the cochlear partition as a function of the distance x from the oval window (Eq 103). The maximum amplitude occurs where the distance derivative of η vanishes, $d\eta/dx = 0$. This condition leads to an expression of the form

$$x = A \ln \frac{(b^2 - 4acf^2)^{1/2} - b}{2af^2}, \quad (146)$$

where A , a , b , and c are positive constants depending on the physical parameters of the cochlea. In order to obtain the distance from the helicotrema, we introduce $x_h = l - x$, with l indicating the length of cochlear canals. Thus

$$x_h = l + A \ln \frac{2af^2}{(b^2 - 4acf^2)^{1/2} - b} \quad (147)$$

The first postulate is satisfied if Eq 147 is identical with Eq 143, which determines the number of jnd steps as a function of frequency. Clearly, this is not the case. The numerical difference between the mel scale and the x_h function may be derived from Fig 28. We must conclude that pitch is not directly proportional to the distance of the locus of maximum oscillation from the helicotrema. Still, the curves of Fig 28 are sufficiently similar to entertain a hypothesis of close correlation between the two variables. In order to clarify the situation somewhat, we may attempt to determine the size of jnd's or mel steps in terms of length intervals along the cochlear

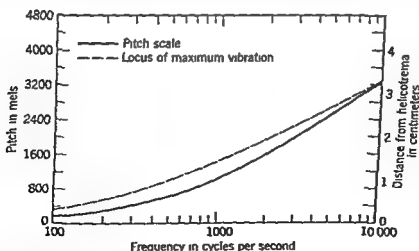


Fig 28 Pitch and the location of the vibration maximum along the cochlear partition as a function of sound frequency

Since the spectral phase relation is the only differentiating parameter between the two kinds of oscillation, Zwicker concluded that the ear is phase sensitive for low modulation frequencies, that is, when the frequency difference between the sidebands is small. Beyond a certain critical spectrum width, the phase sensitivity disappears. A further investigation revealed that the critical spectrum width depends on the carrier frequency in the same way as do the frequency jnd's at low modulation rates and the size of mel steps. This means that the critical spectrum width is equal to a constant number of mels. It is possible to deduce from Zwicker's data that, for 1000 cps tones, this number is approximately 115 mels, independent of sensation level.

The critical spectrum width manifests itself in several kinds of auditory experiments and may be regarded as a fundamental concept in auditory theory (Feldkeller & Zwicker, 1956, Zwicker, Flottorp, & Stevens, 1957). The concept was introduced previously by Fletcher (1940) and called *critical band*. The absolute size of Fletcher's critical band resulted from postulated rather than from empirical data, although a direct measurement was attempted. For this reason, Zwicker, Flottorp, and Stevens (1957) suggested that it be called the *critical ratio* and that the name *critical band* be transformed to the more recently modified concept. Fletcher determined the size of the critical ratio as a function of sound frequency, and he found a similar relationship to that for the critical band. As a consequence, the quotient of the critical band and the critical ratio is practically independent of frequency, it amounts to approximately 2.5.

The similarity between pitch, frequency jnd, and critical band functions suggests the hypothesis that all three have a common physiological substrate. Helmholtz (1863) postulated that pitch is determined by the maximally excited nerve fibers and that this excitation is controlled by the locus of maximum vibration amplitude in the cochlea. The mechanism he invoked to explain the occurrence of such a maximum may have been inaccurate, but v. Békésy (1942, 1943) has demonstrated that the maximum occurs and that its locus changes with frequency in the way Helmholtz predicted. For low frequencies, it is situated near the helicotrema, for high frequencies, in the vicinity of the oval window. Theories linking pitch to location along the cochlear canals are now called *place theories*.

Some simple postulates that may be proposed with respect to a place theory are (1) pitch is directly proportional to the distance between the vibration maximum and the helicotrema and (2) the size of mel steps, jnd's, and critical bands is inversely proportional to the derivative of the distance with respect to sound frequency.

In Sec. 1, the functional relationship between the locus of maximum vibration and sound frequency was derived theoretically on the basis of the

anatomy and the physical constants of the cochlea. It was shown that the obtained relation is consistent with empirical measurements. An expression was found that determines the amplitude of vibration η of the cochlear partition as a function of the distance x from the oval window (Eq 103). The maximum amplitude occurs where the distance derivative of η vanishes, $d\eta/dx = 0$. This condition leads to an expression of the form

$$x = A \ln \frac{(b^2 - 4acf^2)^{1/2} - b}{2af^2}, \quad (146)$$

where A , a , b , and c are positive constants depending on the physical parameters of the cochlea. In order to obtain the distance from the helicotrema, we introduce $x_h = l - x$, with l indicating the length of cochlear canals. Thus

$$x_h = l + A \ln \frac{2af^2}{(b^2 - 4acf^2)^{1/2} - b} \quad (147)$$

The first postulate is satisfied if Eq 147 is identical with Eq 143, which determines the number of jnd steps as a function of frequency. Clearly, this is not the case. The numerical difference between the mel scale and the x_h function may be derived from Fig 28. We must conclude that pitch is not directly proportional to the distance of the locus of maximum oscillation from the helicotrema. Still, the curves of Fig 28 are sufficiently similar to entertain a hypothesis of close correlation between the two variables. In order to clarify the situation somewhat, we may attempt to determine the size of jnd's or mel steps in terms of length intervals along the cochlear

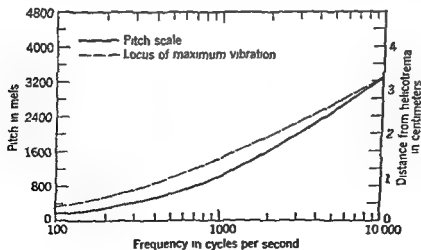


Fig 28 Pitch and the location of the vibration maximum along the cochlear partition as a function of sound frequency

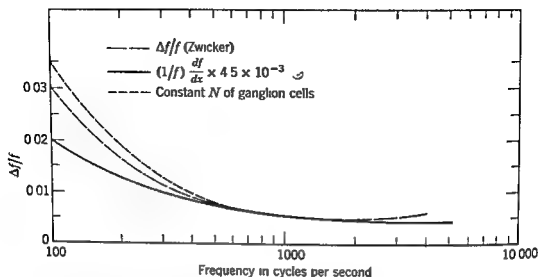


Fig 29 Relative frequency jnd and the relative frequency change for a constant shift of the vibration maximum along the cochlear partition and for a shift of the maximum by a constant number of ganglion cells

partition For this purpose, we take the derivative of Eq 147 with respect to f

$$\frac{dx_h}{df} = A \left[\frac{4af}{2af^2} - \frac{8acf}{2(b^2 + 4acf)^{1/2}} \frac{1}{(b^2 + 4acf^2)^{1/2} - b} \right] \quad (148)$$

This determines the distance that the maximum of vibration is shifted along the cochlear partition when the sound frequency changes by one cycle. By taking $1/(dx_h/df)$, which is equivalent to df/dx_h , we obtain the frequency change per unit distance. The quotient divided by the sound frequency and multiplied by $4.5 \cdot 10^{-3}$ cm is plotted in Fig 29 together with the relative jnd, $\Delta f/f$. It is apparent that at frequencies above 500 cps

$$\frac{\Delta f}{f} = \frac{1}{f} \frac{df}{dx_h} \times 4.5 \times 10^{-3}, \quad (149)$$

so that, letting $df = \Delta f$, we obtain

$$\Delta x_h = 4.5 \times 10^{-3} \text{ cm}, \quad (150)$$

the distance equivalent to one jnd. Accordingly, the jnd, pitch, and critical band values are approximately proportional to distance increments along the cochlear partition, and the second postulate is satisfied. Below 500 cps, no such proportionality exists, however, so that the relationship $\Delta f/\Delta x_h = \text{constant}$ at higher frequencies must be considered coincidental rather than of basic significance.

A more significant relation is found when the distribution of the

stimulus is plotted in terms of the distribution of nerve endings rather than in terms of distances from the helicotrema. Such a procedure seems justified since, in the end, the frequency information must be transmitted by the nervous system.

The sensory cells in the cochlea are innervated by approximately 30,000 neurons which have their cell bodies in the ganglion spiralis. Wever (1949) made a thorough count of these ganglion cells, determining their density per millimeter of the basilar membrane. In the basal and middle turns of the cochlea, the neural density appears to be nearly constant and amounts to approximately 1150 ganglion cells per millimeter. In the apical turn, the density decreases toward the helicotrema according to a monotonic function. This function may be expressed in the form

$$\gamma = \frac{\sigma_x}{\sigma_m}, \quad (151)$$

with σ_x denoting the local innervation density and σ_m , the average inner-
vation density in the basal and middle turns. If now the expression $(1/f)(df/dx_h) \cdot 4.5 \cdot 10^{-3}$ is divided by γ , the dotted curve in Fig. 29 results. This curve agrees considerably better with the relative jnd curve than does the plot of the expression $(1/f)(df/dx_h) \cdot 4.5 \cdot 10^{-3}$. The remaining difference can easily be accounted for by the errors of the psychophysical as well as of the anatomical and physiological measurements. We conclude therefore that equal distances on the pitch scale correspond to equal numbers of peripheral neurons. More specifically

$$1 \text{ mel} = \sim 12 \text{ neurons,}$$

$$1 \text{ jnd} = \sim 52 \text{ neurons,}$$

$$1 \text{ CB} = \sim 1300 \text{ neurons,}$$

where CB denotes critical band.

This conclusion is encouraging to those who attempt to relate the subjective scales to innate properties of the organism rather than view them entirely as a result of learning.

In order to visualize what a critical band means in terms of the vibration pattern in the cochlea, consult Fig. 12. The relative position of the two curves that indicate the amplitude distribution along the cochlear partition corresponds to two critical bands. A distance equivalent to one jnd is about fifty times smaller. Such sensitivity is astounding, and it is difficult to conceive how it arises without assuming an additional mechanism that emphasizes the amplitude differences. Such a mechanism is discussed in Sec. 5.

4 THRESHOLD OF AUDIBILITY

The concept of threshold is a controversial matter in psychophysics. It has even been suggested that, in a strict sense, psychophysical thresholds do not exist (see Chapter 3, Vol I). Nevertheless, it is possible to define a threshold as a stimulus intensity that, under given conditions, produces a certain probability of signal detection. A strong auditory stimulus can be detected practically 100% of the time when no interfering noise is present. When its intensity is decreased, the probability of detection becomes gradually smaller and finally approaches zero. Conventionally, the threshold intensity is defined as the intensity that produces a 50% chance of detection. This intensity may vary from situation to situation, and we shall analyze some of the physical and physiological parameters involved. The higher order psychological factors are discussed in Chapter 3, Vol I.

The threshold of audibility, as previously defined, depends on the stimulus duration and on the energy distribution along the frequency continuum. Since the latter relationship depends in part on the first, it appears reasonable to discuss them in that order.

4.1 Threshold of Audibility as a Function of Temporal Stimulus Pattern

When the duration of a short tone burst is increased, its threshold of audibility decreases at an approximate rate of 3 db per doubling of duration. This is true up to a duration of approximately 200 msec at which point the threshold curve flattens and slowly approaches an asymptote.

A departure from a 3 db slope may also occur at extremely short durations. This phenomenon is probably due to changes in the frequency spectrum (Garner, 1947). An infinitely long tone has only one spectral line, but a tone burst has a spectrum whose major portion occupies a frequency range

$$\Delta f = \frac{2}{\tau},$$

where τ is the burst duration. Under such conditions, the sound energy is distributed over the whole spectrum, and the threshold may be raised or lowered depending on the sensitivity variation of the auditory system with frequency. Garner (1947) and Scholl (1961) have analyzed the phenomenon in detail from different points of view, but we shall neglect it in the following considerations where it is of secondary importance.

Empirical data obtained in three independent studies for a 1000 cps tone are shown in Fig 30 (Garner, 1947, Feldtkeller & Oettinger, 1956; J. Zwislöcki, 1960) For durations greater than 10 msec, they can be approximated quite well by a curve obeying the equation

$$\frac{I_t}{I_\infty} = \frac{1}{1 - e^{-\alpha t}}, \quad (152)$$

where I_t denotes the threshold intensity of the burst, I_∞ , the threshold intensity of a continuous tone, and t , the burst duration. The equation describes a simple energy integrator with a time constant $1/\alpha \approx 200$ msec (Munson, 1947, Feldtkeller & Oettinger, 1956). As a consequence, we assume such an integrator as a model and investigate how well it predicts the threshold of audibility for various time patterns of the stimulus.

Other, probabilistic, models have been suggested, but their validity was not tested beyond the function of threshold intensity versus duration (Garner & Miller, 1947, Green, Birdsall, & Tanner, 1957).

Physiological studies suggest that, from the point of view of the peripheral nervous system, each wave of sinusoidal vibration may be considered as an independent stimulus. Thus, when the duration of a tone is varied, the number of elemental stimuli is varied simultaneously. These variables may be separated by using pulse sequences and changing the number of pulses while keeping the duration of the sequence constant, or vice versa (Zwislöcki, 1960). A drawback in such experiments is the spectral distribution of energy. The major portion of the spectrum of a pulse that is τ sec long is included between 0 and $1/\tau$ cps. The pulse repetition rate determines the density of spectral components. The difficulty may be overcome by avoiding very short pulses and high repetition rates (Zwislöcki, Hellman, & Verrillo, 1962). This is particularly true when the pulses are transmitted through a selective network emphasizing

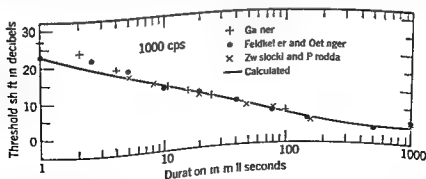


Fig 30 Threshold of audibility for 1000 cps tone bursts as a function of their duration. The points indicate several series of experimental data, the curve is theoretical. Adapted with permission from Zwislöcki (1960 p 1053)

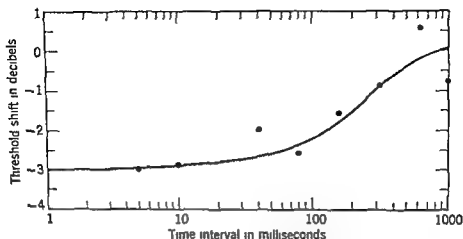


Fig 31 Threshold of audibility of pulse pairs as a function of time interval between the pulses. The points are experimental, the curve is theoretical. Adapted with permission from Zwislocki (1960 p 1047)

a relatively narrow band of frequencies. The same goal may be achieved by using sequences of short tone bursts when the tone frequency is sufficiently high (Scholl, 1961).

In one experiment the threshold of pairs of pulses was measured as a function of time interval between the pulses (Zwislocki, 1960). The closed circles in Fig 31 show the experimental data. They are fitted quite well by the curve drawn in the same figure, which obeys the equation

$$TS = 10 \log (1 + e^{-st}) \quad (153)$$

This equation can be rewritten in the form

$$I_1 = I_2 + I_2 e^{-st}, \quad (154)$$

in which I_1 denotes the threshold intensity of a single pulse, I_2 the threshold intensity of the pulse pair, and t the time interval between the pulses. In this form, it suggests a simple neural mechanism which may be described in the following way (Zwislocki, 1960).

Certain portions of the nervous system function in a quantal manner and produce the so-called all or-none responses. The response of other portions is graded. Axon cylinders exhibit the first kind of response, dendrites, synaptic junctions, and muscle endplates the second. In the presence of a graded response, the resultant excitation appears to be a simple sum of elemental excitations produced by incoming stimuli. After the stimulation has been terminated, the excitation decays approximately according to an exponential function

$$\eta = \eta_0 e^{-at} \quad (155)$$

If we postulate the graded response of the nervous system to be the mechanism controlling the threshold of audibility as a function of the temporal stimulus pattern, we can write for the pulse pair

$$\eta = \epsilon_1 e^{-at} + \epsilon_2, \quad (156)$$

where η is the total excitation immediately after the second pulse has occurred, ϵ_1 the excitation produced by the first pulse, and ϵ_2 the excitation produced by the second pulse. The equation describes correctly the measured threshold of pulse pairs when $\epsilon_1 = \epsilon_2$ and $\epsilon = CI$. These requirements should be in agreement with the known functioning of the auditory nervous system. It is possible to show that they are.

Without much doubt, it should be possible to express the functional relationship between the excitation and the stimulus intensity I by means of a series expansion of the form

$$\epsilon = a_1 I + a_2 I^2 + a_3 I^3 + \quad (157)$$

According to physiological evidence, the peripheral auditory system exhibits considerable spontaneous activity (Tasaki, 1957). We can consider this activity as if it were produced by a stimulus intensity I_0 . Then, denoting the intensity of the extrinsic stimulus by I_e , we obtain

$$\epsilon = a_1(I_0 + I_e) + a_2(I_0 + I_e)^2 + \quad (158)$$

This series can be rewritten in the form

$$\epsilon = a_1 I_0 \left(1 + \frac{I_e}{I_0}\right) + a_2 I_0^2 \left(1 + \frac{I_e}{I_0}\right)^2 + \quad (159)$$

On the basis of masking experiments, which are discussed in Sec. 5, we can assume that $I_e \ll I_0$ in the immediate vicinity of the threshold of audibility so that

$$\epsilon \cong a_1 I_0 \left(1 + \frac{I_e}{I_0}\right) + a_2 I_0^2 \left(1 + \frac{2I_e}{I_0}\right) + \quad (160)$$

or

$$\epsilon \cong a_1 I_0 + a_2 I_0^2 + (a_1 + 2a_2 I_0 + \quad) I_e \quad (161)$$

Thus, independent of the exact relationship between the excitation and the stimulus intensity, the excitation produced by the extrinsic stimulus can be written in the form

$$\epsilon_e = E I_e, \quad (162)$$

where $E = a_1 + 2a_2 I_0 + \quad = \text{constant}$. The direct proportionality between the excitation and stimulus intensity holds only near the threshold, at higher intensities more complex phenomena seem to take place. They are discussed in Sec. 6.

Now, it is necessary to demonstrate that the requirement $\epsilon_1 = \epsilon_2$ is consistent with direct observations. McGill and Rosenblith (1951, 1952) have investigated neural responses to pairs of short pulses. They found that, in general, a preceding pulse decreases the response to the following one. This does not happen at very small intensities, however. The phenomenon can be explained in terms of neural response probability. Although at high intensities a considerable percentage of available neural units respond to the first pulse, only a few units respond at intensities near the threshold. Denoting the probability of response to the first pulse by p , we obtain for the number of available units at the time of occurrence of the second pulse

$$N_2 = N[1 - p + \kappa(t)p], \quad (163)$$

where N is the total number of available units and $\kappa(t)$ is the fraction of units that have had time to recover before the second stimulation. If p is very small, $N_2 \cong N$. This demonstration is consistent with the assumption that $I_e \ll I_0$, since under such conditions I_0 , not I_e , controls the number of available units.

We now investigate whether the theory developed for a pulse pair holds for any number of pulses and also for sinusoidal oscillations (Zwislocki, 1960). For n pulses with a repetition rate $\nu = 1/\Delta t$, we obtain

$$\eta_n = \epsilon_n(e^{-\alpha(n-1)\Delta t} + e^{-\alpha 2\Delta t} + e^{-\alpha \Delta t} + 1), \quad (164)$$

which, for $n \rightarrow \infty$, is a series expansion of

$$\eta_\infty = \frac{\epsilon_\infty}{1 - e^{-\alpha \Delta t}} \quad (165)$$

The ratio between the excitation at the end of n pulses and the asymptotic excitation follows to

$$\frac{\eta_n}{\eta_\infty} = \frac{\epsilon_n}{\epsilon_\infty} (1 - e^{-\alpha \Delta t}) \sum_{\kappa=1}^n e^{-\alpha(\kappa-1)\Delta t} \quad (166)$$

It does not seem unreasonable to assume that $\eta_n = \eta_\infty = \text{const}$ at the threshold of audibility. Consequently, with $\epsilon = CI$, we obtain

$$\frac{I_n}{I_\infty} = \frac{1}{(1 - e^{-\alpha \Delta t}) \sum_1^n e^{-\alpha(\kappa-1)\Delta t}}, \quad (167)$$

which can be simplified to

$$\frac{I_n}{I_\infty} = \frac{1}{1 - e^{-\alpha n \Delta t}}, \quad (168)$$

or in decibel notation

$$TS \approx 10 \log I_n - 10 \log I_\infty = -10 \log (1 - e^{-at}), \quad (169)$$

where TS means threshold shift and $t = n \Delta t$ is the duration of the pulse sequence

Equations 168 and 169 are analogous to Eq 152, so that the theory describes accurately the threshold of audibility of a 1000 cps tone as a function of duration. Experiments have shown that the shape of the threshold curve does not change with frequency as long as the main portion of the frequency spectrum is confined to one critical band (see Secs 3 and 5). As a consequence, the theory holds independent of sound frequency.

In the two-pulse experiment the duration of the sequence is varied without varying the number of elemental stimuli. When the procedure is reversed and the number of pulses is varied in a constant time interval, then, by letting n in Eq 168 vary from 1 to some higher number, we can write

$$\frac{I_n}{I_1} = \frac{1 - e^{-at/n}}{1 - e^{-at}} \quad (170)$$

The closed circles in Fig 32 show experimental data for a sequence duration $t = 80$ msec. The crosses in the same figure indicate the theoretical predictions. The largest difference is 1.5 db (at $n = 9$), and that is within the range of experimental error.

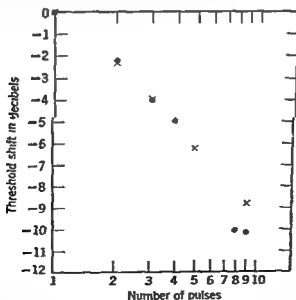


Fig 32 Threshold of audibility for short pulses equally spaced within a time interval of 80 msec, plotted as a function of the number of pulses. The circles are experimental and the crosses theoretical. Adapted with permission from Zwillocki (1960, p 1052)

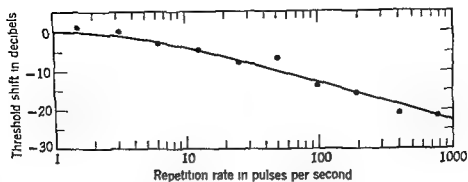


Fig 33 Threshold of audibility for a practically infinite series of short pulses as a function of pulse repetition rate. The circles are experimental, and the curve is theoretical. Adapted with permission from Zwislocki (1960, p. 1052)

When the duration of the sequence is made very long ($t \gg 1/\alpha$), the expression $e^{-\alpha t}$ approaches zero, and Eq. 170 can be written in the form

$$\frac{I_2}{I_1} = 1 - e^{-\alpha/\nu}, \quad (171)$$

where $\nu = n/t$ is the repetition rate. We then obtain a prediction of the threshold as a function of the pulse repetition rate. The theoretical and experimental data are compared in Fig. 33, in which the points indicate the experimental results. The pulses were 0.5 msec wide. For shorter pulses, the agreement becomes less good because of spectral interference. A still better agreement can be obtained when pulses are replaced by short tone bursts.

In order to find a mathematical expression for sequences of tone bursts we go back to Eq. 164 and rewrite it in the form

$$\begin{aligned} \frac{\eta}{\epsilon} = & 1 + e^{-\alpha \Delta t} + e^{-\alpha 2 \Delta t} + \dots + e^{-\alpha(\kappa+1)\Delta t} + \dots + e^{-\alpha(\kappa+\sigma)\Delta t} \\ & + e^{-\alpha(\kappa+\sigma+1)\Delta t} + \dots + e^{-\alpha(2\kappa+\sigma)\Delta t} \\ & + e^{-\alpha(2\kappa+\sigma+1)\Delta t} + \dots + e^{-\alpha(2\kappa+2\sigma)\Delta t} + \dots, \end{aligned} \quad (172)$$

where κ and σ are arbitrary numbers and $\Delta t = 1/f$.

By suppressing all pulses between the numbers κ and $\kappa + \sigma$, between $2\kappa + \sigma$ and $2\kappa + 2\sigma$, and so forth, an intermittent stimulus is generated. It is on during the time $T_p = (\kappa + 1)\Delta t$ and off during the time $T_i = (\sigma - 1)\Delta t$. Under these conditions, Eq. 172 becomes

$$\begin{aligned} \frac{\eta_i}{\epsilon_i} = & (1 + e^{-\alpha \Delta t} + e^{-\alpha 2 \Delta t} + \dots + e^{-\alpha \kappa \Delta t}) \\ & (1 + e^{-\alpha(\kappa+\sigma)\Delta t} + e^{-\alpha 2(\kappa+\sigma)\Delta t} + \dots + e^{-\alpha(n-1)(\kappa+\sigma)\Delta t}) \end{aligned} \quad (173)$$

For steady state, that is, for $n \rightarrow \infty$, Eq 173 may be simplified according to Eq 165

$$\eta_i = \epsilon_i (1 - e^{-a(\kappa+\sigma)\Delta t})^{-1} \sum_{i=0}^{\kappa} e^{-a\Delta t} \quad (174)$$

Since $f = 1/\Delta t$ is the carrier frequency of the intermittent tone, and $\nu = 1/(\kappa + \sigma)\Delta t$ is the burst repetition rate, we can also write

$$\eta_i = \epsilon_i (1 - e^{-a/\nu})^{-1} \sum_{i=0}^{\kappa} e^{-a i/f} \quad (175)$$

Using Eqs 165 and 175, it is possible to compare the excitations produced by the intermittent and continuous tones

$$\frac{\eta_i}{\eta_{\infty}} = \frac{\epsilon_i}{\epsilon_{\infty}} (1 - e^{-a/f})(1 - e^{-a/\nu})^{-1} \sum_{i=0}^{\kappa} e^{-a i/f} \quad (176)$$

This can be simplified to

$$\frac{\eta_i}{\eta_{\infty}} = \frac{\epsilon_i}{\epsilon_{\infty}} (1 - e^{-a(\kappa+1)/f})(1 - e^{-a/\nu})^{-1} \quad (177)$$

Assuming again that, at the threshold, $\eta_i = \eta_{\infty}$, and introducing $T_B = (\kappa + 1)\Delta t$, we obtain

$$\frac{I_i}{I_{\infty}} = \frac{1 - e^{-a/\nu}}{1 - e^{-aT_B}}, \quad (178)$$

or

$$TS = 10 \log (1 - e^{-a/\nu}) - 10 \log (1 - e^{-aT_B}) \quad (179)$$

Garner (1947a) measured the threshold of audibility of an intermittent 1000 cps tone for various burst repetition rates and burst durations. His data are compared to the theoretical predictions in Fig 34. The agreement

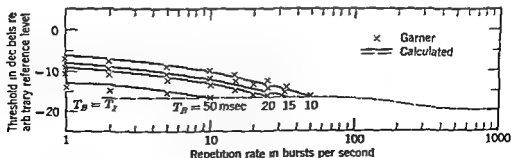


Fig 34 Threshold of audibility for an intermittent 1000 cps tone as a function of repetition rate and burst duration. The points are experimental, the curves are calculated. Adapted with permission from Zwisllock (1960 p 1054)

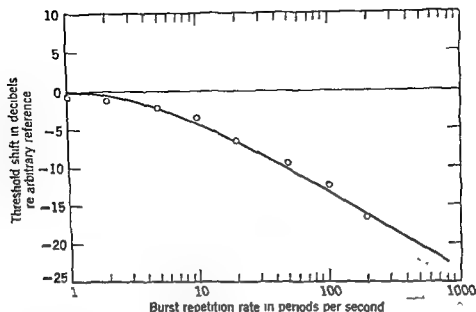


Fig. 35. Threshold of audibility for 5 msec bursts of a 2000 cps tone as a function of burst repetition rate. The circles indicate experimental values; the curve is calculated. Adapted with permission from Zwislöck et al (1962, p. 1651).

is reasonably good with respect to the slope of the curves as well as with respect to their position. In Eq. 175, the sum

$$\frac{\eta_B}{\epsilon_B} = \sum_{i=0}^{\infty} e^{-a_i/\tau} \quad (180)$$

determines the excitation caused by one burst, so that the ratio between the excitations by an infinite number of bursts and by one burst amounts to

$$\frac{\eta_t}{\eta_B} = \frac{\epsilon_t}{\epsilon_B} (1 - e^{-a/\tau}). \quad (181)$$

The corresponding threshold shift is

$$TS = 10 \log (1 - e^{-a/\tau}), \quad (182)$$

in agreement with the threshold shift for pulses (Eq. 171). Figure 35 compares theoretical data to those obtained by Scholl (1961) with 5 msec bursts of a 2000 cps tone.

Many more experiments on the threshold of audibility as a function of stimulus duration were performed. No experiment has yet been reported that contradicts the theory of temporal summation previously discussed. On the contrary, its validity was extended to sounds heard in presence

of interfering noise (Zwicker & Wright, 1963) and to suprathreshold stimuli. In the application of the theory, it is necessary to pay attention to the frequency spectrum, however. The threshold of a broad-band random noise, for instance, varies less with duration than does the threshold of a narrow-band stimulus. The phenomenon was explained by Scholl (1961) in terms of critical band formation. Here, we may mention that the theory of temporal summation also applies to other sense modalities and motor activity (Zwislocki, 1960). It is possible that we are dealing with a general psychophysical law that, for the threshold of detectability, may be expressed by the equation

$$\frac{I_{S_2}}{I_{S_1}} = \frac{\sum_{\kappa=1}^n (I_{\kappa}/I_{S_1})e^{-\alpha t_{\kappa}}}{\sum_{\mu=1}^m (I_{\mu}/I_{S_2})e^{-\alpha t_{\mu}}}, \quad (183)$$

where I_{S_1} means the threshold intensity of a stimulus sequence, I_{S_2} the threshold intensity of another stimulus sequence, I_{κ} and I_{μ} the intensities of elemental stimuli in each sequence, t_{κ} and t_{μ} the time intervals between elemental stimuli and the effective end of the sequence, and $1/\alpha$ a time constant. It should be emphasized that α is a system constant which does not depend in any way on the stimulus. Neither the intensities of the elemental stimuli nor the time intervals between them need be constant. The term "effective end of the sequence" means the time at which the excitation η reaches its maximum. For certain functions $I(t)$, this may happen before the stimulation ends. Thereby, the assumption is made that the maximum excitation determines the threshold.

Equation 183 may also be written in the form

$$\sum_{\mu=1}^m I_{\mu} e^{-\alpha t_{\mu}} = \sum_{\kappa=1}^n I_{\kappa} e^{-\alpha t_{\kappa}} \quad (184)$$

When the intervals $t_{\kappa} - t_{\kappa+1}$ and $t_{\mu} - t_{\mu+1}$ are very small, the sums of Eq 184 may be replaced by convolution integrals, and we obtain

$$\int_0^{T_2} I_2(\tau) e^{-\alpha(t-\tau)} d\tau = \int_0^{T_1} I_1(\tau) e^{-\alpha(t-\tau)} d\tau, \quad (185)$$

with T_1 and T_2 denoting the effective sequence durations. It should be noted that, for the purpose of determining these durations, the maximum excitation is assumed to occur at the time for which

$$\frac{d\eta}{dt} = 0 \quad \text{and} \quad \frac{d^2\eta}{dt^2} < 0$$

4.2 Threshold of Audibility as a Function of Sound Frequency

The threshold of audibility for pure tones strongly depends on sound frequency. An indication that temporal summation contributes to this dependence has already been given in the first part of this section. The transmission characteristic of the ear has an even stronger effect. A threshold curve measured in a free sound field is plotted by means of the solid line in Fig. 36. It is an average of three independent studies in which the listeners faced the sound source (Sivian & White, 1933, Fletcher, 1953, Zwislocki, 1957). In all three studies, the sound pressure was measured at the center of the listener's head. The intermittent line shows a partly theoretical and partly experimental transmission characteristic of the ear, which was determined in the following manner:

In Sec. 2, we determined the transmission characteristic as the ratio between the sound pressure in a free field and the maximum volume displacement of the cochlear partition. It is unlikely that the volume displacement itself controls the stimulation of the sensory cells. A comparison of electrophysiological measurements of Tasaki, Davis and Legoux (1952) with the mechanical measurements of v. Bekesy (1943) leads to the conclusion that cochlear microphonics are approximately proportional to the amplitude of displacement. Since cochlear microphonics are regarded as the trigger for neural discharges, it is probably more correct to deal with the displacement amplitude of the cochlear partition than with its volume displacement. The width of the basilar membrane,

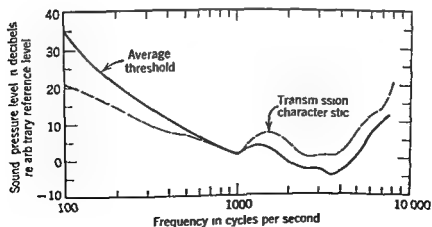


Fig. 36. An average threshold of audibility measured under freefield conditions and the transmission characteristic of the ear as a function of frequency.

which supports the sensory cells, increases from the stapes to the helicotrema by a factor of five (Wever, 1938). As a consequence, the ratio between the relevant displacement amplitude and the volume displacement of the partition changes from one end of the cochlea to the other by approximately the same amount. Because of the correlation between sound frequency and location along the cochlear canal, this results in a slight improvement of sound transmission for high frequencies relative to the low ones. A ratio of five for the amplitude corresponds to an increment of 13 db. We shall assume that the ratio between displacement amplitude and volume displacement increases in direct proportion to distance from the helicotrema and therefore in nearly direct proportion to the logarithm of frequency. Since the effect is small, the knowledge of the exact function is not necessary.

The transmission characteristic of Fig 36 is that derived in Sec 2 corrected by the ratio of the displacement amplitude of the basilar membrane to its volume displacement. Theoretically, it determines the amplitude of cochlear microphonics for a given free-field sound pressure measured at the location of the center of the listener's head. There is an obvious difference between the transmission characteristic and the threshold curve, which means that the transmission properties of the ear are not the only controlling factor. The decibel difference between the two curves is shown in Fig 37 by closed circles. Over the frequency range of 200 and 8000 cps, the points cluster around a regression line with a slope of 3 db per octave. This line agrees almost exactly with the threshold change predicted from the theory of temporal summation (Eqs 171 and 182, and Figs 33 and 35). Only at the lowest frequencies is there a significant departure of the points from the regression line. It

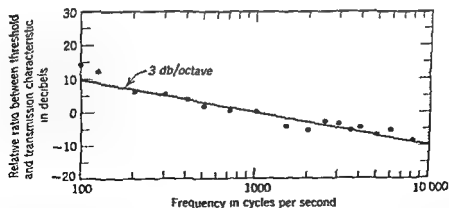


Fig 37 The decibel difference between the threshold of audibility and the transmission characteristic of Fig 36 (circles), and a theoretical curve resulting from temporal summation

coincides with the region in the cochlea where the density of innervation begins to decrease. A correlation between the density of innervation and the sensitivity is well known for both the eye and the skin. We make the simple assumption, therefore, that the threshold intensity is proportional to the density of innervation. According to Wever (1949), there are about half as many ganglion cells per millimeter of length of the basilar membrane at the place of vibration maximum for 100 cps as there are at locations corresponding to frequencies above 500 cps. This corresponds to a threshold increment of 3 db, in approximate agreement with the position of the 100 cps point relative to the regression line.

It is possible to conclude on the basis of Figs. 36 and 37 that the threshold of audibility as a function of sound frequency is controlled in the main by the transmission characteristic of the ear, the density of ganglion cells as projected on the basilar membrane, and the temporal summation in the nervous system. This appears to be true for pure tones, wide band noise introduces an additional factor.

Fletcher (1940) assumed that spectral energy of noise was integrated over the bandwidth of a critical ratio. According to Feldtkeller and Zwicker (1956) the summation occurs over the entire critical band (Secs. 3 and 5). If we assume a random noise with a uniform spectral energy density σ , then the energy in a critical band is $\sigma \kappa$. It was shown in Sec. 3 that the critical bandwidth is proportional to the frequency f and, therefore, depends on frequency according to the equation

$$\kappa = \alpha(Kf + 1), \quad (186)$$

which is the same as Eq. 141 except for the multiplicative constant α approximately equal to 25. The equation can also be written in the form

$$\kappa = Kf + \kappa_0, \quad (187)$$

where $K = 10^{-1}$ and $\kappa_0 = 63$ cps.

Because of energy summation, the threshold of audibility for wide band noise decreases with frequency faster than does the threshold for pure tones. The ratio between the two is expressed by κ .

5 MASKING, CRITICAL BAND, AND CONTRAST PHENOMENA

A complex sound containing two or more sinusoidal components may be heard as one event or, within certain limits, each component may be listened to separately. This analytical property of the auditory system is ascribed in part to the wave pattern in the cochlea. In Sec. 2, it was

shown that for each audible frequency there is a maximum of vibration in a different portion of the cochlea. Physiological evidence indicates that the location of the vibration maximum determines the group of maximally excited nerve endings which, as we assume, leads to pitch perception. We saw that the vibration maximum is broad and, in itself, would permit only a very coarse sound analysis. The distribution of excitation in the peripheral nervous system is considerably sharper, however, so that the amazing frequency sensitivity of hearing may be explained (Galambos & Davis, 1943, Tasaki, 1954, Katsuki, et al., 1958). The sharpening mechanism has been a subject of debate for a long time (v. Bekésy, 1929, 1960a,b, Huggins & Licklider, 1951), and we shall discuss one possibility later in this section.

No analyzer has an infinite resolving power, and frequency components can interfere with each other to a greater or lesser extent, depending on their spacing along the frequency scale. One aspect of such interference in hearing is called *masking*. It means that a stronger component may prevent a weaker one from being heard. The masking effect is measured most frequently as a threshold elevation of one or a group of frequency components produced by other stronger frequency components. The masking sound is often called a "masker." Masking effects produced by a pure tone of 400 cps and by a narrow band of noise of 90 cps width centered at 410 cps are shown in Fig. 38 (Egan & Hake, 1950). The threshold elevation in decibels over the threshold obtained in the absence of the masker is plotted as a function of frequency of the exploring pure tone. Although the sound-pressure level of both maskers was the same,

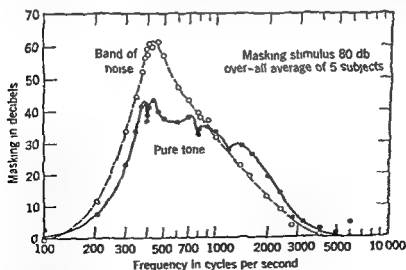


Fig. 38 Masking effects of a 400 cps tone and a 90 cps wide random noise centered at 410 cps. Adapted with permission from Egan & Hake (1950, p. 628).

the narrow band noise produced a greater maximum effect. The pure tone masking curve is flatter and spreads somewhat further toward high frequencies. It also shows several relative maxima and minima. For a long time, the complex shape of the pure tone masking curve was regarded as a result of nonlinearities in the mechanical system of the ear. More recently, v. Békésy (1957) suggested a different explanation. We shall not discuss these secondary effects here, except to state that differences in phase relations and amplitude patterns are probably the main reasons for the different masking effects of random noise and pure tones. These conditions also depend on the noise bandwidth, so that the masking effect of a wide band noise cannot be completely predicted from the masking effects of narrow band components.

One of the most interesting aspects of the masking phenomenon is that the threshold elevation at a given frequency is proportional to the noise energy within the critical band centered on that frequency. Fletcher (1940) introduced it first as a postulate and defined the critical band so that a pure tone is just masked when its power is equal to the power of the noise band. Later, Zwicker (1954) demonstrated that the power was summed over a band approximately 2.5 times wider than Fletcher assumed, and consequently, that the noise power had to be 4 db above that of the just masked tone. Fletcher's band was renamed "critical ratio," and the name "critical band" was applied to the spectral width discovered by Zwicker. To be sure, we are dealing here with the same critical bandwidth that was described in Sec. 3 for phase sensitivity and in Sec. 4 for power summation at the threshold of audibility. In the next section, we shall see still another aspect of the critical band concept.

Since the critical band increases with frequency, a broad band noise with uniformly distributed spectral power produces a greater masking effect at high than at low frequencies (Hawkins & Stevens, 1950). Specifically, denoting by I_T the threshold intensity in quiet, by I_M that in presence of the masking noise, by I_S the spectral intensity of noise, we obtain

$$\frac{I_M}{I_T} \cong 0.4\kappa \frac{I_S}{I_T}, \quad (188)$$

where κ denotes the critical bandwidth. In terms of threshold shift, Eq. 188 becomes

$$TS \cong 10 \log \frac{I_S}{I_T} + 10 \log \kappa - 4, \quad (189)$$

or introducing for the critical band its expression from Eq. 187

$$TS \cong 10 \log \frac{I_S}{I_T} + 10 \log (Kf + \kappa_0) - 4 \quad (190)$$

If we are interested in the masked threshold rather than in the threshold shift, we can write

$$T_M \cong 10 \log \frac{I_S}{I_0} + 10 \log (Kf + \kappa_0) - 4, \quad (191)$$

where $I_0 = p_0^2/\rho c$ and p_0 is the effective reference sound pressure, usually taken as 0.0002 dyne/cm². Replacing intensities by sound pressures in Eq. 191 we obtain

$$T_M \cong 20 \log \frac{p_S}{p_0} + 10 \log (Kf + \kappa_0) - 4 \quad (192)$$

Equations 191 and 192 indicate that the threshold of a pure tone in presence of a uniform random noise is frequency independent for very low frequencies and increases in direct proportion to frequency in the high frequency region.

All equations derived so far in this section show that the masked threshold increases in direct proportion to the spectral intensity. This is not quite true when the noise intensity per critical band only slightly exceeds the threshold of audibility in quiet. Figure 39 illustrates the situation when the noise intensity is plotted per critical ratio, as defined in terms of TS by Eq. 189 (Hawkins & Stevens, 1950). For critical ratio levels exceeding 10 db, the masked threshold is directly proportional to

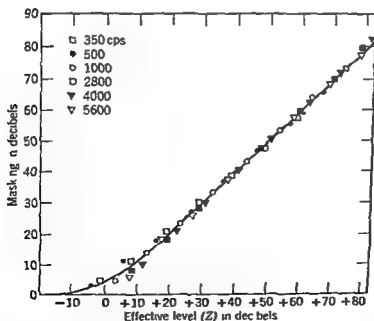


Fig. 39 Masking of pure tones by random noise. The threshold shift is plotted as a function of the effective noise level. Adapted with permission from Hawkins & Stevens (1950 p. 11).

the noise level. However, below 10 db the curve flattens and asymptotically reaches the threshold level in quiet. This agrees with the assumption made in Sec. 4 that there is a considerable amount of spontaneous activity in the auditory nervous system and that this activity may be introduced as an equivalent stimulus intensity I_N . Extending the assumption to mean that I_N produces a masking effect which, in fact, determines the threshold in absence of an extrinsic masker, we can correct Eqs. 189 and 190 by adding $I_{NS} = I_N/\kappa$ to I_S and write

$$TS \cong 10 \log \frac{I_S + I_{NS}}{I_T} + 10 \log (Kf + \kappa_0) - 4 \quad (193)$$

In absence of extrinsic maskers, Eq. 193 becomes

$$TS \cong 10 \log \frac{I_{NS}}{I_T} + 10 \log \kappa - 4 = 0, \quad (194)$$

indicating that

$$10 \log \kappa I_{NS} = 10 \log I_T + 4, \quad (195)$$

that is, the stimulus intensity per critical band that is equivalent to spontaneous neural activity is 4 db above the threshold of audibility. With the numerical value of κI_{NS} so determined, Eq. 193 correctly describes the curve of Fig. 39. As a consequence, Eqs. 191 and 192 have to be rewritten as follows

$$T_M = 10 \log \frac{I_S + I_{NS}}{I_0} + 10 \log (Kf + \kappa_0) - 4, \quad (196)$$

and

$$T_M = 20 \log \frac{(P_S^2 + P_{NS}^2)^{1/2}}{P_0} + 10 \log (Kf + \kappa_0) - 4 \quad (197)$$

The masking effect of a sound as a function of frequency has often been regarded as an indicator of the distribution of excitation in the nervous system. If this is approximately true, a comparison of Fig. 38 with Fig. 12 shows that the neural excitation has a much sharper maximum than the vibration amplitude in the cochlea, in agreement with neurophysiological recordings. The sharpening effect appears to belong to the same class of phenomena as the well known contrast effects in vision. When, for instance, a uniform gray field is seen on a darker surround, its edge appears lighter than its central portion and the surround appears darkest in the immediate vicinity of the boundary. The phenomenon may be schematized as shown in Fig. 40. The upper trace indicates an idealized distribution of light intensity along a diameter of a light field presented against a darker surround, and the lower trace shows the resulting apparent brightness.

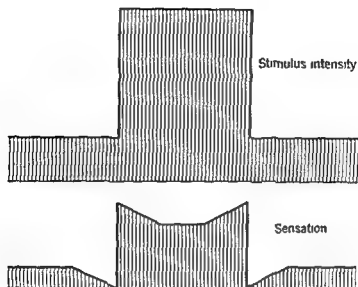


Fig 40 A schematic representation of sensory contrast phenomena

A considerable amount of experimental and theoretical work has been done on contrast phenomena. These effects are generally ascribed to lateral neural inhibition, which was clearly demonstrated by Hartline and his co-workers in experiments on the limulus eye (Hartline, 1949, Hartline & Ratliff, 1957, Ratliff, Hartline, & Miller, 1963). Since such an inhibition may occur at several neural levels and its mechanism is not known in all necessary detail, v. Békésy suggested a fictitious neural unit as a sensation element. The unit could also be called a response unit since it directly relates the sensation to the stimulus. It consists of an area of sensation surrounded by an area of inhibition as shown in Fig. 41 (v. Békésy, 1960b). The upper drawing (a) schematizes an introspective impression of sensation distribution in a one-point experiment on the skin, the lower drawing shows a simplification of (a) for computational purposes. Because of the smallness of the unit, the simplification does not materially affect the computational results. The height of the sensation area of the unit in Fig. 41b is made equal to the stimulus intensity at a given point of the receptor area. Otherwise, the unit is defined by three parameters: the extent of each of the inhibition areas r , the extent of the sensation area s , and the ratio between the sensation area and the inhibition areas S/R . The extent of the sensation area is not critical. v. Békésy determined the parameters for the eye and the skin (touch) and constructed sensation distributions for various distributions of stimulus intensity. Figure 42 (v. Békésy, 1960b) shows as an example a stimulus increasing according to a constant gradient bounded at both ends by areas of constant stimulus intensity. It can be seen that a depression of sensation is associated with a

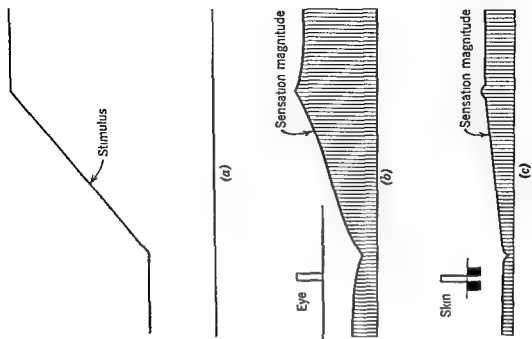


Fig 42 Computed sensation distributions for the eye (b) and the skin (c) obtained by superposition of response units according to the stimulus distribution at the top (a) Adapted with permission from R. A. 1.2

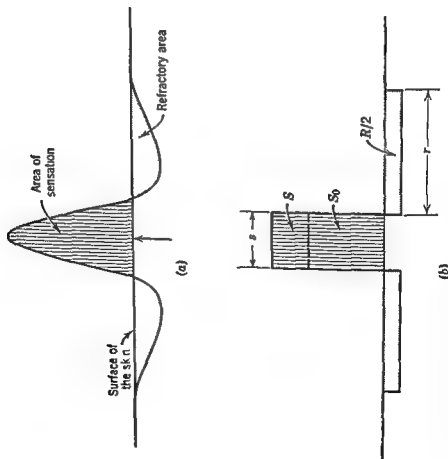


Fig 41 Sensation and inhibition areas due to pressure k of a single point on the surface of the skin (a) The lower drawing shows a simplification of the sensation distribution for computational purposes Adapted with permission from v. Bekesy (1960b, p. 1061)

positive second derivative of stimulus intensity and an enhancement with a negative second derivative. The extent of the areas of suppression and enhancement correlates with the extent of the elementary area of inhibition, which is much greater for the eye than for the skin. More exactly, the extent of these areas is equal to $2r$.

The numerical values of the parameters of the response unit can be determined from psychophysical experiments. For the determination of the elementary inhibition area, v Békésy used a trapezoidal distribution of stimulus intensity, as shown in Fig 43. When the length of the top of the trapezoid is shorter than or equal to r , the sensation distribution is flat topped. When the length r is exceeded, however, the sensation magnitude shows a maximum at each of the two upper edges of the stimulus distribution.

The extent of the sensation area s may be determined, according to v Békésy, by means of a two-point stimulation. When the points are close to each other, a unimodal distribution of sensation magnitude is obtained. When the separation exceeds the width s , two sensation maxima become apparent.

The ratio S/R was determined by v Békésy with the help of a trial and-error method. He first found a distribution of stimulus intensity producing a uniform distribution of sensation magnitude, as shown in Fig 44. In the next step, he matched this distribution by varying the S/R ratio.

v Békésy's response unit should be applicable to hearing, although the corresponding computations have not been made as yet. The distribution of the stimulus magnitude over the sensory cells in the cochlea is controlled by the frequency spectrum at the oval window and by the pattern of vibration of the basilar membrane. A pure tone produces an amplitude pattern of volume displacement as shown in Fig 12. It can be transformed

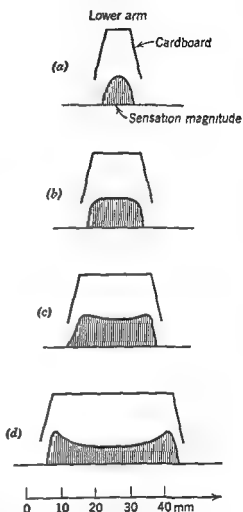


Fig 43 Sensation distribution on the skin as a function of the width of a trapezoid stimulus. Adapted with permission from v Békésy (1960b p 1066)

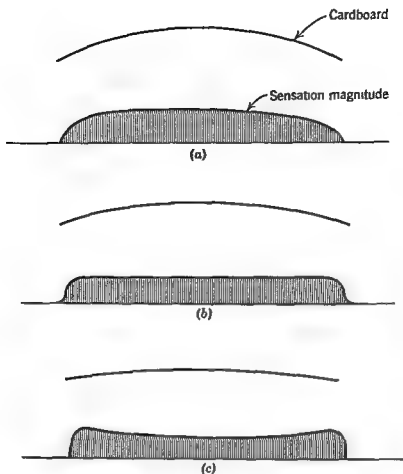


Fig 44 Sensation distribution for various curvatures of a wide stimulus The curvature under (b) produces an even distribution Adapted with permission from v Békésy (1960b p 1067)

into a pattern of displacement amplitude of the basilar membrane by taking into account the width of the membrane, according to the procedure outlined in Sec 4 When two or more frequency components are introduced, the resulting effective amplitude amounts to

$$Y = (Y_1^2 + Y_2^2 + \dots + Y_n^2)^{1/2} \quad (198)$$

The distribution of the effective amplitude as a function of the distance from the stapes is shown in Fig 45 for two tone complexes in the frequency range between 500 and 1000 cps The solid line corresponds to a frequency separation of one critical band between the two components, and the intermittent line to a separation of two critical bands It can be seen that, in the first instance, the amplitude distribution has a plateau whose width is one-half of a critical band, in the second instance, the plateau is widened to a whole critical band

According to v. Bekesy's procedure, a flat-topped, approximately trapezoidal distribution of stimulus magnitudes makes it possible to determine the extent of the inhibition area. When the flat top is less than half the extent of the total inhibition area of a response unit, the distribution of the sensation magnitude is unimodal, when it exceeds one-half of the inhibition area, the sensation magnitude shows two maxima coinciding with the edges of the flat top. For a flat top equal to one-half the inhibition area, the sensation distribution has a flat top also.

Greenwood (1961) performed masking experiments with two-tone complexes and found that the distribution of the masking effect with respect to sound frequency is unimodal as long as the spacing between the components is less than one critical band. For greater frequency differences, the distribution becomes bimodal. Characteristically, the appearing maxima are closer together than the frequency components of the masking complex.

By comparing Greenwood's results (Fig. 46) to the amplitude distribution of Fig. 45, we can see that the transition from unimodal to bimodal masking distribution occurs when the flat top of the amplitude distribution has a width of one-half a critical band. If we agree to consider the masking effect as a measure of neural excitation, we can conclude using v. Bekesy's procedure that the critical band is equal to the extent of the inhibition area of the response unit,

$$\kappa \approx 2r \quad (199)$$

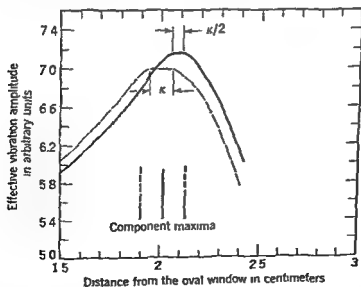


Fig. 45 Distribution of the effective vibration amplitude along the cochlear partition for two-tone complexes in the frequency range from 500 to 1000 cps. The solid lines correspond to a frequency separation between the components of one critical band, and the intermittent lines correspond to a separation of two critical bands.

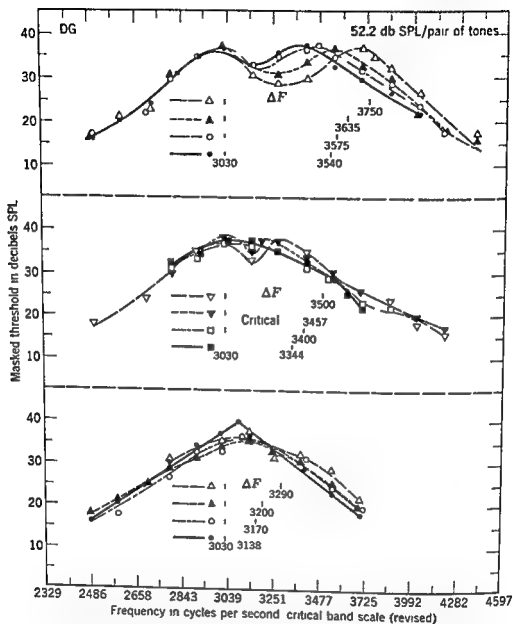


Fig 46 Masking patterns produced by two-tone complexes for various frequency separations between the components Adapted with permission from Greenwood (1961, p 498)

Since the critical band does not change with sound-pressure level, the same is true for the extent of the unit inhibition area. It appears to be a system constant determined by the neural network. This constancy does not necessarily apply to the ratio S/R of the sensation and inhibition areas, but there is no simple way to investigate this at present.

Although v. Békésy's procedure is essentially graphical, he indicates a possible analytical method. It is based on the superposition theorem. In a

first step, the stimulus function is approximated by unit step functions, as shown in Fig. 47. Next, the sensation area of the response unit is divided into two components, S_0 such that $S_0/R = 1$, and the remainder of the area, $S - S_0$. With the modified response unit $S_0 - R$, the sensation distribution is constructed for the step function. This distribution has values different from zero only in the neighborhood of the step. It is used as a

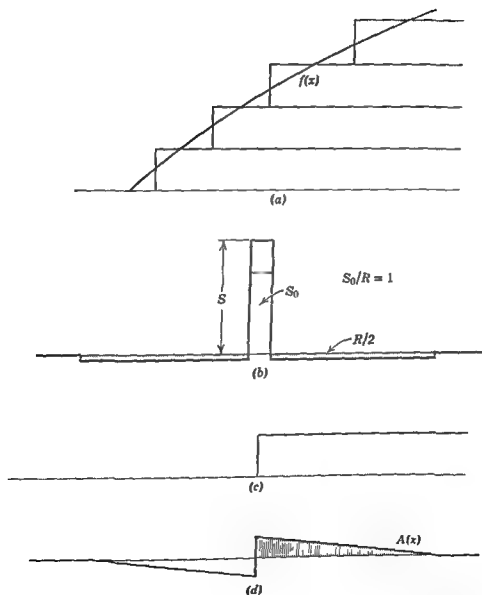


Fig. 47 Analytical method for calculation of sensation distribution by means of superposition of v. Békésy's response units. (a) approximation of stimulus distribution by means of step functions, (b) response unit, (c) step function, (d) response distribution for the step function under (c) obtained by means of superposition of response units. Adapted with permission from v. Békésy (1960b, p. 1069).

new elementary function $A(x)$ According to the superposition theorem, the sensation distribution for $f(x)$, based on S_0 - R units, is

$$y_0(x) = \int_{-\infty}^{+\infty} A(x - \lambda) f'(\lambda) d\lambda, \quad (200)$$

where x is the coordinate at which the sensation magnitude is being determined and λ is the distance between x and the origin of each step function Thus the sensation distribution based on the entire response unit S - R is

$$y(x) = (S - S_0)f(x) + y_0(x) \quad (201)$$

It is too early to evaluate the usefulness of v Bekesy's response unit Its disadvantage is that, in general, the ratio S/R changes with stimulus intensity Nevertheless, it has already fulfilled the purpose of indicating the extent of unit inhibition areas in the eye, in the skin for touch, and in the ear

6 THE LOUDNESS FUNCTION

Although pitch may be regarded as a qualitative attribute of sound, loudness is an intensive one Pitch changes by substitution and was classified by Stevens (1957) as a metathetic continuum Loudness changes by addition and belongs therefore to the class of prothetic continua

The question of loudness quantification has been a subject of psychophysical controversy since Fechner's time, and today psychophysicists are still divided on the subject (Stevens, 1960) One of the main issues appears to be the subjective size of just noticeable differences, another is whether loudness can be obtained by integrating the just noticeable steps Fechner assumed that the size of the jnd's was constant and that the total magnitude of sensation was proportional to their number On the basis of a large number of scaling experiments, Stevens (1961) maintains that neither postulate holds true Nevertheless, Zwicker (1958) has been able to show that the integral of intensity jnd's is equal to total loudness, provided that the assumption is made that not the subjective size of the jnd's but its ratio to the total magnitude of sensation is constant

We shall not go deeper into the controversy, but demonstrate, instead, that direct scaling experiments resulting from Steven's work lead to a construct that is coherent and consistent with the general knowledge of the auditory system To do that, we have to assume that observers can function as null indicators and correctly detect deviations from sensation identity or, in other words, that it is possible to establish sensation

identity Such an axiom has never been seriously questioned, and loudness matches are accepted as standard experimental and clinical procedures They are even used for earphone calibration

Perfect stimulus identity is practically impossible to achieve, but it can be closely approximated when a sound with a certain frequency and phase spectrum is compared in loudness to another sound with the same spectrum Such a comparison is trivial, of course unless the relation between the sound pressure and loudness of one sound is different from that of the other The necessary variation may be produced by means of a masking noise or as a result of certain pathological changes in the auditory system

A considerable number of experiments were performed in which the loudness of a masked pure tone was compared to that of the same tone in the absence of masking, unfortunately, only a few of these experiments were carefully balanced A monaural experiment designed especially to validate the loudness scaling procedures was performed in the following general way (Hellman & Zwislocki, 1964) A 1000 cps tone was presented to one ear together with a random noise whose sound-pressure level was varied parametrically The observers estimated the loudness of the tone by assigning numbers according to the procedure of magnitude estimation without a designated standard and, later, by adjusting the loudness to match given numbers according to the method of magnitude production, also without a designated standard The combined method is called "numerical magnitude balance" By presenting the tone intermittently with the noise in an auxiliary experimental series, small biases produced by listening to one signal in the presence of another were eliminated Next, the loudness of the masked tone was compared by a method of adjustment to the loudness of the same tone in the absence of masking In order to minimize biases, the masked tone was adjusted in one series and the unmasked in the other The final results of direct loudness matches are compared to those derived from numerical magnitude balance in Fig 48 The closed circles indicate the direct results, the curves, the derived ones The evident agreement between them could not have been achieved unless the observers judged correctly the loudness ratios in the procedure of numerical magnitude balance The conclusion that observers can express loudness magnitudes in numbers appears inescapable It is reinforced by the agreement of the direct loudness matches with results of similar experiments performed by other investigators who used balanced procedures Loudness matches between one normal and one pathological ear obtained on a large number of patients have led to similar data (Miskolczy-Fodor, 1960)

The direct results obtained by means of numerical magnitude balance are plotted in Fig 49 as a function of sound pressure level The highest

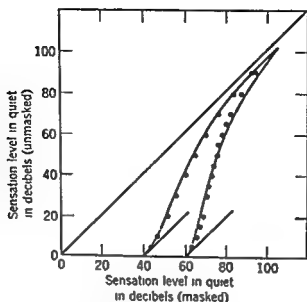


Fig 48 Sensation level of an unmasked 1000 cps tone versus the sensation level (in quiet) of the same tone in presence of masking noise when both produce the same loudness. The circles indicate the results of direct loudness matches, the curves were derived from numerical magnitude balancing. Adapted with permission from Hellman & Zwislacki (1964).

loudness values were obtained in the absence of the masking noise. The remaining two curves show the course of the loudness function when the masking effect amounts to 40 db and 60 db respectively. Open circles on the abscissa axis indicate the corresponding threshold levels. At high intensities, all curves approach asymptotically a straight line obeying the power function

$$L = Kp^{2\theta} \quad (202)$$

This function is an expression of Steven's (1957) power law, which appears to have a general validity for prosthetic sensory continua. The constant K depends on the choice of units and also on the sense modality and certain stimulus parameters. The exponent θ is grossly characteristic of each sense modality, but it also appears to depend slightly on stimulus parameters. In Fig 49, $K \cong 3.1$ ($\text{dynes}^2 \text{cm}^{-4}$)^{-2\theta} and $\theta \cong 0.27$. A binaural loudness curve for 1000 cps is somewhat steeper (Reynolds & Stevens, 1960, Hellman & Zwislacki, 1963). Usually, the units are so chosen that the curve passes through the point 40 db sensation level and one sone, which is the unit of loudness (Stevens, 1936).

We now can attempt to derive a general mathematical expression for the loudness function as exemplified by the family of curves in Fig 49 (Hellman & Zwislacki, 1964). This may be done with the help of the power law and an auxiliary psychophysical phenomenon which appears well established

The loudness of a sound signal consisting of more than one frequency component depends not only on the sound pressure of the components but also on the frequency spacing between them (Zwicker & Feldtkeller, 1955; Bauch, 1956; Zwicker, Flottorp, & Stevens, 1957). When the frequency intervals are so large that there is little interference among the components, the total loudness is equal to the sum of component loudnesses (Fletcher, 1940). When the frequency spectrum is limited to one critical band, the loudness depends on the total power, irrespective of its

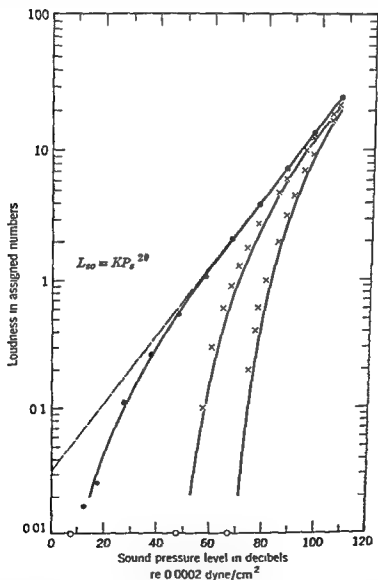


Fig. 49. Loudness of a monaurally presented 1000 cps tone in quiet and in presence of a random masking noise. The curves indicate averages of experimental data, and the open circles the corresponding threshold levels. The closed circles and crosses indicate the theoretical loudness values.

frequency distribution. Thus we can write

$$L = K \left(\sum_i^{1+CB} P^2 \right)^{\theta} \quad (203)$$

for the loudness function at sufficiently high sound-pressure levels. The complex sound may consist, for instance, of a band of random noise and a sinusoid in its center. For such conditions, Eq. 203 may be rewritten in the form

$$L = K(P_S^2 + P_V^2)^{\theta}, \quad (204)$$

where P_S means the effective pressure of the sinusoid and P_A that of the random noise.

Because of the analytic properties of our auditory system which go beyond the phenomena described by the place theory (Licklider, 1959), we are able to listen either to the total acoustic event, to the random noise alone, or to the sinusoid alone. Figure 49 indicates what happens to loudness in the latter event. It is quite clear that the loudness of the random noise is disregarded since the perceived loudness approaches zero at 47 or even 67 db, where the noise is well above its threshold of audibility.

Probably the simplest way of representing mathematically the phenomenon of selective listening is to subtract the loudness of noise from the total loudness. When the spectrum is limited to one critical band, this leads to the equation

$$L_S = K(P_S^2 + P_A^2)^{\theta} - L_V \quad (205)$$

for the loudness of the sinusoid. From Eqs. 202 and 203,

$$L_N = K P_N^{2\theta}, \quad (206)$$

hence we have

$$L_S = K[(P_S^2 + P_N^2)^{\theta} - P_N^{2\theta}] \quad (207)$$

In Secs. 4 and 5, the existence of an intrinsic physiological noise was mentioned. In Sec. 5, it was assumed that the threshold of audibility in quiet is determined by the masking effect of this noise. This assumption, together with the critical band concept, permits a calculation of the equivalent stimulus power of the intrinsic noise, with the result that

$$P_{NI}^2 = 2.5 P_T^2, \quad (208)$$

where P_{NI} is the equivalent sound pressure of the intrinsic noise and P_T is the threshold sound pressure. The noise sound pressure in Eq. 207 may be regarded, therefore, as if it consisted of two components: an extrinsic component P_{NE} and an intrinsic component P_{NI} . Consequently,

$$P_N^2 = P_{NE}^2 + P_{NI}^2, \quad (209)$$

and Eq 207 can be written in the form

$$L_S = K [(P_S^2 + P_{NE}^2 + P_{NI}^2)^0 - (P_{NE}^2 + P_{NI}^2)^0] \quad (210)$$

When no extrinsic noise is introduced, Eq 210 becomes

$$L_S = K [(P_S^2 + P_{NI}^2)^0 - P_{NI}^{20}], \quad (211)$$

or introducing the value of P_{NI}^2 from Eq 208

$$L_S = K [(P_S^2 + 2.5 P_T^2)^0 - (2.5 P_T^2)^0] \quad (212)$$

This result is similar to Zwicker's (1958), which was obtained in a different way. The closed circles in Fig 49 show that Eq 212 accurately describes the monaural loudness function for a 1000 cps tone presented in quiet. A certain point in the derivation remains to be clarified, however. It was assumed at the outset that the spectral power is limited to one critical band. This is not true for the intrinsic noise, and the contributions of bands outside the critical band centered at 1000 cps have to be considered. According to Feldtkeller and Zwicker (1956) and other investigators, the total loudness of an acoustic event of low power whose spectrum exceeds one critical band is simply an arithmetic sum of loudness contributions of all critical bands included in the spectrum. Consequently, Eq 211 for the loudness of a sinusoid in the presence of intrinsic noise, or any other weak noise, has to be modified as follows

$$L_S = K[(P_S^2 + P_{NI}^2)^0 - P_{NI}^{20}] + \sum L_{NI} - \sum L_{NI} \quad (213)$$

The sums $\sum L_{NI}$ exclude the band containing the sinusoid. Since they are identical but of opposite sign, they cancel each other out, and we see that Eq 213 is identical to Eq 211.

At higher sound pressure levels, the situation is more complex. Even a pure tone excites more than one critical band, and the excitations of different bands partially cancel each other. Under such conditions, it is still possible to write for the loudness contribution of a critical band to the loudness of a sinusoid heard in presence of random noise

$$L_{S\kappa} = K_{\kappa}[(P_{S\kappa}^2 + P_{N\kappa}^2)^0 - P_{N\kappa}^{20}], \quad (214)$$

where $P_{S\kappa}$ is the effective equivalent sound pressure produced by the sinusoid in the critical band κ and $P_{N\kappa}$ is the effective sound pressure produced by the masking noise in the same band. The total loudness of the sinusoid amounts to

$$L_S = \sum K_{\kappa}[(P_{S\kappa}^2 + P_{N\kappa}^2)^0 - P_{N\kappa}^{20}], \quad (215)$$

in which the sum can be limited to those bands for which $P_{S\kappa} > 0$. As in Eq 213, terms contributed by other bands cancel each other out. Except

in the immediate vicinity of the masked threshold of audibility,

$$(P_{S\kappa}^2 + P_{N\kappa}^2)^0 \cong P_{S\kappa}^{20},$$

so that for high and moderate sensation levels, Eq 215 may be approximated by

$$L_S = \sum K_{\kappa}(P_{S\kappa}^{20} - P_{N\kappa}^{20}), \quad (216)$$

or separating the sum terms

$$L_S = \sum K_{\kappa}P_{S\kappa}^{20} - \sum K_{\kappa}P_{N\kappa}^{20} \quad (217)$$

In this equation, the first sum represents the total loudness of the sinusoid in absolute quiet and the second sum represents the partial loudness of noise that is contributed by the critical bands excited by the sinusoid. The number of these bands is limited to one at the threshold of audibility and increases with the sound pressure level of the sinusoid. For noise bands of moderate width and for sufficiently high sensation levels of the tone, all bands that are excited by the noise are also excited by the tone, and so the second sum of Eq 217 is equal to the total loudness of the noise when presented alone. Under such conditions, it is possible to write

$$L_S = L_{S0} - L_{N0}, \quad (218)$$

where the subscripts 0 indicate the loudness of each sound presented separately. More accurately, according to Eq 207, we have

$$L_S = K[(P_S^2 + P_N^2)^0 - P_N^{20}] \quad (219)$$

The masked curves of Fig 49 were obtained with a band of noise of approximately one octave. Such a noise centered at 1000 cps should produce a signal to noise ratio of approximately -10 db at the threshold of audibility. For these conditions Eq 219 may be rewritten in the form

$$L_S = K[(P_S^2 + 10P_T^2)^0 - 1.9P_T^{20}] \quad (220)$$

Loudness values calculated by means of Eq 220 are plotted in Fig 49 by crosses. They follow the shape of the experimental curves quite accurately, but are displaced by 1 to 2 db toward smaller sound pressures. Of course, a nearly constant error of this size is quite insignificant. It may stem from a somewhat inaccurate sound pressure measurement, an experimental bias, as well as from a small interaction not included in the mathematical derivation. The general agreement between the theoretical and experimental data in Fig 49 is sufficient to validate the theory, at least as a first order approximation.

The loudness equations derived earlier, especially Eq 212, lead to an interesting conclusion concerning the threshold of audibility as defined

for the purposes of this chapter At the threshold, $P_S = P_T$, and Eq 212 becomes

$$L_S = K [(3.5 P_T^2)^0 - (2.5 P_T^2)^0], \quad (221)$$

which means that $L_S > 0$ This is as it should be By definition, the signal can be perceived 50% of the time, and since there is no negative loudness, the average loudness must be greater than zero

The derivations of this section are based on the tacit assumption that the primary loudness function,

$$L = KP^{2.0},$$

is determined by a stimulus transformation in the end organ, and that the operations of higher centers are limited to algebraic summation This postulate has been validated indirectly by the experimental verification of the obtained equations If it is not disproved by further research, it may become of fundamental importance in understanding the functioning of the brain

As a corollary to the conclusion that higher centers are limited to algebraic summation, we should expect loudness to be proportional to the peripheral neural activity Physiological experiments of Katsuki and his co workers (1962) provide an opportunity for a preliminary check in this respect Katsuki investigated the neural activity in the peripheral and central auditory system of monkeys He made a statistical evaluation of thresholds in neurons of the first order and correlated these thresholds with the dynamic response characteristics Of particular interest is his investigation of responses to preferred frequencies that is, to frequencies that are associated with the lowest threshold of a given neuron Katsuki found that these minimum thresholds are normally distributed within two populations, a large population with low and medium thresholds and a small population with high thresholds In neurons with low thresholds, the firing rate increases with the sound-pressure level slowly, and in those with higher thresholds, more rapidly On the basis of Katsuki's data, a smooth curve can be drawn indicating the rate at which the firing increases with sound pressure level as a function of the threshold level Using this curve, a firing characteristic can be drawn for each class of neurons A somewhat idealized family of such characteristics is plotted in Fig 50 It shows a striking regularity The responses grow approximately according to a logarithmic function of sound pressure, and the slope increases with the sound pressure level at the threshold This sound pressure level is indicated by the points where the curves cross the abscissa axis The preferred frequencies of the neurons, whose characteristics are approximated in Fig 50, were included within the range of 600 and 1000 cps The numbers on the curves indicate the percentage of neurons in each

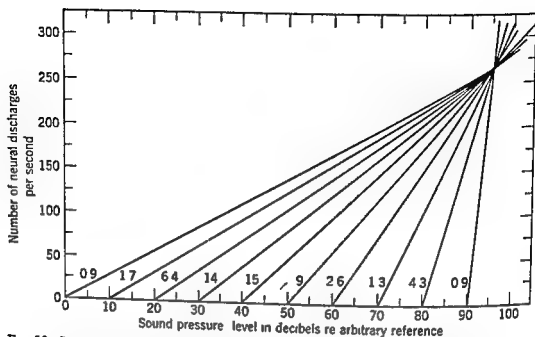


Fig 50 Simplified firing characteristics of the first-order afferent neurons in the auditory system. The crossings of the firing curves with the abscissa axis indicate the corresponding threshold levels. The numbers on the curves indicate the percentage of neurons associated with each characteristic.

threshold category, they follow Katsuki's distribution curves. The spontaneous neural activity has been omitted.

Figure 50 makes it possible to investigate the growth of neural firing in the group of units responding maximally to a given sound frequency. At low and medium intensities, the total response may be considered roughly proportional to that of these fibers. The necessary integration may be performed numerically by selecting a sound-pressure level and summing all the response rates associated with it. In so doing, the firing rate represented by each curve has to be multiplied by the percentage of neurons associated with it. A normalized result of the calculation is shown in Fig 51 by crosses. The solid curve in the same figure follows the previously discussed loudness function for 1000 cps. When the reference sound pressure is selected so that 10 db in Fig 50 corresponds to 0.0002 dyne/cm^2 , and the response rates are divided by approximately 10^4 , the rate function of neural discharges coincides with the loudness function at low and moderate sound-pressure levels. At higher levels, the calculated neural response increases less rapidly. This has to be expected, since there is an increase in the contribution of neurons that respond maximally to frequencies other than that of the stimulus, and the total response progressively deviates from the calculated one. The coincidence between the two functions of Fig 51 over a range of 50 db is remarkable, especially since the physiological response was recorded on monkeys and the loudness

function was produced by human observers. In order to show that the coincidence is unlikely to be fortuitous, we can introduce two additional points of information. First, the sound-pressure level identified with the threshold of audibility agrees with the level for which the response of the most sensitive neurons in Katsuki's experiments begins to rise above the spontaneous activity of these neurons. Second, the numerical constants of the loudness function (Eq. 212) appear reasonable when applied to physiological measurements. Assuming that the discrepancy between the calculated neural response and the loudness curve at higher sound-pressure levels stems from the omission of a portion of active fibers, we can conclude that the total rate of neural firing obeys the same function as loudness. Consequently, Eq. 212 should hold with the exception of the numerical value of K , which is about 10^4 times larger than for loudness. We obtain for the total rate of neural firing

$$N_S = K^*[(P_S^2 + 2.5P_T^2)^6 - (2.5P_T^2)^6], \quad (222)$$

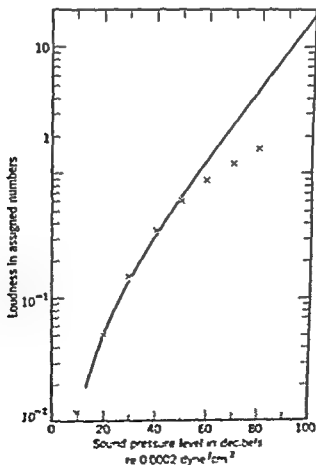


Fig. 51. A comparison between the unmasked loudness curve of Fig. 49 and a computed rate of neural firing.

where $K^*(2.5P_T^2)^0$ should be equal to the total spontaneous activity N_{NI} of the considered group of fibers. If the threshold is assumed to be the same as for human observers, the threshold sound pressure is 7 db above 0.0002 dyne/cm², that is, at $4.5 \cdot 10^{-4}$ dyne/cm². With this value, $K^* = 3.1 \cdot 10^4$ and $\theta = 0.27$, the spontaneous activity in a group of 100 fibers is

$$N_{NI} = 530 \text{ sec}^{-1}$$

This result appears to be of the same order of magnitude as the spontaneous activity observed by Katsuki. Some of the most sensitive fibers investigated by him fired spontaneously at a rate of 70 per second, and it is possible to estimate from his diagrams that 10 to 20% of the fibers exhibited spontaneous activity. Thus Eqs. 212 and 222 describe accurately the perceived loudness as well as the peripheral neural activity, and the prior conclusion that the auditory nervous system above the level of the end organ behaves like a linear system is confirmed. A further verification can be found in an analysis of loudness as a function of stimulus duration or repetition rate.

It is known that the loudness of short noise or tone bursts increases with burst duration (v. Bekesy, 1929, Munson, 1947, Miller, 1948, Garner, 1948). The auditory system behaves as if it integrated acoustic power in a similar way to that found at the threshold of audibility. The only clear difference can be found in the time constant of integration which appears to decrease with sensation level (Miller, 1948). Thus, loudness as a function of stimulus duration could be described by a set of equations similar to that derived for the threshold. A more careful scrutiny of the problem establishes, however, that the situation is more complex, and that loudness is not always proportional to the time integral of power. A more generally valid theory emerges from the analyses performed for the threshold of audibility as a function of stimulus duration and for loudness as a function of sound pressure provided that one additional piece of physiological information is added. At the same time, all postulates accepted in earlier parts of this chapter and the resulting theorems maintain their validity. In particular, the steady-state expression for the loudness function and the time constant of temporal summation remain unchanged. These rules mean that the elementary neural excitation below the level of temporal integration is given by

$$\epsilon = K[(P_S^2 + 2.5P_T^2)^0 - (2.5P_T^2)^0], \quad (223)$$

and a short stimulus produces at the level of integration an excitation

$$\eta = \epsilon e^{-\alpha t} = K[(P_S^2 + 2.5P_T^2)^0 - (2.5P_T^2)^0]e^{-\alpha t}, \quad (224)$$

which decays exponentially with a time constant $1/\alpha = 200 \text{ msec}$. In

Sec. 4, we derived an equation for intermittent stimuli and expressed it as a ratio of excitations produced by an infinite series of bursts and one burst. This equation has the form

$$\frac{\eta_i}{\eta_B} = \frac{\epsilon_i/\epsilon_B}{1 - e^{-a/v}}, \quad (225)$$

or in terms of sound pressure,

$$\frac{\eta_i}{\eta_B} = \frac{(P_{Si}^2 + 2.5P_T^2)^\theta - (2.5P_T^2)^\theta}{[(P_{SB}^2 + 2.5P_T^2)^\theta - (2.5P_T^2)^\theta](1 - e^{-a/v})} \quad (226)$$

For sufficiently high sound-pressure levels, it can be simplified to

$$\frac{\eta_i}{\eta_B} = \frac{P_{Si}^{2\theta}}{P_{SB}^{2\theta}(1 - e^{-a/v})} \quad (227)$$

Equality of loudness is considered to be identical with the equality of cumulative excitation, so that $\eta_i = \eta_B$. Under these conditions

$$\frac{P_{SB}^{2\theta}}{P_{Si}^{2\theta}} = \frac{1}{1 - e^{-a/v}}, \quad (228)$$

or in decibels,

$$20 \log \frac{P_{SB}}{P_{Si}} = \frac{1}{\theta} 10 \log \left(\frac{1}{1 - e^{-a/v}} \right) \quad (229)$$

Since $\theta = 0.27$, $1/\theta = 3.7$, and the sound-pressure level changes 3.7 times as rapidly with increasing repetition rate as at the threshold of audibility. This is in contradiction to direct experiments that result in a considerably smaller slope. The disagreement is due to an important omission in the derivation of Eq. 225. This derivation was based on the assumption that, for equal elemental stimuli, the elemental excitations are equal. Such an assumption is reasonable for near-threshold sound pressures, but not for suprathreshold stimulation. McGill (1952) has demonstrated that a preceding click depresses the neural response to a following one. Peake, Goldstein, and Kiang (1962) measured the magnitude of peripheral response as a function of repetition rate of short noise bursts and found that it decreased rapidly with repetition rate as well as with burst duration and sound pressure. Their results can be described quite well by the empirical formula

$$\epsilon_v = \epsilon_R(1 - e^{-\phi/v}), \quad (230)$$

where the constant ϕ depends on sound pressure and burst duration. Peake's results for one set of parameters are compared to values obtained by means of Eq. 230 in Fig. 52. Agreement with other of his results is

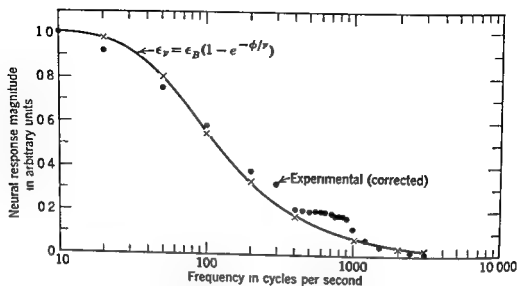


Fig 52 Amplitudes of averaged N_1 responses in the auditory nerve as a function of the repetition rate of short noise bursts (Peake et al 1962). The crosses and the curve show a mathematical approximation of the data

equally good or better. Under such conditions, Eq 228 must be corrected to

$$\frac{P_{SB}^{\text{eq}}}{P_{St}^{\text{eq}}} = \frac{1 - e^{-\phi/v}}{1 - e^{-a/v}}, \quad (231)$$

and we obtain in decibels

$$20 \log \frac{P_{SB}}{P_{St}} = \frac{10}{\theta} \log \left(\frac{1 - e^{-\phi/v}}{1 - e^{-a/v}} \right) \quad (232)$$

Provided that the parameter ϕ is known, Eq 232 determines values of P_{SB}/P_{St} and v producing equal loudness. The results of the calculation may be checked against data obtained by Pollack (1957), who investigated the loudness level of an intermittent white noise as a function of burst duration and interburst interval. The observers matched the loudness of the intermittent noise to that of a continuous one maintained at a constant sound-pressure level of 82 db. Pollack's results are shown in Fig 53 in terms of difference in sound-pressure level between the intermittent and the continuous noise; they are plotted as a function of burst repetition rate for three burst durations. Curves in the same figure indicate theoretical predictions for various values of ϕ . Unfortunately, Peake and his co-workers used only two burst durations and lower sound pressure levels than Pollack, so that an accurate determination of the parameter ϕ is not possible for Pollack's conditions. Nevertheless, it can be estimated by extrapolation. Thus, for instance, ϕ should lie between 40 and 50 for a sound pressure level of approximately 80 db and 0.1 msec bursts. In the vicinity of 110 db, ϕ should decrease to about 30. Eq 232 shows

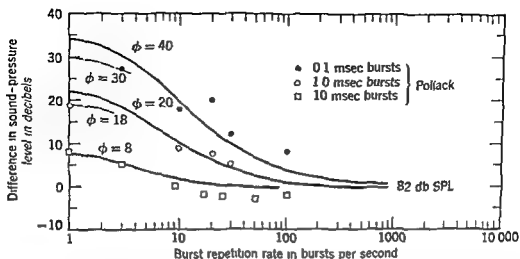


Fig 53 Loudness level of an intermittent white noise as a function of burst repetition rate and burst duration. The points have been obtained experimentally by Pollack (1958), the curves are calculated. Adapted with permission from Pollack (1958 p 182)

that these values produce differences in sound-pressure level that are in reasonable agreement with those obtained by Pollack. Furthermore, it should be noted that the position and the slope of the theoretical curves are related to each other in approximately the same way as the corresponding parameters of the experimental results. For low repetition rates, the theoretical sound-pressure level decreases at a rate of approximately 3 db for each doubling of burst duration. This also is in agreement with the experimental findings. The only effect that is not accounted for theoretically is the slightly negative level difference at moderately high repetition rates and for burst durations equal to or greater than 1 msec. The effect may be due to a slow neural adaptation whose existence was ascertained by several investigators and which was omitted from the theory. For the purposes of this chapter, the effect is of little interest. We can conclude, therefore, that the theoretical and the experimental data of Fig 53 are mutually consistent. When more relevant empirical results are accumulated, the theory discussed in this section should make it possible to calculate loudness as a function of spectral sound pressure and temporal pattern, that is, for any acoustic stimulus.

References

- Bauch, H. Die Bedeutung der Frequenzgruppe für die Lautheit von Klängen. *Acustica*, 1956, 6, 40-45.
- Beck, J., & Shaw, W. A. Magnitude estimations of pitch. *J. acoust. Soc. Amer.*, 1962, 34, 92-98.
- v. Békésy, G. Zur Theorie des Hörens, Die Schwingungsform der Basilar membran. *Physik Zeits.*, 1928, 29, 793-810.

- v Bekesy, G Zur Theorie des Horens, Ueber die Bestimmung des einem Reinen Tonempfinden Entsprechenden Erregungsgebietes der Basilmembran Vermitteltst Ermundungserscheinungen *Physik Zeits*, 1929, 30, 115-125
- v Bekesy, G Zur Theorie des Horens, Ueber die Eben Merkbaren Amplituden und Frequenzänderungen eines Tones, Die Theorie der Schwebungen *Physik Zeits*, 1929, 30, 721-745
- v Bekesy, G Ueber die Messung der Schwingungsamplitude der Gehörknochenchen Mittels einer Kapazitiven Sonde *Akust Zeits*, 1941, 6, 1-16
- v Bekesy, G Ueber die Elastizität der Schneckentrennwand des Ohres *Akust Zeits*, 1941, 6, 265-278
- v Bekesy, G Ueber die Schwingungen der Schneckentrennwand beim Präparat und Ohrenmodell *Akust Zeits*, 1942, 7, 173-186
- v Bekesy, G Ueber die Resonanzkurve und die Abklingzeit der verschiedenen Stellen der Schneckentrennwand *Akust Zeits*, 1943, 8, 66-76
- v Bekesy, G The variation of phase along the basilar membrane with sinusoidal vibrations *J acoust Soc Amer*, 1947, 19, 452-460
- v Bekesy, G Sensations on the skin similar to directional hearing, beats, and harmonics of the ear *J acoust Soc Amer*, 1957, 29, 489-501
- v Békesy, G *Experiments in hearing* New York McGraw-Hill, 1960 (a) P 745
- v Bekesy, G Neural inhibitory units of the eye and skin Quantitative description of contrast phenomena *J opt Soc Amer*, 1960, 50, 1060-1070 (b) Also in R D Luce R R Bush, J E Galanter (Eds) *Readings in mathematical Psychology*, Vol II New York Wiley, 1964 Pp 229-309
- v Bekesy, G, & Rosenbluth, W A The mechanical properties of the ear In S S Stevens (Ed), *Handbook of experimental psychology* New York Wiley, 1951 Pp 1075-1115
- Beranek, L L *Acoustic measurements* New York Wiley, 1949 P 914
- Davis, H Excitation of auditory receptors Vol 1, Sec 1 *Handbook of physiology* Washington D C American Physiological Society, 1959 Pp 565-584
- Egan, J P, & Hake, H W On the masking pattern of a simple auditory stimulus *J acoust Soc Amer*, 1950, 22, 622-630
- Feldtkeller, R, & Oetinger, R Die Hörbarkeitsgrenzen von Impulsen Verschiedener Dauer *Acustica Akust Beih* 1956, 6, 489-493
- Feldtkeller, R, & Zwicker, E *Das Ohr als Nachrichtenempfänger* Stuttgart S Hirzel, 1956 P 89
- Fletcher, H Auditory patterns *Rev Mod Physics*, 1940, 12, 47-65
- Fletcher, H On the dynamics of the cochlea *J acoust Soc Amer*, 1951, 23, 637-645
- Fletcher, H *Speech and hearing in communication* Princeton, New Jersey Van Nostrand, 1953 P 461
- Galambos, R, & Davis, H The response of single auditory nerve fibers to acoustic stimulation *J Neurophysiol*, 1943, 6, 39-57
- Garner, W R Auditory thresholds for short tones as a function of repetition rate *J acoust Soc Amer*, 1947, 19, 600-608 (a)
- Garner, W R The effect of frequency spectrum on temporal integration of energy in the ear *J acoust Soc Amer*, 1947, 19, 808-815 (b)
- Garner, W R The loudness of repeated short tones *J acoust Soc Amer*, 1948, 20, 513-527
- Garner, W R, & Miller, G A The masked threshold of pure tones as a function of duration *J exp Psychol*, 1947, 37, 293-303
- Green, D M, Birdsall, T G, & Tanner, W R, Jr Signal detection as a function of signal intensity and duration *J acoust Soc Amer*, 1957, 29, 523-531

- Greenwood, D D Auditory masking and the critical band *J acoust Soc Amer*, 1961, 33, 484-502
- Hartline, H K Inhibition of activity of visual receptors by illuminating nearby retinal elements in the limulus eye *Federation Proc*, 1949, 8, 69
- Hartline, H K, & Ratliff, F Inhibitory interaction of receptor units in the eye of limulus *J gen Physiol*, 1957, 40, 357-376
- Hawkins, J E, Jr, & Stevens, S S Masking of pure tones and speech by noise *J acoust Soc Amer*, 1950, 22, 6-11
- Hellman, R P, & Zwislocki, J Monaural loudness function at 1000 cps and interaural summation *J acoust Soc Amer*, 1963, 35, 856-865
- Hellman, R P, & Zwislocki, J Loudness function of a 1000 cps tone in presence of a masking noise *J Acoust Soc Amer*, 1964, 36, 1618-1627
- Helmholtz, H L F *Die Lehre von den Tonempfindungen als Physiologische Grundlage für die Theorie der Musik* Brunswick, Germany Vieweg Verlag, 1863 English translation by A J Ellis, *Sensations of tone* (3rd Ed) New York Dover Publications, 1954 P 576
- Huggins, W H, & Licklider, J C R Place mechanisms of auditory frequency analysis *J acoust Soc Amer*, 1951, 23, 290-299
- Jahnke, E, & Emde, F *Tables of functions with formulae and curves* New York Dover Publications, 1943 Pp 126-266
- Katsuki, Y, Suga, N, & Kanno, Y Neural mechanism of the peripheral and central auditory system in monkeys *J acoust Soc Amer*, 1962, 34, 1396-1410 Also in R D Luce, R R Bush, and E Galanter (Eds) *Readings in mathematical Psychology*, Vol II New York Wiley, 1964 Pp 349-363
- Katsuki, Y, Sumi, T, Uchiyama, H, & Watanabe, T Electric responses of auditory neurons in cat to sound stimulation *J Neurophysiol*, 1958 21, 569-588
- Kinsler, L E, & Frey, A R *Fundamentals of acoustics* New York John Wiley, 1950 P 516
- Kinsler, L E, & Frey, A R *Fundamentals of acoustics* (Rev 2nd Ed) New York John Wiley, 1962, pp 524
- Lamb, H *Hydrodynamics* New York Dover Publications, 1945, pp 1-738
- Licklider, J C R Basic correlates of the auditory stimulus In S S Stevens (Ed), *Handbook of experimental psychology* New York Wiley, 1951 Pp 985-1039
- Licklider, J C R Three auditory theories In S Koch (Ed), Vol 1, *Psychology a study of a science* New York McGraw Hill, 1959 Pp 41-144
- Lindsay, R B *Mechanical Radiation* New York McGraw Hill, 1960 P 415
- McGill, W J *A statistical description of neural responses to clicks recorded at the round window of the cat* Ph D Thesis, Harvard Univer, 1952 P 57
- McGill, W J, & Rosenbluth, W A Electrical responses to two clicks a simple statistical interpretation *Bull Math Biophysics*, 1951, 13, 69-77
- Miller, G A The perception of short bursts of noise *J acoust Soc Amer*, 1948, 20, 160-170
- Miskolczy Fodor, F Relation between loudness and duration of tonal pulses III Response in cases of abnormal loudness function *J acoust Soc Amer*, 1960, 32, 486-492
- Møller, A R Network model of the middle ear, *J acoust Soc Amer*, 1961, 33, 168-176
- Morton, J Y, & Jones, R A The acoustical impedance presented by some human ears to hearing aid earphones of the insert type, *Acustica*, 1956, 6, 339-345
- Munson, W A The growth of auditory sensation *J acoust Soc Amer*, 1947, 19, 584-591

- Olson, H F *Acoustical engineering* Princeton, New Jersey Van Nostrand, 1957 P 718
- Olson, H F *Dynamical analogies* Princeton, New Jersey D Van Nostrand, 1958 P 278
- Onchi, Y Mechanism of the middle ear *J acoust Soc Amer*, 1961, 33, 794-805
- Papoulis, A *The Fourier integral and its applications* New York McGraw-Hill, 1962 P 318
- Peake, W T, Goldstein, M H, Jr, & Kiang N Y-S Responses of auditory nerve to repetitive acoustic stimuli *J acoust Soc Amer*, 1962, 34, 562-570
- Peake, W T, Kiang, N Y-S, & Goldstein, M H, Jr Rate functions for auditory nerve responses to bursts of noise effect of changes of stimulus parameters *J acoust Soc Amer*, 1962, 34, 571-575 Also in R D Luce, R R Bush, & E Galanter (Eds) *Readings in mathematical psychology*, Vol II New York Wiley, 1964 Pp 372-376
- Pollack, I Loudness of periodically interrupted white noise *J acoust Soc Amer*, 1958, 30, 181-185 Also in R D Luce, R R Bush, & E Galanter (Eds) *Readings in mathematical psychology*, Vol II New York Wiley, 1964 Pp 377-381
- Ranke, O F Das Massenverhältnis Zwischen Membran und Flüssigkeit im Innenohr *Akust Zeits*, 1942, 7, 1-11
- Ratliff, F, Hartline, H K, & Miller, W H Spatial and temporal aspects of retinal inhibitory interaction *J opt Soc Amer*, 1963, 53, 110-120
- Reynolds, G S, & Stevens, S S Binaural summation of loudness *J acoust Soc Amer*, 1960, 32, 1337-1344
- Scholl, H *Das Dynamische Verhalten des Gehörs im Bereich der Hörschwellen* Dr Ing Thesis, Technische Hochschule, Stuttgart, 1961 P 82
- Schouton, J F The perception of pitch *Philips Tech Rev*, 1940, 5, 286-294
- Shower, E G, & Biddulph, R Differential pitch sensitivity of the ear *J acoust Soc Amer*, 1931, 3, 275-287
- Sivian, L J, & White, S D On minimum audible sound fields *J acoust Soc Amer*, 1933, 4, 288-321
- Stevens, S S A scale for the measurement of a psychological magnitude loudness *Psychol Rev*, 1936, 43, 405-416
- Stevens, S S On the psychophysical law *Psychol Rev*, 1957, 64, 153-181
- Stevens, S S On the new psychophysics *Scand J Psychol*, 1960, 1, 27-35
- Stevens, S S To honor Fechner and repeal his law *Science*, 1961, 80, 133, 80-86
- Stevens, S S, & Davis, H Hearing, its psychology and physiology New York Wiley, 1938 P 489
- Stevens, S S, & Galanter, E H Ratio scales and category scales for a dozen perceptual continua *J exp Psychol*, 1957, 54, 377-411
- Stevens, S S, Volkman, J, & Newman, E B A scale for the measurement of the psychological magnitude pitch *J acoust Soc Amer*, 1937, 8, 185-190
- Stevens, S S, & Volkman, J The relation of pitch to frequency A revised scale *Amer J Psychol*, 1940, 53, 329-353
- Tasaki, I Nerve impulses in individual auditory nerve fibers of guinea pig *J Neurophysiol*, 1954, 17, 97-122
- Tasaki, I Hearing *Annual Reviews of Physiology*, 1957, 19, 417-438
- Tasaki, I, Davis, H, & Legoux, J P Space-time pattern of the cochlear microphonics (guinea pig), as recorded by differential electrodes *J acoust Soc Amer*, 1952, 24, 502-519
- Tonndorf, J Fluid motion in cochlear models *J acoust Soc Amer*, 1957, 29, 558-568

- Wever, E. G. The width of the basilar membrane in man *Ann Otol Rhinol and Laryngol*, 1938, 47, 37-47
- Wever, E. G. *Theory of hearing* New York Wiley, 1949
- Wever, E. G., & Lawrence, M. *Physiological acoustics* Princeton, New Jersey Princeton Univer Press, 1954 P 454
- Wiener, F. M., & Ross, D. A. The pressure distribution in the auditory canal in a progressive sound field *J acoust Soc Amer*, 1946, 18, 401-408
- Zwicker, E. Die Grenzen der Horbarkeit der Amplitudenmodulation und der Frequenzmodulation eines Tones *Acustica Akust Beih*, 1952, 3, 125-133
- Zwicker, E. Die Verdeckung von Schmalbandgeräuschen durch sinustone *Acustica, Akust Beih*, 1954, 4, 415-420
- Zwicker, E. Ueber Psychologische und Methodische Grundlagen der Loutheit, *Acustica, Akust Beih*, 1958, 6, 237-258
- Zwicker, E., & Feldtkeller, R. Ueber die Lautstärke von Gleichförmigen Geräuschen *Acustica*, 1955, 5, 303-310
- Zwicker, E., Flottorp, G., & Stevens, S. S. Critical band width in loudness summation *J acoust Soc Amer*, 1957, 29, 548-557 Also in R. D. Luce, R. R. Bush & E. Galanter (Eds) *Readings in mathematical psychology*, Vol II New York Wiley, 1964 Pp 406-415
- Zwicker, E., & Wright, H. N. Temporal summation for tones in narrow band noise *J acoust Soc Amer*, 1963, 35, 691-699
- Zwislocki, J. Ueber die Mechanische Klanganalyse des Ohres *Experientia*, 1946, 2, 415-417
- Zwislocki, J. Theorie der Schneckenmechanik *Acta Oto-Laryng*, 1948 Suppl 72, p 76
- Zwislocki, J. Theory of the acoustical action of the cochlea *J acoust Soc Amer*, 1950, 22, 778-784
- Zwislocki, J. Review of recent mathematical theories of cochlear dynamics *J acoust Soc Amer*, 1953, 25, 743-751
- Zwislocki, J. In search of the bone-conduction threshold in a free sound field *J acoust Soc Amer*, 1957, 29, 795-804
- Zwislocki, J. Some impedance measurements on normal and pathological ears *J acoust Soc Amer*, 1957, 29, 1312-1317
- Zwislocki, J. Effect of the transmission characteristic of the ear on the threshold of audibility *J acoust Soc Amer*, 1958, 30, 430-432
- Zwislocki, J. Theory of temporal auditory summation *J acoust Soc Amer*, 1960, 32, 1046-1060
- Zwislocki, J. Analysis of the middle-ear function Part I input impedance *J acoust Soc Amer*, 1962, 34, 1514-1523
- Zwislocki, J., & Feldman, A. S. Post mortem acoustic impedance of human ears *J acoust Soc Amer*, 1963, 35, 104-107
- Zwislocki, J., Hellman, R. P., & Vernillo, R. T. Threshold of audibility for short pulses *J acoust Soc Amer*, 1962, 34, 1648-1652

*Theoretical Treatments of
Selected Visual Problems¹*

Leo M. Hurvich
Dorothea Jameson
and David H. Krantz
University of Pennsylvania

*1 Preparation of this chapter was supported in part by Grant NB 04342,
National Institutes of Health*

Contents

1	Spatial Variations of Stimulation	102
2	Temporal Variations of Stimulation	105
3	Qualitative Variations of Stimulation	113
3 1	Stimulus and perceptual domains,	113
3 2	Helmholtz's line-element,	120
3 3	Sinden's reevaluation of the line-element,	124
3 4	Stiles' increment threshold and the line-element,	125
3 5	MacAdam discrimination ellipses,	130
3 6	Munsell notation,	133
3 7	Opponent-process model,	135
3 8	Some characteristics of the psychophysical experiments,	144
3 9	Metric scaling model based on similarity judgments,	150
	References	154

Theoretical Treatments of Selected Visual Problems

The reader familiar with the visual literature knows that this is an area of many laws and little order. Quantitative data on a variety of psychophysical relations are plentiful, and the field is consequently an inviting one for the systematist or model builder. In this chapter we shall review a sample of the problems and the approaches that seem to us, for one reason or another, to be of potential interest to the mathematical psychologist. The review is by no means comprehensive nor is it intended as a substitute for an introductory treatment for readers totally unfamiliar with the area. The selection and presentation of material in this chapter are intended rather to illustrate, for the reader who is not a specialist in vision, issues for which, at best, only partial solutions have been developed.

For the first section on spatial variations of stimulation, we have selected treatments of the Mach band effect, which is a special case of spatial contrast, because it represents an instance in which the mathematical treatment of a particular effect ought to have, but typically has not had, a general influence on analyses of the whole class of brightness contrast phenomena to which it belongs. It is, moreover, an example of an old phenomenon that is currently of much interest, and one for which the mathematical description of a hundred years ago (Mach) can be compared and contrasted with some contemporary approaches and formulations.

The second section of the chapter focuses on the problem of visual flicker perception. This problem was selected to provide a good context in which to compare points of view that concentrate, respectively, on photochemical events at the peripheral receptor level (Hecht), on the lawful biological characteristics of the total organism (Crozier), on specific properties of neural behavior (Bartley and Pieron), and, finally, on the translation of sensory problems to the language and analytical techniques of systems engineering (Kelly).

The third section deals with different approaches toward a general transformation that will describe the basic relations between light stimulation that varies along the two physical continua of energy and wavelength or that varies in spectral composition and the three-dimensional psychological domain of perceived color space. This topic provides a broad

framework in which to examine a variety of matters of fundamental importance to the model builder in the area of color vision and to look somewhat more critically at the kinds of assumptions, interpretations, and experiments that are involved in the development of visual theories

1 SPATIAL VARIATIONS OF STIMULATION

Mathematical analyses of visual problems may be categorized in any of a number of ways, for example, in terms of the kind of model employed, the class of phenomena considered, the kinds of data incorporated, and so on. To take a specific example, "Mach bands" represent a striking instance of the class of visual brightness phenomena more generally described as contrast and edge effects that depend on certain characteristics of the spatial distribution of stimulus luminance. The term "Mach bands" refers to the perception of sharp brightness reversals (contours) in spite of the fact that the associated stimulus variation is a monotonic spatial gradient of luminance. Ernst Mach (1865, 1866a, 1866b, 1868a, 1868b, 1906), who called attention to the phenomenon, analyzed the effect in terms of a visual brightness response that followed the second derivative of the stimulus distribution, which does indeed show sharply defined maxima and minima at the locations where bright or dark contours are perceived visually.

As an approximate descriptive relation, Mach first (1865) suggested the following formula for the perceived brightness e of a surface whose illumination i varies only in the x direction of the x, y retinal stimulus plane²

$$e = a \log \left\{ \frac{1}{b} \pm \frac{k(d^2 i / dx^2)^2}{i} \right\}$$

In the formula, the Fechnerian psychophysical law is assumed, and a and b are constants. This description implies that the local brightness depends only on the curvature of the physical light distribution at that spatial location, and it makes no provision for an influence on the local brightness of stimulus variations in more remote parts of the field. To provide for such an influence, which decreases with distance r from the local surface element of stimulus intensity j , Mach proposed (1866a) the following approximation

$$e = j \left/ \left[\frac{(i_1/r_1) + (i_2/r_2) + (i_3/r_3) + \dots}{(1/r_1) + (1/r_2) + (1/r_3) + \dots} \right] \right. = j \frac{\sum_k i_k/r_k}{\sum_k 1/r_k}$$

² In the following equations Mach's original notation is retained

In a fourth paper, (1868a) he stated the more general relation

$$e = i \frac{\sum \phi(r) \Delta v}{\sum i' \phi(r) \Delta v}$$

Here, e represents the excitation at the position of stimulus intensity i , Δv represents each element of the remaining surface, ϕ is a function of the distance r of each element from the e position, and i' is the stimulus intensity associated with each element Δv . The distance function $\phi(r)$ can be thought of as made up of two components, one that drops very rapidly with distance over short distances and the other that has a slower rate of change over larger separations. We do not reproduce here Mach's mathematical development to derive the specific form of this distance function, but it remains the most detailed analysis of the problem to date.

Mach saw clearly, moreover, the broader implications of the phenomena that he was attempting to analyze. They implied to him, first, that spatial distributions of perceived brightness are not geometrically similar to spatial distributions of luminance, and second, that the observed psychophysical transformations require mechanisms of physiological interaction among excited elements of the visual tissue. This was almost a hundred years ago, but only recently has there been a renewed interest in the Mach bands and edge effects and along with this interest a new series of attempts at their mathematical description and generalization. Among the more recent suggestions, Ludvigh (1953) has proposed that, in addition to the dependence of the center of the perceived band or doublet on the second derivative of the stimulus distribution $i = f(x, y)$ at the retina, there is a dependence of the width of the band on the fourth derivative of this function.

Lowry and dePalma (1961) have used the Mach band phenomenon to study the transfer function, or sine wave response function, of the visual system. The concept of sine-wave response is used to evaluate the fidelity of optical systems and entails a convolution of the object luminance distribution function with the line spread functions of the image-forming elements. It presumably represents a more useful measure than the older techniques that depend on determinations of the limits of resolution of the system. The psychophysical correlate of the older technique would be the determination of threshold contrast sensitivity or visual acuity. The Mach band phenomenon, however, is a suprathreshold effect that relates more directly to the characteristics of optical imagery that the newer transfer function approach is intended to define. In essence, the procedure used by Lowry and dePalma involves the use of a specially constructed visual photometer to match the variations in apparent brightness at a series of positions along a known spatial gradient of luminance that gives

rise to the Mach effects. The matching, or apparent, luminance values define the image distribution function $I(x')$ generated by the objective luminance distribution $O(x)$ through the transfer function of the visual system. Mathematical analysis of the data is accomplished by taking the Fourier transforms, $O^*(\nu)$, and $I^*(\nu)$, of the object and image distributions. The sine-wave response, $A^*(\nu)$, is derived by dividing $I^*(\nu)$ by $O^*(\nu)$. The amplitude function is $|A^*(\nu)|$. The function determined in this way reflects the characteristics of the whole visual system, that is, the optical characteristics of the eye and the properties of the physiological response mechanisms. To what extent the function determined by this kind of analysis can be given more general application to describe the characteristics of visual contour perception remains to be determined.

Bliss and Macurdy (1961) compared two alternative models to account for the Mach band phenomenon, a "physiological" one based on Hartline and Ratliff's (1958) study of the laws of inhibitory interaction in the eye of the Limulus, and a "psychological" one based on v. Békésy's (1960) concept of the "neural" unit and a "funneling" process in the skin, ear, and visual systems. They apply the techniques of linear-system analysis to relate the two models as discrete systems, using two-sided z -transforms for the space functions of excitation, response, inhibition, and neural unit. The development shows that the transform of the neural unit is the reciprocal of the transform of the inhibition function. Related continuous systems are described by Fourier transforms, and Gaussian blurring functions are assumed to convert the objective light distribution to the stimulus distribution at the retina. The spatial distribution of brightness response derived from this approach for a given object luminance distribution is shown by Bliss and Macurdy to exhibit the main features of the Mach edge effect when compared with the response distribution that is generated by the use of Mach's early (1865) formula.

These examples are cited mainly as illustrations of a current trend to apply concepts and procedures derived from other disciplines (physiology, optics, systems engineering) to the analysis or mathematical description of particular perceptual phenomena. In these instances, and in many others that could be cited, basic properties of the visual system are evaluated by analysis of a single phenomenon or of data obtained for an expressly restricted set of experimental conditions.

The laws of visual behavior, however, are influenced by many parameters of the light stimulus including luminance, size, spatial distribution, temporal distribution, and wavelength composition. These stimulus variables can be manipulated in conjunction with a variety of organismic variables, such as locus of retinal stimulation and state of visual adaptation, and empirical psychophysical relations that interrelate many of these

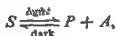
variables for a variety of criterion responses have been determined and redetermined to create a literature of enormous proportions

Broadly comprehensive models are clearly needed to integrate this material, but such treatments are rare. Comprehensive treatments stated with sufficient precision to handle the data quantitatively are still more rare.

2 TEMPORAL VARIATIONS OF STIMULATION

The best known attempt to provide a precisely stated comprehensive theoretical model of vision is that of Selig Hecht. Hecht's theory, which was first presented in detail in the early 1930's (Hecht, 1934), was based almost exclusively on the use of equations intended to describe the photoreceptor process. Many specific details are now of historical interest only. Nonetheless, Hecht's work continues to deserve careful study, for it contains a summary of major psychophysical relations that still need to be accounted for and provides detailed examples of both the advantages and shortcomings of this kind of analytical approach. Other photochemical formulations have been presented, but Hecht's extensive and systematic work has unquestionably been the most influential.

Hecht summarized the events underlying the photoreceptor process as composed of primary and secondary reactions in a coupled system that involve, essentially, a photochemical change with absorption of light, the triggering of a nerve impulse by the energy release, and a replenishment of the photosensitive substance. In simplest paradigm form, the first part of the receptor process can be represented as a completely reversible system



where S is the primary photosensitive substance and P and A are photolytic products. Since the system is, in principle, a reversible one, continuous steady illumination establishes a stationary "equilibrium" state where the rate of photolysis of S is balanced by the regenerative, dark formation of S from P and A . The equation of this steady state is

$$KI = \frac{x^n}{(a - x)^m},$$

where I is light intensity, a is the initial concentration of the photosensitive substance S , x is the concentration of photoproducts or the fraction of S that is changed to P and A at equilibrium, and K is the

equilibrium constant of the system and represents the ratio of k_1 , the photochemical velocity constant, to k_2 , the dark-reaction velocity. The exponents m and n relate, respectively, to the order (monomolecular, bimolecular, etc.) of the primary photochemical process and of the dark reaction.

The essential hypothesis for Hecht's whole development was that a particular criterion response is associated with a fixed change in photochemical concentration in the sensory receptors from one steady state to another. Thus, for example, brightness discrimination between a given intensity I and a second $I + \Delta I$ is determined by a constant difference $x_2 - x_1$ in the two steady-state equations

$$KI = \frac{x_1^n}{(a - x_1)^m}, \quad K(I + \Delta I) = \frac{x_2^n}{(a - x_2)^m}$$

On this kind of conceptual basis, Hecht was able to derive theoretical functions which gave, on the whole, rather good fits to a variety of psychophysical data relating stimulus intensity to brightness discrimination, instantaneous thresholds, visual acuity, and critical flicker fusion, for the time course of bright and dark adaptation for rod and cone vision, and also, with additional assumptions and the postulation of a specific set of spectral absorptions for three different cone substances, for data of color mixture and color discrimination.

This integrative account of many of the data of vision and Hecht's derivation of their quantitative properties from an analysis of the initial photoreceptor processes was an impressive achievement. It gave impetus to further development and refinements of photochemical models (Jahn, 1946, Moon & Spencer, 1945) and the impact of this kind of approach still continues (Cornsweet, 1962).

Hecht's conceptual approach has, however, been subjected to considerable criticism over the years. The predominant role assigned to photochemical events has especially been questioned, and neural aspects of adaptation and discrimination have received increasing emphasis (Bouman & ten Doesschate, 1953, deVries, 1943, Holway & Hurvich, 1938, Lythgoe, 1940, Schouten & Ornstein, 1939, Thomson, 1950). Probably the earliest and, in any event, the most formidable of Hecht's critics was W. J. Crozier. His objections were many.

He argued, for one thing, that since behavioral measures of discrimination involve the complete reacting organism, the quantitative properties of visual performance are not likely to be determined exclusively by the properties of the peripheral retina and the retinal excitatory process (Crozier, Wolf, & Zerrahn-Wolf, 1938c). Furthermore, since the quantitative properties of intensity discrimination are very much the same for

different modes of sensory excitation in vision, audition, kinesthesia, somesthesia, etc (Crozier, 1936, Crozier & Holway, 1937, 1938, 1939; Holway & Crozier, 1937), and since this is true for quite different organisms (Crozier, Wolf, & Zerrahn-Wolf, 1937a), Crozier was most dubious that the specific physiochemical mechanism of excitation is derivable from the discrimination functions (Crozier, Wolf, & Zerrahn-Wolf, 1937b, 1937c). Hecht's success at curve fitting for visual data was attributed by Crozier to his *ad hoc* manipulation of constants. If these constants were allowed to vary only in ways consistent with their presumed photochemical significance, then, Crozier pointed out, they failed to predict correctly the outcomes of specific experimental manipulations (Crozier, Wolf, & Zerrahn-Wolf, 1937d).

To apply his steady-state equation to flicker data, for example, Hecht assumed that the stationary state is reached at critical fusion frequency. Since the luminance of an intermittently flashing stimulus that appears continuous is decreased by the fraction of dark-phase time in a single cycle and since by simply adjusting the luminance of the flash proportionately, we can make a brightness match to a continuous stimulus of the original luminance, time and intensity are clearly reciprocal and interchangeable. In the flicker experiments, when critical fusion frequency is reached at a given luminance level and flicker has disappeared, we have a steady state. At this equilibrium point the rapidly alternating light and dark reactions of equal duration are in balance, what has been decomposed during the light period is regenerated during the dark period. The steady-state equation

$$KI = \frac{x^n}{(a - x)^m}$$

applies here. With the additional assumption that flicker frequency for fusion depends directly on the concentration x of the photoproducts during a light flash, the flicker data can be described by the relation

$$KI = \frac{F^n}{(F_{\max} - F)^m}$$

Here, the critical flicker rate F for fusion at any luminance is substituted for the photochemical concentration x at that luminance, and the maximal critical flicker frequency F_{\max} for the given parameters is substituted for the maximal amount of photosensitive material, a . In this relation, we remember that K does have biological significance since it represents the ratio k_1/k_2 of the photochemical and dark-reaction velocity constants. The constant k_1 for the photochemical change must have a low temperature coefficient, whereas k_2 for the dark reaction must have a much larger

temperature coefficient Crozier sought to evaluate Hecht's system experimentally by studying the critical flicker frequency dependence on stimulus intensity when the temperature of the tested organism is varied (Crozier, Wolf, & Zerrahn-Wolf, 1937b, 1937c) With a decrease in temperature, k_2 decreases and the ratio k_1/k_2 must increase If F_{\max} is independent of temperature (an experimentally established fact), then at any fixed frequency F the critical illumination I must vary in the same sense as the temperature variation Instead, the data show that the relation is an inverse one Flicker measures on the insect *Anax*, a vertebrate sunfish *Enneacanthus*, and the turtle *Pseudemys* all show constancy of the F_{\max} value, an invariant form of the CFF dependence on $\log I$, but a displacement of the functions toward lower rather than higher intensities with an increase in temperature (Crozier, Wolf, & Zerrahn-Wolf, 1937b, 1937c, 1938b) Other discrepancies occur in evaluating Hecht's model with respect to variations in the light-to-dark time ratios of the flickering stimulus (Crozier, Wolf, & Zerrahn-Wolf, 1938c), and all in all Crozier was able to amass a rather impressive body of data to justify his assertions that Hecht's use of quantitative formulas to describe flicker functions fell far short of demonstrating that the functions reflected in any simple way the quantitative properties of retinal photochemistry

In the last decade or so considerable additional evidence has cast further doubt on some of the basic premises of photochemical theories of vision There have been large strides in photopigment analysis (Dartnall, 1957), advances in electrophysiological techniques (Granit, 1962), new developments in reflection densitometry (Rushton et al., 1955, Weale, 1953), and increased concern with the quantum aspects of the visual stimulus (Baumgardt, 1949, 1959, Pirenne, 1962)

In essence, no simple relation has been demonstrated between pigment concentration and measured visual sensitivities as was originally assumed These discrepancies between photopigment concentrations and discrimination performance (recorded behaviorally or electrophysiologically) have been found both in man and in lower organisms, and the discrepancies are large (Granit, Munsterhjelm, & Zewi, 1939, Rushton & Cohen, 1954)

Crozier's own systematic treatment of visual data is in interesting contrast to Hecht's photochemical model analysis Crozier's concern with visual problems stemmed from a broad biological interest in the sensory control of oriented behavior and in the specification of those invariant features laid down by the heritable biological characteristics of the organism (Crozier & Hoagland, 1934) He sought to develop a functional analysis of behavior by determining relations that could be described by rational equations with dimensionally significant variables and constants Thus, for Crozier, the functional relations might, in some

stances, reveal fundamental differences that had their basis in different properties of the peripheral sensory receptors, as in the "rod" and "cone" branches of many visual functions, but other features of these same relations might have nothing to do with the specific properties of the sensory end organs (Crozier & Wolf, 1938, 1944)

The concept crucial to Crozier's analytical approach is that of biological variability, thus any given stimulus value is associated with a population of excitatory effects in the neural centers, and any measure of sensory discrimination between two stimulus values is determined by the characteristics of the two associated population distributions. The biological response variability is, of course, measured by the dispersion of stimulus values for a constant criterion response, and hence Crozier's analyses of psychophysical discrimination functions always involved measures of dispersion as well as of mean stimulus values (Crozier, 1935, 1936)

When the analysis is applied, say, to the problem of critical flicker frequency, the relevant data are not simply the mean flicker values F_m related to mean stimulus light intensities I_m for marginal flicker recognition or fusion. They are, rather, the measures $F_m \pm \sigma_F$ or $I_m \pm \sigma_I$ that determine the band margins of the flicker contour. Response to flicker is regarded as a limiting case of intensity discrimination involving the statistical comparison of neural effects of excitation during the light phase with their aftereffects during the dark phase, and the function relating critical flicker frequency to light intensity has the characteristics of a log Gaussian probability integral

$$F = \frac{F_{\max}}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\log I} \exp \left[-\frac{1}{2\sigma^2} (\log I - \log I_0)^2 \right] d(\log I)$$

(Crozier & Wolf, 1940b) F is the critical flicker frequency, F_{\max} the maximal frequency at which the function becomes asymptotic, I the critical illumination at frequency F , I_0 is the critical illumination at the inflection point, where F has a value equal to $\frac{1}{2}F_{\max}$, and $\sigma = \sigma_{\log I}$ is the standard deviation for flicker recognition. The dispersion of the critical illumination for flicker recognition is a variability measure of special interest for, like other visual functions that depend on intensity discrimination, this dispersion is directly proportional to the mean difference in illumination required for the criterion response. In Crozier's view, it reflects the spread of excitabilities of the central response system and defines a band of constant width $\pm \log I_m$ in the functional relations between critical illumination I and flicker frequency F . The F_{\max} parameter is taken to be related to the total population of elements of sensory effect that are capable of involvement in the discrimination of flicker, and F is proportional to the additive effects of elemental excitations at a given level

of illumination I . The I_0 parameter (also represented by the symbol τ' in many of Crozier's publications) is interpreted as a measure of relative mean excitability.

Crozier's flicker studies were designed to establish the characteristics of the flicker contour for various animal species and to analyze the effects of specific experimental manipulations on the various parameters of the flicker function. All these parameters may differ from one animal species to the next, and in a given individual they differ for stimulation of separate, nonhomogeneous receptor populations, that is, the rods and cones of duplex retinas (Crozier, Wolf, & Zerrahn-Wolf, 1938c). For a given ratio of light-to-dark time in the flicker cycle, Crozier was able to demonstrate the "organic invariance" of the $\sigma_{\log I}$ and the F_{\max} parameters as specifically heritable attributes subject to genetic variation as shown by cross-breeding experiments (Crozier, Wolf, & Zerrahn-Wolf, 1937a, Crozier & Wolf, 1939). The $\sigma_{\log I}$ and the F_{\max} parameters are interpreted as stable, limiting characteristics of the organism and consequently not susceptible to manipulation by changes in the momentary state of a given organic system. Thus, for example, temperature variations that alter the excitation levels of the response elements do not affect either the spread or the total population of excitable effects, and $\sigma_{\log I}$ and F_{\max} are invariant with temperature changes. The I_0 parameter, however, which is interpreted as an inverse measure of relative excitability, is systematically dependent on temperature changes (Crozier, Wolf, & Zerrahn-Wolf, 1938b).

At a given temperature, as the light-dark ratio is changed, $\sigma_{\log I}$ remains unchanged, but both F_{\max} and I_0 vary, which is consistent with the interpretation of the F_{\max} parameter as an index of the total population of elements "capable of involvement in" the discrimination (Crozier & Wolf, 1940a, 1941). Here Crozier was making use of the concept of "neural availability" proposed by Holway and Hurvich (1937, 1938) to account for their discrimination data in both kinesthesia and vision. Thus, F_{\max} increases with a decrease in relative light time "because the succession of briefer flashes and longer dark time enables more elements to be 'caught' in an excitable state, just as with larger visual areas" (Crozier, Wolf, & Zerrahn-Wolf, 1938d). Moreover, since temperature influences excitation essentially by influencing the velocities of reactions in the neurochemical systems, the dependence of the flicker contour on the ratio of light-to-dark time in the flicker cycle also varies systematically with temperature.

An account of Crozier's many ingenious empirical manipulations of the parameters of the flicker function and of the complexities that emerge in the analysis of interspecies differences and in the human data for a variety of stimulus variables is beyond the scope and purpose of this chapter.

Reports of most of this work can be found in the *Journal of General Physiology* in a series of papers published between 1940 and 1945. Our intent here is simply to indicate the nature of his approach and the potential of this kind of probability model as a powerful and broadly applicable analytical tool in the study of visual data (Crozier & Wolf, 1942).

Much more typical in the visual literature is the kind of analysis that attempts to "explain" the psychophysical data in terms of specific characteristics of neural response. Bartley (1959) probably best exemplifies this approach with respect to visual flicker data. He has been particularly concerned with the so called brightness enhancement effect that occurs at relatively low stimulus frequencies, and he has attempted to integrate the neurological findings as related to the various flicker data in terms of his major conceptual formulation, "the alternation of response theory." Bartley's experimental work on the relation of the light-dark ratio to critical flicker frequency has led him to question the usual assumption that flicker is eliminated at only one critical light-dark ratio for a given stimulus intensity and flicker rate. He accounts for his findings with the assumption that "off responses" may trigger flicker perceptions under some conditions (Bartley, 1961).

Pieron (1961) has also recently given an interpretative theoretical account of the complex visual flicker situation in relation to neurophysiological evidence at various levels in the optic pathway. The perceptual uniformity or nonuniformity of an intermittent stimulus is, in his view, determined by the behavior of the slow excitatory retinal potential and the activity of what Piéron calls the "on-with" discharge units whose frequency is proportional to this potential. Whether these potentials are uniform, oscillatory, or discrete depends on whether the photons or light quanta (regardless of their time distribution) fall within the critical duration τ (the duration within which time and intensity are reciprocally related). If they do, the slow potential is uniform and there is no perceptible flicker, since this critical duration decreases with a logarithmic increase in the stimulus luminance, critical flicker frequency increases directly with log luminance in accordance with the well known Ferry-Porter law $F = k \log I + a$. If τ is exceeded and there is only partial time intensity integration, the slow potential oscillates and the neural impulse frequency is such that flicker is perceived. If the pulsing stimuli are still longer and exceed the "utilization time" for even partial time intensity integration, then the potentials are independent and discrete light pulses are perceived. Piéron has been particularly concerned with resolving apparent discrepancies in the literature concerned with the dependence of flicker on the light-to-dark ratio. His theoretical account seeks to demonstrate that the "contradictory" empirical data are necessary consequences of the

way the stimulus luminance is controlled in the different experiments, that is, whether the luminance of each pulse is the same for different light-to-dark time ratios or whether compensatory adjustments are made to maintain a constant average luminance for the different pulse durations.

Pieron is sharply critical of the increasing recent trend to emphasize the wave form of the stimulus variations in analysis of visual flicker responses. Since the process of physiological excitation does not measure or mirror the stimulus wave pattern, Pieron regards the application of Fourier analysis to the complex stimulus wave forms and the generated harmonics as irrelevant to an understanding of flicker phenomena.

There is, nevertheless, much of interest in the use of harmonically pure stimulus wave forms to investigate the temporal characteristics of the visual process (deLange Dzn, 1954, Levinson, 1959). The background of this kind of experimental and analytical study lies in engineering. Within this conceptual framework, the problem is translated into one of information processing, with stimuli as "signals," the visual mechanism as a light transducer, and threshold determinations as an application of the "driving-point null" method. A good example of systems analysis applied to the study of visual flicker is contained in Kelly's recent work (1961a, 1961b, 1962a).

In his experiments with time-dependent stimuli, Kelly employed primarily a large 68° "edgeless" Ganzfeld (to minimize spatial interaction effects) and harmonically pure, sinusoidally flickering lights instead of the rectangular pulse trains conventionally used in flicker experiments. Modulation amplitude thresholds are measured at a series of fixed frequencies at constant time-average radiances. Functions relating amplitude sensitivity and modulation frequency for white light at a series of adaptation levels show that amplitude sensitivity tends to rise to a maximum as frequency increases, to fall sharply as frequency is further increased, and to exhibit a gradual shift in sensitivity peak toward higher frequencies as adaptation level is increased.

When plotted as amplitude-luminance curves with modulation frequency as parameter, the data that reflect the visual system's temporal sensitivity exhibit trends markedly similar to spatial contrast sensitivity versus luminance curves obtained by others. The data can also be shown in the form of luminance-frequency curves for constant modulation ratios, in this form they resemble, not surprisingly, the more conventional *CFF* versus luminance functions obtained for rectangular wave forms. Of particular interest, however, is the fact that these plots show that of the three parameters only amplitude is clearly single valued, both low and high frequency thresholds can be found over the entire photopic luminance range for the three smallest modulation ratios.

Individual differences, color, and color adaptation dependencies have also been explored by Kelly (1962b, 1962c) in his analysis of the time-dependent responses. Of major interest, however, is his basic single-channel model proposed to account for the average white light data of his group of eight observers.

Kelly (1961b) seeks to account for his experimental data by postulating signal transformations at two levels in the visual receptor system. The model, which takes account of the neural refractory periods and the "all-or-none" law, is more physiological than the usual electrical network analogs. The first stage is a photoreactive linear one whose filtering action controls the shape of the amplitude sensitivity curve at low frequencies, the second stage, a pulse modulator or encoding stage that converts the first stage output into trains of impulses of identical heights and durations but spaced at varying time intervals, controls the high frequency cut-off. The amplitude response function of the first stage (a transform function of a linear differential response) is combined with the threshold amplitude response function of the pulse-encoding stage and with proper choice of parameters is shown to fit the white light amplitude sensitivity data. The model is also shown to predict reasonably well the transient responses obtained in other types of psychophysical and electrophysiological experiments. The details of the model are given in Kelly (1961b).

3 QUALITATIVE VARIATIONS OF STIMULATION

3.1 Stimulus and Perceptual Domains

Some of the most intriguing and refractory problems of visual psychophysics arise in the analysis of color perception where the relations are three-dimensional.

The psychological dimensions of perceived color may be conceived as hue, saturation, and brightness, the physical dimensions as radiant energy or total light intensity, wavelength, and wavelength distribution. The wavelength distribution specifies for a given light stimulus the relative amounts of energy throughout the visible spectrum. In the event that the stimulus is a so-called monochromatic spectral light, the wavelength distribution is specified as the spectral bandwidth, and the stimuli are often treated as if they were limited to only the single wavelength on which the actual band is centered.

It is, of course, known that there is not a unique, one-to-one correspondence between each uniquely specifiable physical light stimulus and the color perception that it elicits, in general, a variety of physical stimuli can be used to elicit identical color sensations.

The stimulus domain undergoes a first-order transformation to a psychophysical domain by the determination of three-dimensional color-equivalence relations in color-mixture experiments. Three spectral stimuli are selected, each of which is fixed in bandwidth and wavelength, but variable in energy or radiant flux. The amounts and proportions of these stimuli are then varied in order to produce a color equation for each of a finite series of wavelengths taken to represent the visible spectrum. In principle, these equations may be written as^{*}

$$\Gamma_{\lambda} = \Gamma_1 + \Gamma_2 + \Gamma_3,$$

where Γ_{λ} is the arbitrary unit radiant flux of the given spectral wavelength to be matched and Γ_1 , Γ_2 , and Γ_3 are the radiant flux proportions of the three mixture primaries required for the color equation. In practice, the spectral color-match situation is represented by

$$\Gamma_{\lambda} + \Gamma_{123} \equiv \Gamma_1 + \Gamma_2 + \Gamma_3,$$

where Γ_{123} represents the proportion of one of the three mixture-primaries which is combined with the test stimulus Γ_{λ} in order to make the actual color match, represented by the symbol \equiv ³. The expression written in this form takes cognizance of the fact that narrow band spectral test stimuli appear more saturated than do mixtures of the three spectral primaries, and hence the spectral test stimulus must be desaturated by the addition of one of the primaries to establish a complete color match. Under these circumstances, one of the terms in the first equation takes on a negative value. A complete series of matching relations constitutes the experimental data for the unit color equations by means of which any color stimulus of known wavelength distribution and energy content can be specified in terms of its three-variable mixture equivalent⁴.

If the mixture primaries are evaluated in terms of their relative luminances rather than in percentage units of radiant flux, the color equation becomes

$$L_{\lambda} = L_1 + L_2 + L_3$$

In this equation luminances are implied to be strictly additive. That is, in the color match the radiant energy of the spectral test stimulus λ evaluated by the spectral luminosity function is stated to be identical to the summed luminances of the three mixture primaries λ_1 , λ_2 , and λ_3 when their energies are also evaluated by the same luminosity function. The extent to which this additivity requirement is strictly met in experimental test situations is currently of considerable interest and in some

^{*} This is the standard symbol that has been adopted to represent visual color matches.

⁴ Color matching relations can also be expressed in vector notation. See Hering (1887), and Judd and Wyszecki (1963).

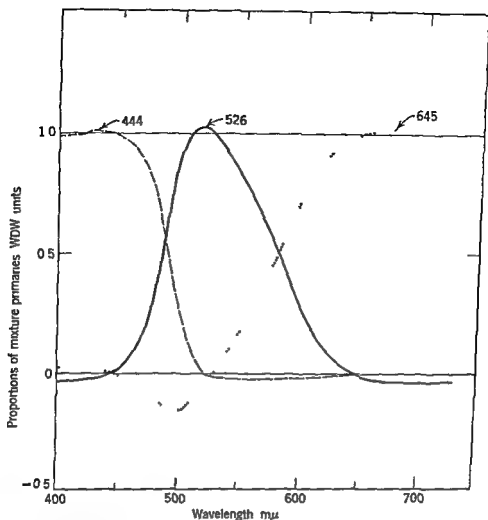


Fig. 1 Color equivalence relations in unit coordinates WDW units. Drawn with permission from Stiles & Burch (1955)

doubt (Graham, 1959, Judd, 1955). For the present purpose, we shall accept the equation as an approximately valid expression of color equivalence in luminance terms.

Treatments of color-mixture data do, in fact, rely completely on the validity of the additivity principle, and many experimenters have followed the practice of specifying the stimulus amounts in arbitrary scale units together with some measure of relative luminances but with no direct calibration of the physical energy units of the mixture primaries.

Figure 1 shows a set of unit coordinate color equivalence relations reported by Stiles and Burch (1955). Here the units are based on the amounts of the primaries required for two selected spectral matches, a convention first proposed by W. D. Wright (1928-1929) and known as the WDW unit coordinate system. Figure 2 shows the same color-mixture

The stimulus domain undergoes a first-order transformation to a psychophysical domain by the determination of three-dimensional color-equivalence relations in color-mixture experiments. Three spectral stimuli are selected, each of which is fixed in bandwidth and wavelength, but variable in energy or radiant flux. The amounts and proportions of these stimuli are then varied in order to produce a color equation for each of a finite series of wavelengths taken to represent the visible spectrum. In principle, these equations may be written as^{*}

$$\Gamma_\lambda = \Gamma_1 + \Gamma_2 + \Gamma_3,$$

where Γ_λ is the arbitrary unit radiant flux of the given spectral wavelength to be matched and Γ_1 , Γ_2 , and Γ_3 are the radiant flux proportions of the three mixture primaries required for the color equation. In practice, the spectral color-match situation is represented by

$$\Gamma_\lambda + \Gamma_{123} \equiv \Gamma_1 + \Gamma_2 + \Gamma_3,$$

where Γ_{123} represents the proportion of one of the three mixture-primaries which is combined with the test stimulus Γ_λ in order to make the actual color match, represented by the symbol \equiv ³. The expression written in this form takes cognizance of the fact that narrow band spectral test stimuli appear more saturated than do mixtures of the three spectral primaries, and hence the spectral test stimulus must be desaturated by the addition of one of the primaries to establish a complete color match. Under these circumstances, one of the terms in the first equation takes on a negative value. A complete series of matching relations constitutes the experimental data for the unit color equations by means of which any color stimulus of known wavelength distribution and energy content can be specified in terms of its three-variable mixture equivalent⁴.

If the mixture primaries are evaluated in terms of their relative luminances rather than in percentage units of radiant flux, the color equation becomes

$$L_\lambda = L_1 + L_2 + L_3$$

In this equation luminances are implied to be strictly additive. That is, in the color match the radiant energy of the spectral test stimulus λ evaluated by the spectral luminosity function is stated to be identical to the summed luminances of the three mixture primaries λ_1 , λ_2 , and λ_3 when their energies are also evaluated by the same luminosity function. The extent to which this additivity requirement is strictly met in experimental test situations is currently of considerable interest and in some

^{*} This is the standard symbol that has been adopted to represent visual color matches

⁴ Color matching relations can also be expressed in vector notation. See Hering (1887), and Judd and Wyszecki (1963)

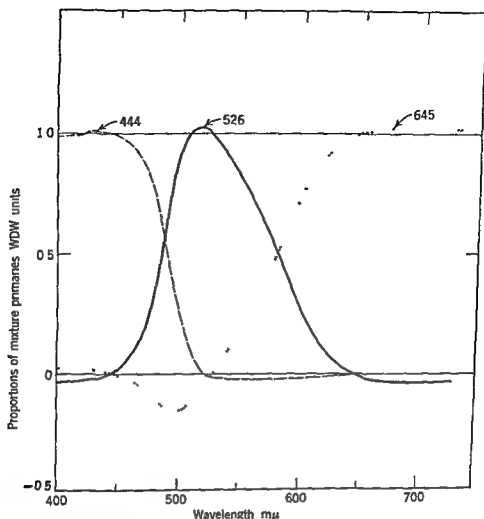


Fig 1 Color equivalence relations in unit coordinates WDW units Drawn with permission from Stiles & Burch (1955)

doubt (Graham, 1959, Judd, 1955) For the present purpose, we shall accept the equation as an approximately valid expression of color equivalence in luminance terms

Treatments of color-mixture data do, in fact, rely completely on the validity of the additivity principle, and many experimenters have followed the practice of specifying the stimulus amounts in arbitrary scale units together with some measure of relative luminances but with no direct calibration of the physical energy units of the mixture-primaries

Figure 1 shows a set of unit coordinate color equivalence relations reported by Stiles and Burch (1955) Here the units are based on the amounts of the primaries required for two selected spectral matches, a convention first proposed by W. D Wright (1928-1929) and known as the WDW unit coordinate system. Figure 2 shows the same color-mixture

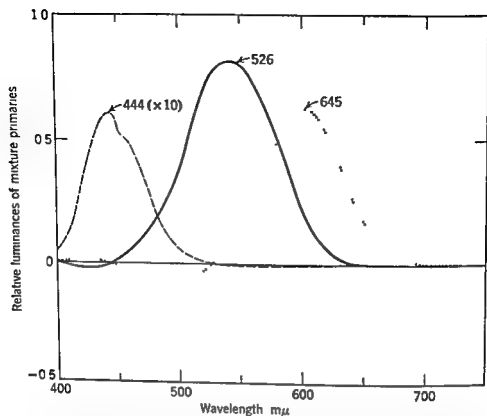


Fig 2 Spectral mixture functions for spectral primaries 645 $m\mu$, 526 $m\mu$, and 444 $m\mu$. Units of relative luminance for an equal energy spectrum (Function for 444 $m\mu$ primary is plotted as ten times the actual values). The three ordinate values at each wavelength sum to yield the spectral luminosity function. Drawn with permission from Stiles & Burch (1955).

data evaluated in terms of relative luminosities for a spectrum of unit energy

Figure 3 shows a plane in chromaticity space plotted from the unit coordinate data of Fig 1. If we assume that the viewing conditions (for example, stimulus size, surround stimulation, state of adaptation, and retinal locus) are held constant, this space has the following properties: any stimulus of known physical characteristics can be represented within it, all stimuli within the triangle formed by the loci of the three primaries ($r = g = 0$, $r = 0$, $g = 1$, $r = 1$, $g = 0$) can be matched by positive mixtures of the three, all stimuli located along any straight line in the space can be matched by positive binary mixtures of two stimuli between whose loci they are encompassed. All physical stimuli represented at the same point in this chromaticity space have the same mixture equivalent in terms of proportional amounts of the specified spectral primaries expressed in WDW units. Since the absolute amounts of the mixture equivalents may not be identical for all such stimuli, such a chromaticity space does not specify that all stimuli having identical loci are identical in

perceived color, but it does say that they can be made to appear identical by a simple adjustment in amount, with no change in their spectral distributions. If, furthermore, a luminance specification is added to the information contained in the chromaticity space, then we can imagine a series of spaces identical to that shown in Fig 3, each one for a specified, constant level of luminance. Under these circumstances all stimuli that have the same locus in the chromaticity space of constant luminance must be identical in color appearance.

A chromaticity chart for a plane of constant luminance is, then, a form of three-dimensional psychophysical relation for perceived color, but only in the sense of specifying perceptual identities. The locus of a stimulus in this space provides no information whatsoever about the hue of the color or how saturated or bright it appears to be. We know only that the hue, saturation, and brightness are the same, respectively, for all those stimuli that have the same chromaticity and luminance specifications, nor does the distance separating two distinct loci in this chromaticity space provide any information concerning the magnitude of the perceived color difference between them. Pairs of stimuli whose chromaticity loci are equally distant need not look equally different in color. Had the basic color mixture experiments been carried out with spectral mixture primaries different from those used by Stiles and Burch (1955), the appearance and the spacing

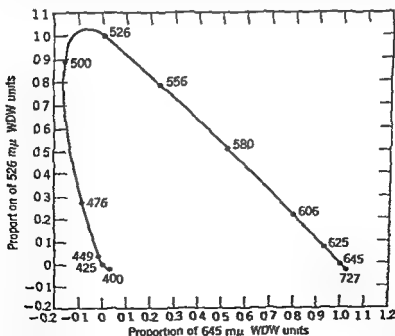


Fig 3 Chromaticity chart with spectrum locus in WDW units for spectral mixture primaries 645 mμ, 526 mμ, and 444 mμ. Drawn with permission from Stiles & Burch (1955)

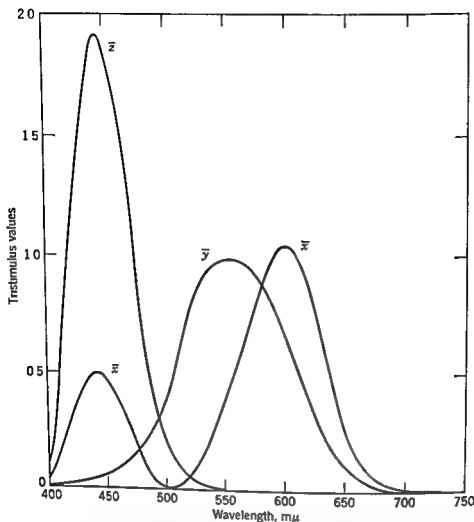


Fig. 4. Stiles and Burch color-mixture data transformed to CIE tristimulus values. All values are positive, the \bar{y} function is the spectral luminosity function, and the \bar{x} , \bar{y} , and \bar{z} primaries are not physically realizable stimuli. Drawn with permission from Stiles & Burch (1955).

If the chromaticity space plotted in Fig. 3 would also have been different. Indeed, once a system of color equivalents has been established experimentally by the use of any three arbitrarily selected primaries, the data may be expressed in terms of any other arbitrarily selected set by means of a simple linear transformation of the form

$$\Gamma'_1 = a_{11}\Gamma_1 + a_{12}\Gamma_2 + a_{13}\Gamma_3,$$

$$\Gamma'_2 = a_{21}\Gamma_1 + a_{22}\Gamma_2 + a_{23}\Gamma_3,$$

$$\Gamma'_3 = a_{31}\Gamma_1 + a_{32}\Gamma_2 + a_{33}\Gamma_3.$$

Figure 4 shows a transformation of the Stiles and Burch data to the \bar{x} , \bar{y} , \bar{z} primaries that have been agreed upon by the International

Commission of Illumination (known by the initials CIE for Commission Internationale de l'Éclairage) as a conventional standard for chromaticity specification, and in Fig 5 the same data are represented in CIE chromaticity space. All stimuli that have a common locus in Fig 3 transform to a new common locus in Fig 5, but stimulus pairs separated by equal distances in Fig 3 are separated by unequal distances in Fig 5. The CIE chromaticity space has certain advantages of convenience (all values are positive) and convention to recommend it, but it has neither more nor less perceptual significance than does the original on which it is based.

The importance of a chromaticity space that permits the unique specification of physical color stimuli is not to be underestimated. It is perfectly clear, however, that color mixture and luminosity data alone provide nothing more than a system of identification relations for metameric colors, that is, for light stimuli of different spectral distributions that evoke identical color perceptions under standard specified viewing conditions. Further steps are needed to supplement the color equivalence, identity relations with a system of ordering relations in order to represent the perceived color differences associated with particular stimulus differences.

For this purpose, any one of a variety of approaches may be followed. Some involve specific theoretical assumptions about the nature of the color vision mechanism, whereas others depend minimally on a priori notions about visual processes but make important assumptions about the statistical properties of the relevant data and the geometrical properties of the perceptual color space. (For a summary of proposed CIE transformations and color difference equations see Judd & Wyszecki, 1963.)

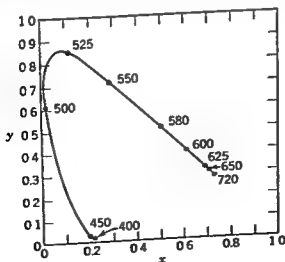


Fig 5 Spectrum locus of Stiles and Burch color mixture data represented in CIE chromaticity chart. Drawn with permission from Stiles & Burch (1955)

In the following sections, we shall discuss some of these different approaches and in so doing try to present a reasonably representative picture of the logical and mathematical handling of some of the basic issues in color vision. We shall neither attempt to provide a comprehensive historical survey nor limit the discussion exclusively to current work. We shall, however, consider the early approaches where they seem essential as background for the later developments.

3.2 Helmholtz's Line-Element

An early attempt to handle perceived color space in quantitative fashion was made by Helmholtz and it is reported in the second edition of his treatise on physiological optics (1896). The data on which his treatment depended include color-mixture equivalence relations, measures of minimally discriminable differences among adjacent wavelengths of constant luminance, and measures of minimally discriminable intensity differences for white light and for a series of spectral wavelengths.

Helmholtz assumed in accordance with Young's theory of color vision that the total gamut of color sensations is determined by three fundamental hue sensations, S_1 , S_2 , and S_3 , in varying amounts and proportions. The magnitudes of the three separate sensation differences are written as dS_1 , dS_2 , dS_3 , and dS represents the net difference in perceived color. Since each of these terms represents a difference in sensation, $dS = 0$ only when $dS_1 = dS_2 = dS_3 = 0$. The separate sensation differences were assumed by Helmholtz to combine in the following manner:

$$(dS)^2 = (dS_1)^2 + (dS_2)^2 + (dS_3)^2$$

The threshold value for the perception of a difference is set equal to one, but values less than one for each of the three separate sensations are assumed to contribute to a net sensation difference that has a minimum value of one.

To relate his formula for sensation differences to stimulus measures, Helmholtz stated the following set of relations, which he described as a more precise expression of Fechner's law:

$$\begin{aligned} dS_1 &= X \frac{dx}{x} \cdot \frac{1}{1 + lx + my + nz}, \\ dS_2 &= Y \frac{dy}{y} \cdot \frac{1}{1 + lx + my + nz}, \\ dS_3 &= Z \frac{dz}{z} \cdot \frac{1}{1 + lx + my + nz} \end{aligned}$$

In this expression, x, y, z are amounts of three fundamental excitations, X, Y, Z are functions of x, y, z such that $X = Y = Z = k$ (a constant) at high excitation levels, and at moderate light intensities

$$X = \frac{kx}{a + x}, \quad Y = \frac{ky}{b + y}, \quad Z = \frac{kz}{c + z},$$

where a, b , and c are constants, assumed to represent the separate components of what Helmholtz called the intrinsic light of the retina. The last term $1/(1 + lx + my + nz)$, where l, m , and n are constants, is a glare factor introduced to account for the "damping" of the measured intensity. Thus the sensation difference relations are essentially empirical generalizations based on the data of intensity discrimination but interpreted as relations between fundamental excitations and fundamental sensations (The symbols used here are not to be confused with the CIE notation.)

In developing his logical argument, Helmholtz states that when a color is changed only in intensity, all its components are changed by the same fraction $d\epsilon$, and thus

$$dx = x \, d\epsilon, \quad dy = y \, d\epsilon, \quad dz = z \, d\epsilon$$

The sensation difference becomes

$$dS = \frac{d\epsilon}{1 + lx + my + nz} (X^2 + Y^2 + Z^2)^{1/2},$$

and for high intensities where $X = Y = Z = k$

$$dS = \frac{(3k^2)^{1/2} d\epsilon}{1 + lx + my + nz}$$

Two slightly different colors may be said to have the excitation values x, y, z and $x + dx, y + dy, z + dz$, respectively. If the first color is increased in intensity by a factor $1 + \epsilon$, then the excitation component differences between the two colors become

$$dx - \epsilon x, \quad dy - \epsilon y, \quad \text{and} \quad dz - \epsilon z$$

Their sensation difference becomes

$$dS = \left[X^2 \left(\frac{dx - \epsilon x}{x} \right)^2 + Y^2 \left(\frac{dy - \epsilon y}{y} \right)^2 + Z^2 \left(\frac{dz - \epsilon z}{z} \right)^2 \right]^{1/2},$$

where the glare factor is equal to unity

To determine the value of ϵ so that dS is a minimum, dS^2 is differentiated with respect to ϵ and the result set equal to zero. The value of ϵ becomes

$$\epsilon = \frac{X^2(dx/x) + Y^2(dy/y) + Z^2(dz/z)}{X^2 + Y^2 + Z^2}$$

By substituting this value of ϵ in the sensation difference equation

$$dS^2 = \frac{X^2 Y^2 \left(\frac{dx}{x} - \frac{dy}{y} \right)^2 + Y^2 Z^2 \left(\frac{dy}{y} - \frac{dz}{z} \right)^2 + Z^2 X^2 \left(\frac{dz}{z} - \frac{dx}{x} \right)^2}{X^2 + Y^2 + Z^2}$$

The units in which x, y, z and X, Y, Z are expressed are arbitrary, but they must be the same for both sets of terms

Since the terms x, y , and z , however, presumably represent the fundamental excitations of the three-variable color system they must bear a specific relation to the trivariate color-mixture data. Helmholtz assumed that they could be represented by a simple linear transformation of the color-matching data obtained by König. Two further restrictions were placed on the permissible transformation. Helmholtz assumed that all values must be positive and that the transformed functions must also account for König's empirical measures of minimally discriminable wavelength differences for colors of equal brightness.

The x, y , and z excitation values are solved for by

$$x = a_{11}R + a_{12}G + a_{13}B,$$

$$y = a_{21}R + a_{22}G + a_{23}B,$$

$$z = a_{31}R + a_{32}G + a_{33}B,$$

where x, y, z are positive values and R, G, B are measured amounts of three stimuli used for spectral color equations (or a linear transformation of such measures). The values x, y, z must simultaneously satisfy the following equation

$$\frac{dS}{k} = \frac{\delta\lambda}{\sqrt{3}} \left[\left(\frac{1}{x} \frac{dx}{d\lambda} - \frac{1}{y} \frac{dy}{d\lambda} \right)^2 + \left(\frac{1}{y} \frac{dy}{d\lambda} - \frac{1}{z} \frac{dz}{d\lambda} \right)^2 + \left(\frac{1}{z} \frac{dz}{d\lambda} - \frac{1}{x} \frac{dx}{d\lambda} \right)^2 \right]^{1/2}$$

Here $\delta\lambda$ represents the measured value for the least discriminable wavelength difference at each given region of the spectrum. The level of intensity is assumed to be sufficiently high so that X, Y , and Z are all equal to k .

The spectral distributions of the three excitation functions resulting from Helmholtz's solution are shown in Fig 6, and Fig 7 shows the chromaticity space based on these distributions. The chromaticities of spectral stimuli are represented on the spectrum locus within this space.

To convert this linear transformation of the color-mixture space to a space that represents minimal sensation differences uniformly, Helmholtz

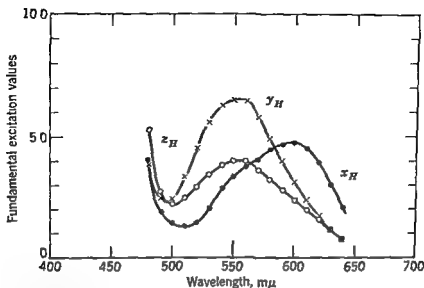


Fig. 6. Helmholtz's fundamental excitation curves. For x_H , y_H , and z_H units, see Helmholtz (1896, p. 455).

stated the following relations between the excitation processes x , y , z and their sensory effects ξ , η , ζ :

$$\log (a+x)=\xi,$$

$$\log (b+y)=\eta,$$

$$\log (c+z)=\zeta.$$

Consequently, for

$$(dS)^2=(dS_1)^2+(dS_2)^2+(dS_3)^2$$

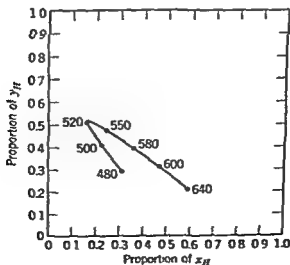


Fig. 7. Helmholtz's chromaticity space based on data in Helmholtz (1896, p. 455).

we write

$$(dS)^2 = (d\xi)^2 + (d\eta)^2 + (d\zeta)^2$$

It is in the logarithmic 3-space for ξ, η, ζ that minimal sensation differences correspond to line-elements within the space

The Helmholtz line-element approach has been presented in this much detail to make clear the extent to which the mathematical treatment depends on specific hypotheses of the underlying theory and to specify the kinds of data actually incorporated in the particular solution developed

Helmholtz's solution was not accepted largely for reasons related to some widely held convictions about the visual mechanism. One of the primary objections to it rested upon an *a priori* notion that fundamental spectral distribution functions, of which Helmholtz's x, y, z are an example, must describe the absorption spectra of the three selective photochemicals assumed to be present in the retinal cones. The broad, double-humped functions (Fig. 6) that resulted from Helmholtz's solution were considered unlikely candidates to represent the spectral absorptions of the visual photochemicals. Helmholtz's curves are also difficult to reconcile with the form of the spectral luminosity function (Schrodinger, 1920)

3.3 Sinden's Reevaluation of the Line-Element

Sinden's (1937, 1938) attempt to improve on Helmholtz's solution was motivated by the idea that the approach was promising but that Helmholtz had been led far afield by the deficient data that were available to him. Sinden was convinced, furthermore, that the arithmetical trial-and-error procedure used by Helmholtz in solving for his basic excitation values involved an unrealistic amount of numerical computation and an inefficient use of the available experimental data. Sinden himself therefore started from the geometry of color space, defined as a three-dimensional rectangular coordinate system in which the coordinates correspond to varying amounts of three primary colors. The locus of colors of like excitation type is a straight line that emanates from the origin (zero intensity) and extends indefinitely. The direction of the line with respect to the coordinate axis defines chromaticity, and the distance between the origin and any point on the line defines the excitation intensity at that point. The axes, designated by x, y , and z , are assumed to represent the natural primaries, or fundamentals, of the color sense mechanism, and all real colors are contained within the positive octant of the geometrical system. The spectral hues are represented by a group of straight lines contained in the positive octant that form a cone-shaped open surface

with apex at the origin. Accepting both Helmholtz's quadratic form of sensation difference equation and the Fechnerian relation between excitation strength and sensation magnitude, Sinden's expression for color difference between equally bright colors is

$$K \Delta S = \sum_{400m\mu}^{700m\mu} \left[\left(\Delta \log \frac{x}{y} \right)^2 + \left(\Delta \log \frac{y}{z} \right)^2 + \left(\Delta \log \frac{z}{x} \right)^2 \right]^{1/4}$$

Sinden devised an ingeniously constructed projection apparatus to project the image of a spectral locus plotted in a color triangle based on three spectral mixture primaries onto a logarithmic triangular coordinate system outlined on a transparent projection screen. By marking off on the spectral locus the varying extents corresponding to experimental measures of minimally discriminable wavelength differences, it was possible to manipulate the projection orientation of the slide containing the spectral locus in order to yield approximately equal distances for the measured $\Delta\lambda$ extents on the projected image in logarithmic coordinates. The geometric transformation relations between the linear coordinates of the stimulus space and the logarithmic coordinates of the projected space were determined approximately by direct measurement in the projection apparatus, and the approximate solution made more precise by numerical calculations.

Although Sinden's transformation yielded a metric that he considered to be an improvement over Helmholtz's solution with respect to data for wavelength discrimination and colorimetric purity discrimination, the solution yielded a result that contradicted one of the fundamental assumptions of the theory on which it was based—the luminosity of one of the three primary colors was found to be negative by Sinden. Although he saw that such a result might be interpreted to mean that there is an independence between the mechanisms for brightness and for hue perceptions, he refrained from making this interpretation since it represents a radical departure from the classical theory of color vision on which the approach was based.

3.4 Stiles Increment Threshold and the Line-Element

Contemporary interest in the line-element is perhaps best represented by the work of W. S. Stiles (1946, 1949a, 1959) that was started in the 1940's and continues to the present. His aim has been to derive the fundamental distribution functions for vision from discrimination data obtained in what he has described as "increment threshold" experiments.

In the standard visual intensity discrimination experiment, light of a

given wavelength or wavelength distribution is used and the measures represent the minimal difference in amount of such light that an observer can detect with a given probability. If Weber's law were to hold exactly, then the discriminable intensity increment would represent a constant fraction of the background intensity $\Delta I = kI$. The form of the function typically found in experiments of this sort is, approximately, $\log \Delta I = k \log (I + c)$. Suppose, however, that the wavelength of the background light is kept constant and that the intensity increments are measured for a series of lights that differ in wavelength from that of the background stimulus. If the eye is equally sensitive to all such wavelengths, then the form of the discrimination function will presumably remain unchanged, and the value of the ΔI increment for any given background intensity will also be the same. If the eye is not equally sensitive to all wavelengths, then we cannot expect the same ΔI values from experiments in which different wavelengths of background and test stimuli are used for the increment threshold measures. Stiles' first and basic assumption is that the form of the function will be the same for all such wavelengths, but that for test stimuli of different wavelengths the different functions will be displaced in a direction parallel to the $\log \Delta I$ ordinate. Thus Stiles anticipated a family of parallel functions from such a series of experiments, with the function for the increment threshold wavelength to which the eye is least sensitive showing maximal upward displacement. He assumed further that the amount of this upward displacement could be used as a measure of the relative sensitivity to the different wavelengths of the increment stimuli.

For the corollary series of experiments, in which the test stimulus wavelength is maintained constant, but the wavelength of the background stimulus is changed from one experiment to the next, Stiles anticipated a second family of functions displaced relative to each other in a direction parallel to the abscissa. In this case, the function for the background to which the eye is least sensitive would be displaced maximally to the left, and again Stiles assumed that the degree of displacement would serve as a measure of the relative sensitivity of the eye to the different background wavelengths. If the form invariance and displacement assumptions are correct, then the sensitivity distribution functions derived by the displacement measures from the two series of experiments should coincide within the uncertainty limits determined by the precision of the data from the different experiments.

The logic of the increment threshold analysis is extended directly to the situation in which we must be concerned not with a single spectrally selective sensitivity distribution, but rather with three such distributions. Suppose that in an experiment the log energy of the least discriminable test stimulus U of wavelength λ is measured as a function of log energy of

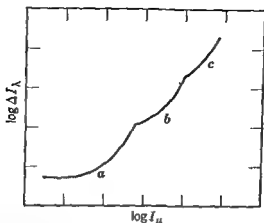


Fig 8 Illustrative increment threshold function with three component mechanisms. Log test field increment (ΔI_λ) plotted against log surround intensity (I_μ). Adapted with permission from Stiles (1949a, p. 158).

the background field W of wavelength μ . Suppose further that the data are described by the function shown in Fig 8, which is apparently made up of three component functions a , b , and c . The analysis assumes that there is a single function of the form shown in Fig 9, $\log \zeta(x)$. Each of the curve components a , b , and c is assumed to be a segment of this common function. The amount by which the common function must be displaced parallel to the ordinate to fit the a , b , and c segments, respectively, is a measure of the relative sensitivities of the three assumed component mechanisms to the test wavelength λ . The displacements parallel to the abscissa that are required to fit the same common function to the three

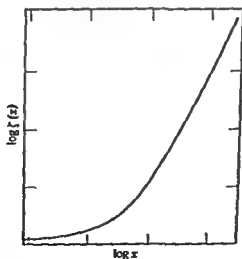


Fig 9 Stiles' theoretical "common component curve" showing the way in which log increment threshold varies with log background intensity for a single color mechanism. Adapted with permission from Stiles (1949b, p. 221).

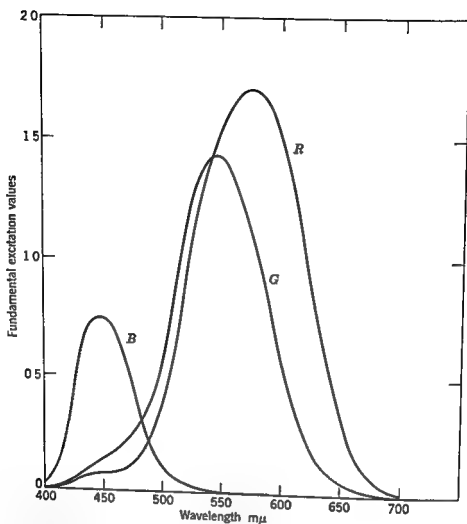


Fig 10 Stiles fundamental excitation curves. Values calculated from transformation equations given in Judd and Wyszecki. Drawn with permission from Judd & Wyszecki (1963 p 281)

component segments is a measure of the relative sensitivities of the three component mechanisms to the background wavelength μ . The curve shown in the last figure is from Stiles and represents this best approximation to a common function derived from component analysis of a very large number of increment threshold functions measured for his own eye.

The spectral distributions of three fundamental mechanisms derived by Stiles in 1949 from increment threshold analysis are shown in Fig 10 (1949b). If these three curves are labeled respectively R , G , B Stiles' (1946) formula for the minimal color difference to be represented as the line-element in uniform color space becomes

$$dS^2 = \left[\frac{dR}{\rho} \zeta(R) \right]^2 + \left[\frac{dG}{\gamma} \zeta(G) \right]^2 + \left[\frac{dB}{\beta} \zeta(B) \right]^2$$

Here, ρ , γ , and β are quantities proportional to the limiting Weber fractions of the three component mechanisms, which are understood by Stiles to differ one from the other. The limiting Weber fraction is greatest (and thus difference sensitivity is least) for the B mechanism, and the G and R mechanisms have limiting fractions that are nearly equal, although the G mechanism is slightly more sensitive than the R one.

The essential form of the color difference formula used by Stiles is directly comparable to that derived by Helmholtz. The specific differences between the two treatments relate to the different limiting Weber fractions for the different mechanisms, the different form of the pseudo-Fechnerian relation, and the different solutions for the three fundamental distributions of the three component mechanisms. There is, moreover, an important methodological difference. Although Helmholtz used his color difference formula in solving for the three fundamental distribution functions, Stiles did not. All of Stiles' discrimination data are analyzed in terms of independently effective component sensitivities. Nowhere does the formula for combined component differences enter in the analysis of his own increment threshold discrimination data. It is important to note that the Stiles procedure and the particular solution that we have been discussing so far represents his work as reported in the late 1940's.

The Stiles approach depends on the validity of the assumption that the increment thresholds are determined by the activities of component mechanisms whose spectral sensitivity distributions are invariant in form. Adaptation to a particular background of specified wavelength and intensity can presumably affect the level of sensitivity of any one component, but not the form of its spectral sensitivity distribution. This assumption is frequently made in analyses of adaptation effects, and its specific formulation as a three-variable coefficient "law" stems from the early work of von Kries (1905). It states that the relative sensitivity levels of three spectrally selective mechanisms will be reduced by exposure to an adapting light in proportion to their initial relative sensitivities to that light.

Stiles continued his increment threshold experiments during the past ten years to include other observers and an extended range of stimulus variables, and he found that the data cannot be accounted for by three such invariant distributions that follow an invariant set of displacement rules. A number of avenues of interpretation would seem to be open for further exploration. The data could suggest that the form invariance assumption is untenable, they could suggest that something other than (or in addition to) a multiplicative change in level is produced by the background stimulus, or that the assumed invariances hold, but that more than the three anticipated component mechanisms are involved. The third possibility seems

to be the only one that Stiles has seriously considered, and his more recent papers (1953, 1959) report the emergence of not three, but at least five and perhaps seven different component sensitivity distributions. Stiles himself is not yet clear on what the implications of these multiple sensitivity distributions may be for the mechanism of color vision, nor is it yet obvious what modifications they will require in the further development of the line element approach to the problem of uniform color space (Graham, 1959).

Both the Helmholtz and the Stiles approaches require the use of data that permit analysis in terms of basic sensitivity or excitation distributions. The color space is then built on these distributions as foundation, and the assumption of a specific formula for minimal color differences defines the line-element of the space. A different approach, but again one that develops a color space on the basis of minimally discriminable color differences, is illustrated by MacAdam's work.

3.5 MacAdam Discrimination Ellipses

MacAdam (1942) set out to determine experimentally the discrimination limits for a constant luminance plane of CIE chromaticity space, and later, with Brown (Brown & MacAdam, 1949), to extend the measures to include combined luminance and chromaticity variations. The chromaticity discrimination measures that he used were standard deviations of color-matching data. In the first set of experiments, all color matches were made using binary mixture-stimuli that were chosen so that they were of constant luminance for all mixture proportions, and so that when a color match to the test stimulus was made, the test and mixture-stimuli could have identical spectral distributions. For each test chromaticity in the sample space, different sets of these binary mixture-stimuli were used to obtain color-matching data for that point in the chromaticity space. The selection of the binary stimuli for physical identity of the mixtures when their proportions are adjusted for a "true" color match is important because it locates the color center at the same chromaticity for all individuals. If the matches were metameric (that is, different spectral distributions of test and matching fields for the condition of identity in color appearance), individual differences in color vision would preclude the assumption of a unique chromaticity center as the "true" color match locus.

MacAdam investigated 25 different test chromaticity centers C , and for each of these points determined the standard deviation of matches along several (5 to 9) straight lines radiating from C . The standard deviations

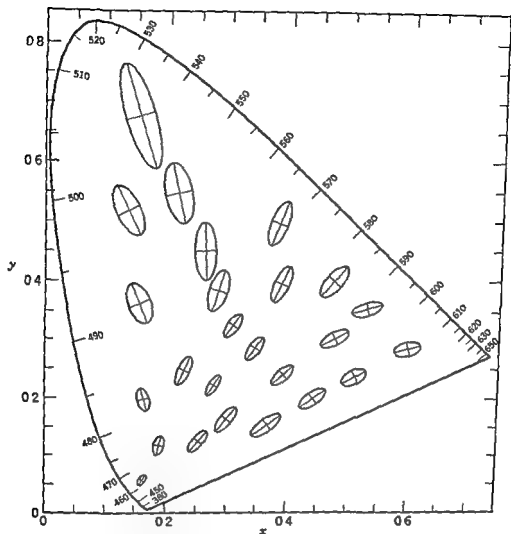


Fig 11 Discrimination ellipses in CIE chromaticity space. Radius of ellipses shown are ten times the experimentally measured standard deviations. Adapted with permission from MacAdam (1949, p. 228).

were based on 50 matches with each binary stimulus combination. MacAdam has fitted ellipses to the discrimination loci, and Fig 11 shows some of these ellipses and the variation in their dimensions and their orientations from one position of C to another.

On the assumption that color space has a Riemannian metric, the elementary distance ds between two points whose chromaticity coordinates are (x, y) and $(x + dx, y + dy)$ can be written in positive quadratic form

$$ds^2 = g_{11} dx^2 + 2g_{12} dx dy + g_{22} dy^2,$$

where $g_{ik} = g_{ki}$. MacAdam's experimental data, determined in the CIE x, y chromaticity space for a plane of constant luminance, are represented

as ellipses and are described by the equation (MacAdam, 1949)

$$g_{11} \Delta x^2 + 2g_{12} \Delta x \Delta y + g_{22} \Delta y^2 = 1.$$

To convert any given ellipse in a given region of the chromaticity diagram to a circle in which minimally discriminable color differences would be represented by equal distances from the center color C , the coordinate network in the region is redrawn with the angle between the x and y coordinate directions given by

$$\cos \omega = \frac{g_{12}}{(g_{11}g_{22})^{1/2}}$$

The units of length along the new x and y directions are proportional to $(g_{11})^{1/2}$ and $(g_{22})^{1/2}$, respectively. MacAdam has worked out such a solution for his data for different regions of the chromaticity diagram, limiting each region to a space within which the g values do not differ by

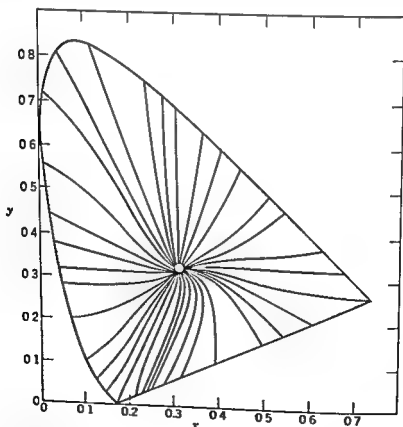


Fig. 12 Geodesics plotted between CIE standard illuminant C and a number of points on the spectrum locus. Geodesics are intended to represent series of colors of constant hue. Adapted with permission from MacAdam (1949, p. 217).

more than about 25%. The total surface is then reconstructed by joining together the part surfaces for which individual solutions have been obtained.

Geodesics of the resulting curved surface should represent series of equiluminous colors that include the least number of just noticeable color steps between any two colors. Figure 12 shows a number of such geodesics plotted on the CIE chromaticity diagram between the chromaticity locus representing the CIE standard illuminant C (artificial daylight) and the chromaticities of a number of spectral wavelengths. Such geodesics presumably represent series of colors that are constant in brightness and constant in hue and that vary only along the saturation dimension.

It is important to note that the uniform chromaticity space developed by MacAdam depends on the assumption that the chromaticities of the color matches can be described by a normal probability distribution in two dimensions, from which the Riemannian metric follows. We shall examine the MacAdam experiments more critically later, but it may be noted here that, although the normality assumption is difficult either to prove or to disprove, LeGrand (1957), for instance, considers it very unlikely. LeGrand proposes as an alternative the hypothesis of a normal probability distribution for each of three receptor excitations (or, near the absolute threshold, a Poisson distribution because of quantum light fluctuations). LeGrand points out that if the elementary difference is not a Riemannian one, the curvatures and other geometrical parameters become only approximate, and there is no *a priori* reason for rejecting any form of color difference equation.

3.6 Munsell Notation

A thoroughly different approach to the problem of establishing a perceptually meaningful ordering for color space is illustrated by the notational system for material color samples developed by A. H. Munsell (1905, 1912, 1941). The starting point here was neither visual theory nor color metrics. It was, rather, an interest in developing a simple, precise, and logical symbolic color language that would serve for the painter much the same function that musical notation does for the musician. Development of the system for practical use required the production of material reference samples that are ordered in a perceptually meaningful way. The differences between samples to be ordered are not the minimal differences of discrimination measures but steps that are many jnds apart. The spacing in this system is determined by a combination of some rather general intuitive notions about the rational basis of perceptual color orderings and

adjustments based on observations of various sorts. Although the system was originally developed primarily for artists and for use in painting instruction, the carefully controlled, specified, stable, and reproducible Munsell color samples have proved highly valuable also in scientific laboratories and commercial color control problems (Judd & Wyszecki, 1963). The Munsell system is, furthermore, frequently referred to as a standard for uniform color spacing (Burnham, 1949, Torgerson, 1958).

In this system, the geometric form of the perceptual color solid was assumed a priori to be a sphere. The continuum of achromatic colors is ordered along the vertical axis with midgray at the center and with samples of increasing reflectance (= increasing "value") in the upper half of the sphere and those of decreasing reflectance (= decreasing "value") in the lower half. The spacing (on a scale from 1 to 10) of the value ordinate was originally a logarithmic function of luminance in accordance with the Fechnerian hypothesis, but it was later modified on the basis of observations to a power function of luminance. The spacing in the present system is a cube root function.

The notion of color balance was at the heart of the original Munsell ordering. Thus any two "neutral" (N) or achromatic samples at equal distances in the two directions from the midvalue sample presumably balance to the midgray ($N/5$) when taken in equal amounts on a color wheel. The same principle is basic to the ordering of the chromatic samples within the color sphere. Five principal "hues" (R, Y, G, B, P) are arranged along the radii of each circular value plane perpendicular to the vertical axis so that they too balance to the achromatic center of the circle when taken in equal amounts on a color wheel. Principal colors of equal value balance to the same value of achromatic sample, and so on. Constant hue samples of varying saturation or "chroma" are ordered along each radius, with color balance again serving as the criterion for uniformity of spacing. Thus the original Munsell system was based on the production of material color samples that fulfill the requirements of specific color-mixture relations that were assumed a priori to characterize the color space, and that fulfill these mixture requirements when ordered within a space (color sphere) whose geometric properties were also assumed a priori.

The system has now been reevaluated for constancy of specified dimensions as well as for perceived uniformity along the hue, value, and chroma continua of color steps corresponding to unit differences in numerical notation (Newhall et al., 1943). The reevaluation by a group of experienced visual scientists was based on their direct observations of the samples as spaced in the *Munsell Book of Colors*. Their estimates were reported as line vectors of sample relocations required to satisfy the perceptual criteria

It is a tribute to A. H. Munsell's analytic and observational powers that the scaling judgments in this reevaluation showed as little need for revision of the spacing as they did. The material samples produced for the *Munsell Book of Colors* are now specified both in terms of Munsell's hue /value/ chroma system of notation and in terms of the CIE system of stimulus specification (Kelly et al., 1943) and the spacing is frequently compared in the latter system with color-difference formulas based on quite different assumptions and data (Judd & Wyszecki, 1963).

It seems obvious, however, that any color space that is tied to a fixed set of material color samples cannot have the same perceptual characteristics for a variety of different illuminations and viewing conditions. If such a color space were invariant in this sense, then we should have complete object color constancy, which is not, in fact, the case. To cite only one example, two Munsell "neutral" samples that differ only in Munsell value (lightness) may indeed appear to be grays of different brightness when viewed against a "neutral" background under some specified quality of daylight illumination. When viewed under a chromatic illumination, the same material samples on the same material background may appear (1) one as neutral and the other as the hue of the illuminant, (2) one as neutral and the other as a hue complementary to that of the illuminant, or (3) one as the illuminant hue and the other as its complementary, depending on the respective values (and thus reflectances) of the two "neutral" samples and the "neutral" background (Helson, 1938). The usefulness of a material sample color space is consequently limited to a limited range of viewing conditions, and both the spacing and the notational system become progressively more questionable with increasing departures of the viewing conditions from those established as "standard" for the system.

3.7 Opponent-Process Model

In the approaches toward development of perceptual color space that have been considered thus far, assumptions about visual processes have been limited to characteristics presumably required to account for the data of color mixture and of specific discrimination functions. We shall now consider a process theory approach that aims to (1) provide testable relations between stimulus variables and apparent hue, brightness, and saturation, relations that are, at the same time, consistent with the stimulus equivalence relations of color matching, (2) encompass some of the dependencies of spectral functions on stimulus luminance, including both the color appearance relations and the spectral discrimination data, and (3) account for some of the changes that occur in apparent color and color

discrimination when the general "field" conditions of visual stimulation are altered.

We may start with a general definition of perceived color which states that in the total visual field, the perceived color of any stimulus element, considered as a discrete element in time and space, is a three-variable complex of responses in the visual system. This complex of responses involves a triple function of the excitation products of the light stimulus energies at different wavelengths times the relative sensitivities of the three processes of different spectral sensitivities, together with the activities engendered by the occurrence of other visual response activities that are closely related in time and/or space. The perceived color relation is expressed as a three-dimensional vector equation

$$C = f[\sum_{\lambda}(e_{\lambda}X_{\lambda}), \sum_{\lambda}(e_{\lambda}\Psi_{\lambda}), \sum_{\lambda}(e_{\lambda}\Omega_{\lambda})] + I,$$

where $f = (f_1, f_2, f_3)$ and $I = (I_1, I_2, I_3)$. Here C is a three-dimensional vector representing the perceived color, e_{λ} the energy distribution of the test stimulus throughout the visible spectrum, X_{λ} , Ψ_{λ} , and Ω_{λ} , the linear wavelength distributions of three variables of the visual system for a stimulus of unit energy at all wavelengths, f is a three-dimensional vector-valued function of the sums throughout the spectrum of the products contained in the expression, and I is a vector representing the incremental activity, not caused by the focal stimulation per se, but induced by other related visual response activities (Jameson & Hurvich, 1959). Specification of the three separate attributes of the perceived color requires, of course, the further statement of three specific relations f_1 , f_2 , and f_3 of the three visual variables. Similarly, the induced activity I should also be thought of as composed of three separate components I_1 , I_2 , and I_3 . In this definition, a general restriction on the forms of the three generalized functions X , Ψ , and Ω is that they should be linear transforms of color-mixture data expressed in terms of relative amounts of three stimulus variables required for complete spectral color equations.

In the specific form of the process theory under discussion, the linear transforms take on two sets of definitions (Hurvich & Jameson, 1955, 1957). The first is a set of three positive functions of wavelength, which we have labeled α , β , and γ , and that presumably represent the initial light absorptions or photoreactions in the retina that transform physical light energies to physiological stimuli. The second set of linear transforms is a set of three paired, bimodal functions that represent the relative responsiveness of the neural visual tissue to the physiological stimulation initiated in accordance with the first set of three positive transforms. It is with these paired, bimodal, linear transforms that we are primarily concerned, since the proposed correlation with perceived color is with respect to response

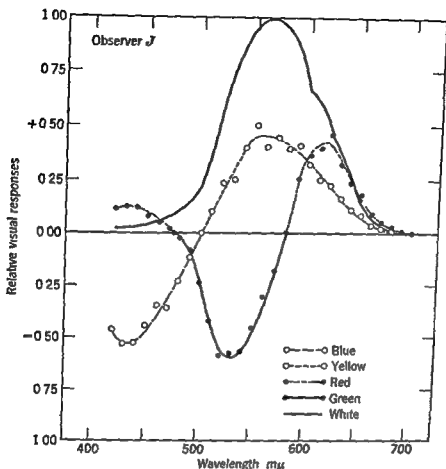


Fig 13 Experimentally measured chromatic and achromatic response distribution functions for one observer

functions of these distributions. The spectral distribution functions are shown in Fig 13 (Jameson & Hurvich, 1955).

These are experimental measures for one individual. The "white" function is taken as the spectral luminosity distribution, the reciprocal of the threshold energy measures for photopic conditions. The distributions of each of the chromatic functions were measured for an equal brightness spectrum by a null technique for hue cancellation. Given each of a series of test stimuli of fixed energy e_t and wavelength λ_t from a region of the spectrum (400 to 500 $m\mu$) that elicits blue hue responses (red-blue, green-blue, blue), determinations were made of the variable energy e_c of a cancellation stimulus of fixed wavelength λ_c (say, 580 $m\mu$) selected from a region of the spectrum that elicits yellow hue responses (red-yellow, green-yellow, or yellow) when the latter is mixed with the test stimulus in proportions such that the additive light mixture is perceived as "neither blue nor yellow" in hue. The same criterion is used for a series of yellow test stimuli (500 to 700 $m\mu$) and a fixed wavelength (say 470 $m\mu$) of blue

cancellation stimulus, and two additional series for red and green hue cancellation complete the measures of relative hue strength on which the chromatic distribution functions are based

The experimental distributions measured in this way for an individual are consistent with the following transforms of the CIE tristimulus distributions for average color-mixture data

$$\begin{aligned}r_{\lambda} - g_{\lambda} &= m\bar{x}_{\lambda} - m\bar{y}_{\lambda}, \\y_{\lambda} - b_{\lambda} &= n\bar{y}_{\lambda} - n\bar{z}_{\lambda}, \\w_{\lambda} - bk_{\lambda} &= p\bar{y}_{\lambda}\end{aligned}$$

The bimodal, positive and negative, characteristics are assumed to represent physiological processes that are opposed in nature, and they are required to account for the mutual exclusiveness of the associated hue qualities in the color perception. The distributions shown are taken to represent the responsiveness of each of the three paired variables and they refer to a set of viewing conditions such that there is no incremental activity I from associated responses caused by directly preceding, or simultaneous but spatially separate, stimulation in other parts of the visual field. For this reason the distribution of opposite sign is not shown for the second member of the achromatic white-black pair, opposed achromatic (blackness) activity is assumed to result exclusively from induced responses.

The linear transformation equations that relate these responses to color-mixture data in tristimulus units specify the psychophysical relations between stimulus and responsiveness in terms of the three independent pairs of response variables: white-black, red-green, and yellow-blue. Relations are also stated to specify the perceived color in terms of the three psychological attributes of brightness, hue, and saturation

$$B = w - bk,$$

$$H_{r-g} = \frac{|r - g|}{|r - g| + |y - b|}, \quad H_{y-b} = \frac{|y - b|}{|r - g| + |y - b|},$$

$$S = \frac{|r - g| + |y - b|}{|r - g| + |y - b| + |w - bk|}$$

In these relations, brightness B is taken as proportional to the net response $w - bk$ in the achromatic, black-white response system. Hue H is expressed as a percentage and relates the net response $r - g$ in the paired red-green system to the total of the chromatic responses in the red-green and yellow-blue systems taken together. In each case, the net response is the appropriate function of the integrated product of energy times responsiveness. The hue is either red or green and/or yellow or blue in

accordance with the sign convention adopted for the particular components of the paired systems. Saturation S is also expressed as a percentage relating the summed net responses of the red-green and yellow-blue chromatic pairs to the sum of the chromatic responses plus that of the achromatic white-black system. It will be noted that this expression for saturation explicitly incorporates saturation changes produced by intermixture with blackness, which is an important aspect of perceived saturation that is almost always lost sight of in theoretical systems that deal with the more limited stimulus variables rather than with the response variables per se.

With relations between stimulus variables and perceived color variables expressible in this quantitative fashion, it becomes clear that this kind of approach can be used to provide the basis for a systematic color metric (Hurvich & Jameson, 1956). In such a system, color differences along any one dimension of perceived color are assumed to be uniform. That is, a unit percentage difference in saturation represents the same saturation difference whatever the hue and whatever the brightness. Similarly, a given percentage difference in hue has the same psychological significance whatever the perceived saturation and brightness of the color, and so on. With respect to differences in perceived color that involve combined differences in more than a single psychological dimension, the model assumes simple additive combinations of hue and saturation differences, with equal weighting of the differences for the different attributes. $\Delta C = \Delta H + \Delta S$ when B is constant. This most simple assumption has yielded rather good agreement with data from wavelength discrimination experiments where apparent brightness is constant but hue and saturation vary concomitantly (Hurvich & Jameson, 1955).

Even in the absence of any variation in pre-exposure or surround stimulation, the stimulus spacing that corresponds to the perceived hue and saturation space for constant brightness shows a strong dependence on the level of constant brightness. The three functions relating response magnitude to stimulus magnitude are assumed to differ among the red-green, yellow-blue, and white-black paired response systems, and thus not only the brightness but also the hue and saturation attributes of the color evoked by a stimulus of fixed spectral distribution all vary systematically with the energy level of the given stimulus. This assumption of the theory is consistent with the empirical data, that is, the Bezold-Brücke spectral hue shifts, the inverted U -shaped functions relating spectral saturation to luminance at any wavelength, and the changing forms of the wavelength discrimination function for different levels of spectral luminance.

Within the size limits of small stimulus areas for which there is summation or some degree of area-intensity reciprocity when stimulus size is

reduced, the perceived color space undergoes changes that are comparable to those associated with reduction in stimulus energy. This assumption of the theory conforms to the observed color appearance and discrimination changes usually described as the phenomena of small field tritanopia, or sometimes by the misnomer of "foveal tritanopia" (Farnsworth, 1955, Hartridge, 1945, Hurvich & Jameson, 1958). The name *tritanopia* is used to categorize a very rare type of color defect in which yellow-blue hue discriminations are lost in contrast to the much more common red-green losses that occur in the usual forms of color blindness or color deficiency. In normal color vision, the term is applied to the situation that prevails with very small stimulus fields, but not to the directly comparable state of affairs found with reasonably large stimulus areas but very low levels of stimulus energy. This inconsistency in terminology derives from theoretical assumptions of no more than historical interest (Konig, 1894), and hence the matter will not be pursued further here.

In an early statement of the opponent-process theory, the precise forms of the three functions that express the three different dependencies of response strength on stimulus luminance were not specified. However, the differential rates were expressed by intensity-dependent multiplicative factors whose numerical values differed for the three paired response systems, and whose ratios to one another differed systematically for different energy levels. The available evidence, especially for brightness relations controlled by the achromatic white-black system, suggests functions of the form

$$f_{w-b} = [\sum_{\lambda} (e_{\lambda} w_{\lambda} - e_{\lambda} b k_{\lambda})]^{n_1},$$

$$f_{y-b} = [\sum_{\lambda} (e_{\lambda} y_{\lambda} - e_{\lambda} b_{\lambda})]^{n_2},$$

$$f_{r-g} = [\sum_{\lambda} (e_{\lambda} r_{\lambda} - e_{\lambda} g_{\lambda})]^{n_3},$$

where n_1 , n_2 , and n_3 are different powers, and n_2 is known to be greater than n_3 .

The threshold values for the three paired systems are also assumed to differ one from the other, and in a way such that

$$t_{w-bk} < t_{r-g} < t_{y-b}$$

Thus, at the absolute threshold for foveal vision, the perceived color space degenerates to the single, paired achromatic dimension, and at near-threshold levels the space is only dichromatic. The achromatic axis of the color space occurs along the locus of values where $f_{r-g} = f_{y-b} = 0$. Since the nonlinearities in the response functions are identical for both members of each single opponent pair, the achromatic balance of the system is not intensity-dependent in terms of its central locus. At any given level of brightness, however, the achromatic zone around the central

axis of achromatic balance varies in shape and area in stimulus space. Thus, in perceived color space, all stimuli, including monochromatic spectral wavelengths, are located at the achromatic center at the absolute photopic threshold. At near-threshold levels only slightly above the absolute threshold, only a small compass of weakly saturated red and green hues is contained in the perceived color space. The total hue gamut, with maximal saturations for the given viewing conditions, occurs at an intermediate level of photopic luminance. As the luminance is increased to very high levels, the hue gamut becomes progressively restricted again because of the differential response rates among the three paired systems. But now the restriction occurs in the compass of red and green hues. At extremely high levels, yellow and blue hues also become strongly desaturated and eventually all stimuli again approach the achromatic axis in perceived color space. This description of perceived color space is somewhat reminiscent of diagrams of the color solid commonly used in textbook illustrations, although the latter do not represent the different paired hue changes as described here. The rough similarity is hardly surprising since these illustrations are intended to describe qualitatively the same phenomena that are handled here in terms of process characteristics.

With the focal stimulus unchanged, the locus of any color stimulus in perceived color space is further modified when the field conditions are changed by the introduction of surrounding or pre exposure stimuli. Under these circumstances, the expression I in the basic definition of perceived color given previously must be evaluated for each different set of viewing conditions. In general, for the surround situation, each of the induced responses $I_{1,2}$, is proportional to but opponent to the corresponding response R_s in the inducing (surround) area (Jameson & Hurvich, 1959, 1961b). For each of the three paired systems, the induced responses are evaluated in terms of sets of simultaneous equations of the following form

$$R_f = f(S_f) - kR_s,$$

$$R_s = f(S_s) - kR_f,$$

In the set of equations, R_f denotes the response of one of the three paired response systems in the area of the visual field designated as focal and R_s , the corresponding response in the area treated as surround. S_f and S_s represent the values of the stimuli impinging on the two areas, respectively, and f is a nonlinear function of the appropriate linear stimulus transform. The induced responses are contained in the incremental expressions $-kR_s$ and $-kR_f$ and, as the constant k indicates, the induced activity is proportional to the response activity in each of the different related visual areas. The interactions occur between or among responses, and not between or among stimuli. The induced responses could be linearly related

to the stimulus values only if the mutual interactions were assumed to occur at the linear, that is, light reception, stage of the visual mechanism. Analysis of data from a variety of brightness constancy and contrast experiments supports the following formulation for induced responses in the achromatic, white-black system (Jameson & Hurvich, 1961a, 1964)

$$R_f = L_f^{1/3} - kR_s,$$

$$R_s = L_s^{1/3} - kR_f,$$

Here the stimulus measures are in luminance units, $e_\lambda w_\lambda$, and the response function f is the cube root

One of the most striking features of contrast interactions, both as observed in experiments and as derived from the formulation discussed here, is the change in perceived color distance that occurs when arrays of stimuli that are not perceived as very different when viewed individually are viewed simultaneously (Hurvich & Jameson, 1960, 1961). A simple effect of this sort is the conversion of a brightness magnitude scale from a simple cube root function of luminance to a curvilinear function of the form

$$y = ax^n - b$$

when the test stimuli are exposed in the presence of an illuminated surround (Jameson & Hurvich, 1959, 1964)

An illustration of increased perceived color gamut and expansion of color differences is given in Fig 14. The points plotted as open circles represent the colors of a series of equally bright stimuli, all of which fall in the red-yellow quadrant of the perceived hue gamut when viewed individually. For these perceptions there are no induced responses. If the same stimuli are viewed simultaneously (points plotted as filled circles), we may, as a rough approximation, consider each to be surrounded by an average color that would have associated responses (without induction) determined by the mean of the isolated color responses to the remaining stimuli in the field. If these mean values for each of the three paired variables of the response system are represented by $r_s (= -g_s)$, $y_s (= -b_s)$, $w_s (= -bk_s)$, and the values for the individual areas by r_f , y_f , w_f , then the individual response values with induction, r'_f , y'_f , w'_f , become

$$r'_f = \frac{r_f - kr_s}{1 - k^2},$$

$$y'_f = \frac{y_f - ky_s}{1 - k^2},$$

$$w'_f = \frac{w_f - kw_s}{1 - k^2}.$$

For the illustration in Fig 14, the value of the induction coefficient k has been set equal to 0.2 for all three response systems (Jameson & Hurvich, 1961b, 1964), and the colors are expressed in hue and saturation terms for a plane of constant brightness (see above)

In the real situation the spatial distribution of the colors in the array would influence the induction strength (the magnitude expressed by k) in terms of contiguities and separations, sizes, and retinal locations. The perceptions may also be influenced by possible dependences on retinal location of the functions f_i , $i = 1, 2, 3$, in the basic definition of perceived color given at the beginning of this section. The model, as developed to date, does not make any statements about this, and these are among the variables that require further experimental study

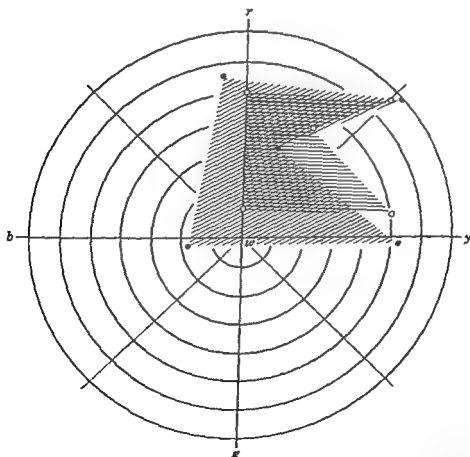


Fig 14 Effect of contrast on perceived color gamut. Open circles represent each of five equally bright stimuli seen in isolation. Note restriction to $r-y$ quadrant. Filled circles represent expanded gamut produced by mutual interactions. Plot is in polar coordinates where hue varies circumferentially and saturation increases radially from center.

A process theory of this sort can, of course, look to physiological data to support its basic hypotheses. Such evidence is available but has not been included in this account since this sort of approach does not depend on a knowledge of the physiology but is based rather on inferences from a wide variety of psychophysical data.

3.8 Some Characteristics of the Psychophysical Experiments

Many experiments in color vision present special problems for measurement theory. This is so in some instances because of the stimulus controls and methods used and in others because of the kinds of judgmental criteria that are involved. Criterion responses in color experiments frequently require the abstraction of a single perceptual attribute. This is sometimes done by using some indirect, but unambiguous, criterion to define the perceptual endpoint, and spectral luminosity determinations provide good examples of this sort. Measurements of relative luminosity for an equal energy spectrum involve the determination of the energies required at a series of spectral wavelengths to meet a specified criterion response (Wright, 1949). The criterion is taken, either directly or indirectly, as an index of constant apparent brightness. The measures may be determinations of the absolute photopic threshold, this criterion presumably implies a uniformly minimal perceptible brightness level. The measures may be determined by a flicker fusion technique in which a standard of fixed intensity and a test stimulus of variable intensity are alternated at a constant rate, there the criterion of fusion is presumably an index of equality of brightness between alternating standard stimulus and test stimulus of specified wavelength. Or the functions may be based on direct brightness matching. Here the criterion of equal brightness is used directly, but the stimuli must be judged to be equal in brightness in the presence of hue and saturation differences (heterochromatic brightness match) between test and comparison stimuli. In all three instances, plots of the energy measures as dependent variable against stimulus wavelength as independent variable provide functions that describe "equal brightness contours." Reciprocals of these contours are, by definition, relative spectral luminosity functions. The different measures obviously involve different stimulating conditions as well as different response criteria, and the resulting functions are, not surprisingly, somewhat different from one method to the next.

To determine contours of equal hue, the variable wavelength of a comparison stimulus of fixed luminance is measured for a hue match to a test stimulus of fixed wavelength at a series of different test luminances.

Hence the measures depend on a judgment of equality in hue in the presence of brightness and saturation differences. The measures of opponent chromatic responses discussed in the preceding section also depend on a specific hue criterion, that is, neither yellow nor blue or neither red nor green for the index response. The appearance of the light mixture for the given index response is thus not constant, but varies from one combination of test and cancellation stimuli to the next.

In all these instances, it is difficult to guarantee that the criterion response for one attribute is not biased or confounded by variations in the other color attributes presumed to be irrelevant.

The same problem arises in connection with measures of discrimination thresholds. There are two frequently used measures of spectral color discrimination, one that relates to discriminability along the spectrum locus, the other to the first discriminable step from a specified "white" point in the color-mixture space along each of a series of lines radiating toward the spectrum locus. Both these types of discrimination experiment also require the abstraction of one attribute of perceived color from the other two. In the wavelength discrimination experiment, the aim is to determine that spectral stimulus, $\lambda + \Delta\lambda$ or $\lambda - \Delta\lambda$, in one-half of a bipartite field that is just discriminably different from the test wavelength λ in the other half when there is no discriminable brightness difference between the two stimuli. In some procedures a judgment of "same" or "different" is required for discrete pairs of stimuli whose energies have been adjusted for equal luminance in terms of a separately determined spectral luminosity function. This procedure makes the decision task a relatively easy one for the observer, but leaves a margin of uncertainty about the "no discriminable difference in brightness" assumption. An alternative to this procedure is to provide the observer with a control knob for varying the energy of the comparison wavelength. The test and comparison stimuli are presented and the observer's task is to attempt to produce a complete color match between the two wavelengths by adjusting the energy (and with it, the brightness) of the comparison wavelength. In this case the observer's judgment is "match" or "no match," and the $\Delta\lambda$ measures represent just discriminable color differences that cannot be eliminated by the elimination of a brightness difference. Such $\Delta\lambda$ experiments are sometimes described as measures of spectral hue discrimination, but the description is erroneous since equally bright spectral stimuli may differ from each other either in hue or in saturation or, more often, in both hue and saturation.

The second kind of spectral discrimination measure relates more directly to the saturation attribute of color. These are the measures of least discriminable colorimetric purity. Here, the aim is to determine the

luminance L of a test wavelength λ that must be intermixed with a "white" light of luminance L_w to produce a just discriminable difference in color between this mixture and the "white" stimulus presented alone in the other half of a bipartite field when there is no discriminable difference in brightness between the two fields. Again, the brightness attribute must be abstracted and made irrelevant to the discrimination, and the procedural problems posed by this requirement are essentially the same as those in the wavelength discrimination experiments.

In both types of experiments, not only must brightness differences be excluded between test and comparison stimuli in the two halves of the bipartite field but the brightness levels should be constant for all stimuli used to determine a given spectral function. This is so because there is sufficient evidence to justify the expectation that the discrimination measures will vary with luminance level for any given spectral region as well as with wavelength for any given luminance level (Hurvich & Jameson, 1955, Purdy, 1929, Weale, 1951). The uniform spectral luminance level requirement is met for measures of purity or "saturation" discrimination where the standard "white" field can be kept constant throughout. Frequently, however, it is not met in wavelength discrimination experiments, largely because of instrumental limitations that provide only relatively low energy levels of the monochromatic bands in the short wave spectral regions (Wright & Pitt, 1934). Ideally, both wavelength discrimination and purity discrimination should be measured and described in terms of families of spectral functions for a series of levels of uniform spectral luminance. In any event, the luminance and wavelength dependencies should not be confounded in a single discrimination function, and the parametric value for which the function was obtained should be reported as an essential datum.

Can we derive discrimination measures from the spread of three-variable spectral color-matching data? How are the errors distributed in these measurements? The average data for spectral color mixture are freely transformed from one set of mixture primaries to another. Can we take the error spread as a measure of discrimination for the same experiments and can it be transformed in the same way and with the same justification as mean color equations?

Before examining the error distribution per se we might examine the consequences of the necessity for desaturating the test stimuli when making spectral color matches. The actual color matches in any three-variable mixture experiment are identity judgments between stimuli located on or within a triangle formed by joining the loci of the three mixture primaries (see Fig 3, p 117). The locus of spectral stimuli falls, for the most part, outside the limits of the mixture triangle, and hence their

mixture equivalents must be derived from matching data in which the test stimulus is a composite made up of the spectral stimulus in question combined with some amount of one of the mixture primaries. The actual locus of the test stimulus in the mixture space consequently depends on the wavelength and energy of the spectral stimulus, the wavelength of the spectral primary used for desaturation, and the amount of this desaturating primary. Thus the test stimuli actually used in a spectral color-matching experiment will, in general, depend on the mixture primaries, and the error measures will be specific for these binary test colors. Hence the spread of the color match data determines the uncertainty limits of the spectral color equation, but obviously cannot serve as a measure of spectral color discrimination.

The error distributions of the matches for the actual test colors used also present some difficulties for analysis as discrimination measures. In such experiments, the method of adjustment is used, and three separate adjustments must be made for each match. The decision process is a continuous one as between "identity" or "nonidentity," and adjustments involve a combination of trial-and-error manipulations, and learned associations between specific manipulations of the stimulus controls and specific color differences to be eliminated that are abstracted from the general nonidentity judgments. Final settings usually involve bracketing between discriminable limits of the stimulus variations, and thus the nature of the instrumental control can influence the final match setting for which the observer tries to center each stimulus variable within the interval of uncertainty or the range of acceptable settings (Stiles, 1955a, 1955b).

In his redetermination of the color-mixture data to serve as a new international standard, Stiles tried to reduce the instrumental error that bracketing with a logarithmic stimulus control introduces. He did this by adding more desaturating stimulus than necessary for particular matches in order to bring the test color to a chromaticity for which the interval of uncertainty corresponds to a small range of stimulus variation along each of the three mixture-stimulus continua. In so doing, Stiles' concern was not with the effect of this source of error on the distribution of the individual match settings, but with the possibility that it contributed an unknown constant error to the arithmetic mean of the data. For metameric color matches there is, of course, no way of evaluating such a constant error in terms of a color equation known to represent a "true" match, since the "accurate" metameric color match depends in each instance on the color vision characteristics of the individual observer.

We have mentioned earlier that in his determinations of discrimination ellipses for equiluminous color matches, MacAdam (1942) selected paired filter primaries along each of a series of intersecting lines in the constant

luminance chromaticity plane, and the selection was such that the intersection locus represented, in each instance, the chromaticity of a mixture stimulus of very nearly the same spectral distribution as the standard. This provides a locus of (approximate) stimulus identity which necessarily coincides with a "true" locus of perceived color identity for any individual observer. MacAdam's experiment also employed a method of adjustment, and the instrumental controls were such that for matches to a given color location along a single binary stimulus continuum, the variation from one endpoint to another was always accomplished by a rotation of 360° of a single knob. If the measures along different continua are to be considered as free from instrumental bias related to the rate of change of the mixture proportions with respect to angular rotation of the knob, it should be demonstrated that the spread of the settings along a single continuum are independent of this rate of change. MacAdam has made such checks (personal communication). The check data, however, were not included in his published report, and it is not clear from the report that the discrimination ellipses are necessarily free from such bias. From the published data for the discrimination limits and the chromaticities of the binary mixture primaries, one can determine for all continua through a given standard color the correlation between the obtained distance that represents unit perceptual distance and the manual rate of change of the instrumental control. We have calculated rank order correlations, based in each case on 6 to 9 pairs, for 25 MacAdam ellipses. Of the 25 correlations, 23 are nonnegative, and of these, 19 are above 0.40 and 7 are above 0.80. Such a rough statistical check is, of course, not conclusive, but it does suggest that manual features of the adjustment task might have introduced an instrumental bias in the discrimination measures described by the MacAdam ellipses.

The theory on which MacAdam's binary mixture experiments and the later three variable color matching experiments of Brown and MacAdam (1949) provide the basis for a color metric starts from the statistical nature of discrimination measures (Silberstein & MacAdam, 1945). When a subject makes a color match by adjusting the amounts of three stimulus primaries in a mixture, data are obtained in the form of triples of scale readings, one triple for each trial. The basic assumption of a probabilistic model is that for a given color and a particular individual, the data triples are observations from a fixed trivariate probability distribution. If we assume, for convenience, that the trivariate distribution for a given standard is unimodal, then it is reasonable to associate the standard color with the mode as a set of coordinates. Thus, for a given observer and a specified set of mixture primaries and instrumental stimulus controls, a one to-one correspondence is established between colors and triples of

real numbers Each triple may be denoted by a single letter, X , Y , etc., where each triple corresponds to a color, if the mixture stimuli and controls are changed, we may denote the triples as X' , Y' , etc., with X and X' , Y and Y' , etc., corresponding to the same color in the original and altered setup, respectively If X is the standard color, the corresponding trivariate density function will be denoted f_X , thus $f_Y(Y)$ denotes the probability density of getting observation Y , for the unprimed conditions, when the standard is such that the modal observation is X

It seems plausible to propose that the *distance* between the colors corresponding to X and Y is some monotonically decreasing function of $f_X(Y)$ that reaches 0 at the maximum of f_X , that is, since X is the mode, the distance from X to itself is 0 Clearly, for such a definition to yield a metric, it is desirable that $f_X(Y) = f_Y(X)$

However, more than the symmetry condition is needed If such a distance is to be meaningful as a perceptual measure, then the density functions f_X , considered as functions representative of the perceived colors, that is, visual effects (rather than the stimulus triples X , Y , etc.), must be independent of the experimental setup Perceptual distances must be independent of the stimulus primaries selected for the matching and of the particular characteristics of the instrumental stimulus controls Formally, we require the condition that

$$f_Y(Y) = f_X(Y)$$

Suppose that g denotes the monotonic decreasing function that transforms probability density into distance, and that $g_X(Y)$ is equal, by definition, to $g[f_X(Y)]$ We also require, of course, that $g_X(Y) = g_X(Y')$ This invariance property suggests strongly that g_Y be regarded as an invariant function of Y , transforming according to tensorial rules with transformations of coordinates The interpretation of $g_X(Y)$ as distance suggests, as an approximation at least, that $g_Y(Y)$ be a quadratic form in Y , with the coordinates of X taken as local origin Thus, if the coordinates of X are (x_1, x_2, x_3) and the coordinates of Y are (y_1, y_2, y_3) , the function $g_Y(Y)$ will be approximated by

$$g_X(Y) \approx \sum_{ij} a_{ij}(y_i - x_i)(y_j - x_j),$$

at least for Y close to X This is the three dimensional form of the Riemannian metric, considered as a local approximation In summary, the argument amounts to this given that the metric is to be coordinate free, the simplest assumption is that it is Riemannian The constants a_{ij} are the components, in the unprimed coordinates, of the metric tensor, and under change of coordinates they transform according to the simple rules governing second-order covariant tensors

If another simple assumption is made, namely, that the transformation from probability density to distance has the simple form $g(t) = -\log t$ plus a constant, then it follows that the density function f_X is approximately normal, and a_{ij} is its covariant matrix. Thus the theoretical argument is essentially that the normal distribution is a good candidate for the distributions f_X since it is unchanged in form under linear transformations of coordinates, and therefore approximately invariant under arbitrary differentiable transformation of coordinates. If the probability density is trivariate normal, approximately, then the metric, defined as minus log probability plus constant, is Riemannian.

This theory, of course, underlies the use of the contour ellipsoids obtained by free matching to estimate the values of the metric tensor in CIE coordinates by Brown and MacAdam (1949) for three variable matches, and by MacAdam (1942) for the ellipses that describe his two variable color-matching data. The theory depends critically on the assumption that the density functions f_X and f_λ satisfy the invariance requirement. The invariance assumption is subject to empirical test, and such tests would seem critical to the use of color-match data in this way as measures of minimal color distances.

3.9 Metric Scaling Model Based on Similarity Judgments

To date, there has been no systematic attempt to obtain empirical measures of both small and large perceived color differences for comparable conditions of observation. Such measurement is important for the development of a *metric* for a color space that represents the similarity and discriminability aspects of the color domain.

A *metric space* is a mathematical structure consisting of a set S and a function d which assigns to each pair, x, y , of elements of S a real number, denoted $d(x, y)$, such that the following axioms are satisfied (x, y , and z are arbitrary members of S)

- (M1) $d(x, y) \geq 0$,
- (M2) $d(x, y) = 0$ if and only if $x = y$,
- (M3) $d(x, y) = d(y, x)$ (symmetry),
- (M4) $d(x, y) + d(y, z) \geq d(x, z)$ (triangle inequality).

Familiar notions of geometry may be introduced in such a mathematical structure, and powerful mathematical tools are available for the analysis of the geometrical properties of a metric space. It is hoped that the similarity aspects of the stimulus domain ultimately may be characterized

by means of geometrical invariants and, as a further step, that the systematic effects of relevant variables on color perception may be characterized by means of geometrical transformations. Such an empirical development would impose specific restraints on theoretical models of the color vision system. It would provide a test for theories such as Stiles' line-element and the opponent colors theories as well as for the various forms of color difference equations that have been proposed from time to time (Judd & Wyszecki, 1963).

The work of MacAdam, Silberstein, and Brown discussed in Secs. 3.5 and 3.8 attempts to develop a metric representation of small color differences. Recent applications of multidimensional scaling to color problems can be found in Torgerson (1952), Messick (1956), Indow and Kanazawa (1960), and Shepard (1962a, 1962b). These investigators have not, however, attempted to encompass both minimal discriminability and large color differences.

Here, we outline a proposed metric scaling model that may provide a fruitful approach to the empirical measurement and representation of perceived color differences. The corresponding experimental procedure entails comparisons of perceived stimulus differences.

One methodological question might be considered at the outset: why make comparisons of stimulus differences rather than scale stimuli directly by magnitude estimation?

In the method of comparisons, the responses to repeated presentations of two stimulus pairs are assumed to be representable as a sequence of (independent) Bernoulli trials with fixed probability. If this assumption is satisfied (see Bush, Galanter, & Luce, 1963) then a large number of such presentations yields a fairly accurate estimate of the probability with which one pair is chosen as exhibiting the lesser difference. In terms of this probability, we can define a measure that can be interpreted as an expression of the relative distances exhibited by the two pairs. With magnitude estimation, however, the mean or median of the numbers emitted by the subject to a pair of stimuli can be taken as the distance measure for that pair after relatively few stimulus presentations. Although magnitude estimation has been used rather extensively to scale a variety of stimulus continua, it has not been used to estimate the very small differences required to subsume the measurement of discriminability as well as global similarity.

There is, however, a more fundamental reason for rejecting the seemingly more efficient method of magnitude estimation. The numbers obtained by magnitude estimation need not satisfy the axioms M1 through M4 and, in particular, one has no intuition, based on the structure of the magnitude estimation task, as to why or whether axiom M4, the triangle inequality,

might or might not hold. Although some transformation of the magnitude estimates rather than the estimates themselves might be the underlying metric, there is no a priori or theoretical reason for choosing any particular transformation. Some restriction on the possible monotonic transformations might be obtainable from a measurement theory for magnitude estimation. Although some investigators are convinced that magnitude estimation does yield "ratio scale measurement" (Stevens, 1957), no measurement theory has been developed for magnitude estimation to justify fully this conviction (Luce & Galanter, 1963b). Another possibility for narrowing down the possible transformations of magnitude estimates is found in the work of Shephard (1962a, 1962b), who has devised a method for obtaining suitable monotonic transformations of "proximity measures" to yield a metric. His method requires, however, that the form of the geometry be assumed in advance to a very great extent, that is, it must be assumed to be Euclidean or some equally strong substitute.

For the method of paired comparisons, the situation is quite different. The development sketched below extends Luce's choice analysis of similarity judgments (Luce, 1961, Luce & Galanter, 1963b). The choice axiom of Luce (1959), which is directly testable, leads in the case of paired comparisons to a ratio scale v defined on pairs of stimuli, such that the probability P that the difference between x and y in the pair (x, y) is less than the difference between z and w in the pair (z, w) is given by the following equation, provided that P is different from 0 and 1

$$P[(x, y), (z, w)] = \frac{v(x, y)}{v(x, y) + v(z, w)} \quad (1)$$

It is intuitively plausible that $v(x, y)$ is a measure of the similarity of x and y and that some monotonic decreasing function of it may be an underlying distance function. The question arises, as it did with magnitude estimation, which transformation shall we choose? In the present case, the measurement theory immediately suggests a restriction: the fact that v is a ratio scale imposes the stringent, yet natural, condition that any transformation suggested for a distance measure should yield the same distances or multiples (in geometric terms, a similarity transform) when the v scale values are multiplied through by a positive constant.

The possible transformations are further narrowed down by the following analysis. Suppose stimuli x and y are so similar that there is an appreciable probability that, on a given trial, the subject will choose the (x, y) difference as less than the (y, y) difference. Then it is plausible that for any third stimulus z the choices will be relatively indifferent between the pairs (x, z) and (y, z) . Thus, when $P[(x, y), (y, y)]$, which is always

$\leq \frac{1}{2}$, becomes appreciably greater than 0, then for any z , $P[(x, z), (y, z)]$ will also be greater than 0 and at least as close to $\frac{1}{2}$ as $P[(x, y), (y, y)]$. That is, the confusability between x and y with respect to a third stimulus z is unlikely to be less than the confusability between x and y with respect to y itself.⁵ Thus we can state as a plausible assumption

$$P[(x, y), (y, y)] \leq P[(x, z), (y, z)] \quad (2)$$

If the probabilities in Inequality 2 are different from 0 and 1, then we may use Eq. 1 to infer that

$$\frac{v(x, y)}{v(y, y)} \leq \frac{v(x, z)}{v(y, z)} \quad (3)$$

Clearly, Inequality 3 is equivalent to

$$\log \frac{\sqrt{v(x, x)v(y, y)}}{v(x, y)} + \log \frac{\sqrt{v(y, y)v(z, z)}}{v(y, z)} \geq \log \frac{\sqrt{v(x, x)v(z, z)}}{v(x, z)} \quad (4)$$

Thus the transformation

$$d(x, y) = \log \frac{\sqrt{v(x, x)v(y, y)}}{v(x, y)} \quad (5)$$

yields a possible definition of a distance measure in terms of the v -scale. It is designed so that the triangle inequality M4 holds at least for x , y , and z that are quite similar to one another. The other metric axioms are also satisfied by the function d , provided that we assume $P[(x, y), (y, y)] \leq \frac{1}{2}$ for all x , y and provided that stimuli x and y such that $d(x, y) = 0$ will be identified as part of the same *metamer class*. The distance is then the distance between metamer classes.

The admission of probabilities of 0 and 1 brings about interesting mathematical problems, but these can all be solved if the assumption of Inequality 2 is extended a bit further. Informally, the required generalization is as follows. Let x , x' , y , and z be four stimuli such that y is closer than z to both x and x' . We assume that the *confusion* of x and x' , when their differences from y are compared, is less than when their differences from z are compared, that is, $P[(x, y), (x', y)]$ is farther from $\frac{1}{2}$ than is $P[(x, z), (x', z)]$. This reduces to Inequality 2 when $x' = y$.

These assumptions about confusability, together with the basic choice axiom underlying v -scale measurement, can be tested directly by experiment. If it turns out that these measurement assumptions are found to be tenable, then we have a solid base for the experimental determination of perceived interstimulus distances that is independent of any a priori notions about the nature of the underlying mechanisms, and even of the

⁵ A priori perceptual analysis suggests that this assumption may not be applicable to exceptional instances of small perceptual distances where y lies between x and z .

validity of any of the so-called laws of visual behavior. Thus, for example, the well-established laws of color mixture that specify the characteristics of metamerism need not be known a priori but would presumably emerge from an analysis of the experimentally determined choice probabilities. The procedures are, moreover, ideally suited to determine, rather than assume, the relations between units of discriminability and units of perceived magnitude along stimulus continua.

The need for and importance of the application of sophisticated measurement theory and metric ideas to the area of color vision do not imply that this kind of empirically determined metric necessarily leads to a better understanding of the underlying processes or that it constitutes the most fruitful approach to such an understanding. In many instances, specific intuitions of potentially fundamental importance can probably be tested directly and efficiently in experiments that are hypothesis oriented. The empirical development of the metric approach should, however, provide a rigorous language in which to develop intuitions, frame hypotheses, and test theories.

References

- Bartley, S. H. Central mechanisms of vision. In H. W. Field, V. E. Magoun & J. Hall (Eds.), *Handbook of physiology*, Vol. I. Washington, D. C.: American Physiological Soc., 1959. Pp. 713-740.
- Bartley, S. H. A clarification of some of the procedures and concepts involved in dealing with the optic pathway. In R. Jung & H. Kornhuber (Eds.), *The visual system: neurophysiology and psychophysics*. Berlin: Springer-Verlag, 1961. Pp. 386-398.
- Baumgardt, E. Les théories photochimiques classiques et quantiques de la vision et l'inhibition nerveuse en vision lumineuse. *Rev. d'Optique*, 1949, 28, 453-478.
- Baumgardt, E. La nature statistique et aléatoire des flux lumineux extrêmement faibles et leur incidence sur la vision de nuit. In *Comm. Intl. de l'Eclairage*. Proc. of XIV Session. Brussels, 1959.
- Beckes, G. Neural inhibitory units of the eye and skin. Quantitative description of contrast phenomena. *J. opt. Soc. Amer.*, 1960, 50, 1060-1070.
- Bliss, J. C., & Macurdy, W. B. Linear models for contrast phenomena. *J. opt. Soc. Amer.*, 1961, 51, 1373-1379.
- Bouman, M. A., & ten Doesschate, J. Nervous and photochemical components in visual adaptation. *Ophthalmologica*, 1953, 126, 222-230.
- Brown, W. R. J., & MacAdam, D. L. Visual sensitivities to combined chromaticity and luminance differences. *J. opt. Soc. Amer.*, 1949, 39, 808-834.
- Burnham, R. W. Comparison of color systems with respect to uniform visual spacing. *J. opt. Soc. Amer.*, 1949, 39, 387-392.
- Bush, R. R., Galanter, E., & Luce, R. D. Characterization and classification of choice experiments. In R. D. Luce, R. R. Bush, & E. Galanter (Eds.), *Handbook of mathematical psychology*, Vol. I. New York: Wiley, 1963. Pp. 78-102.

- Hurvich, L. M., & Jameson, D. Opponent chromatic induction and wavelength discrimination. In R. Jung & H. Kornhuber (Eds.), *The visual system neurophysiology and psychophysics*. Berlin: Springer, 1961. Pp. 144-152.
- Indow, T., & Kanazawa, K. Multidimensional mapping of Munsell colors varying in hue, chroma and value. *J exp Psychol*, 1960, 59, 330-336.
- Jahn, T. L. Visual critical flicker frequency as a function of intensity. *J opt Soc Amer*, 1946, 36, 76-82.
- Jameson, D., & Hurvich, L. M. Some quantitative aspects of an opponent-colors theory I. Chromatic responses and spectral saturation. *J opt Soc Amer*, 1955, 45, 546-552.
- Jameson, D., & Hurvich, L. M. Perceived color and its dependence on focal, surrounding and preceding stimulus variables. *J opt Soc Amer*, 1959, 49, 890-898. Also in R. D. Luce, R. R. Bush, & E. Galanter (Eds.), *Readings in mathematical psychology*. Vol. II. New York: Wiley, 1964. Pp. 340-348.
- Jameson, D., & Hurvich, L. M. Opponent colors theory and physiological mechanisms. In R. Jung & H. Kornhuber (Eds.), *The visual system neurophysiology and psychophysics*. Berlin: Springer, 1961. Pp. 152-161 (a).
- Jameson, D., & Hurvich, L. M. Opponent chromatic induction. Experimental evaluation and theoretical account. *J opt Soc Amer*, 1961, 51, 46-53 (b).
- Jameson, D., & Hurvich, L. M. Theory of brightness and color contrast in human vision. *Vision Res*, 1964, 4, 135-154.
- Judd, D. B. Radical changes in photometry and colorimetry foreshadowed by CIE actions in Zurich. *J opt Soc Amer*, 1955, 45, 897-898.
- Judd, D. B., & Wyszecki, G. *Color in business, science, and industry* (2nd ed.). New York: Wiley, 1963.
- Kelly, D. H. Visual responses to time dependent stimuli I. Amplitude sensitivity measurements. *J opt Soc Amer*, 1961, 51, 422-429 (a).
- Kelly, D. H. Visual responses to time-dependent stimuli II. Single-channel model of the photopic visual system. *J opt Soc Amer*, 1961, 51, 747-754 (b). Also in R. D. Luce, R. R. Bush, & E. Galanter (Eds.), *Readings in mathematical psychology*, Vol. II. New York: Wiley, 1964. Pp. 364-371.
- Kelly, D. H. Visual responses to time dependent stimuli III. Individual variations. *J opt Soc Amer*, 1962, 52, 89-95 (a).
- Kelly, D. H. Visual responses to time dependent stimuli IV. Effects of chromatic adaptation. *J opt Soc Amer*, 1962, 52, 940-947 (b).
- Kelly, D. H. Information capacity of a single retinal channel. *IRE Trans*, 1962, 1, T-8, 221-226 (c).
- Kelly, K. L., Gibson, K. S., & Nickerson, D. Tristimulus specification of the Munsell Book of Color from spectrophotometric measurements. *J opt Soc Amer*, 1943, 33, 355-376.
- König, A. Ueber den menschlichen Schpupur und seine Bedeutung für das Sehen. *Sitzber Akad Wiss Berlin*, 1894, 577-598.
- Kries, J. von. Die Gesichtsempfindungen. In W. Nagel (Ed.), *Handb. Physiol. d. Menschen*, Vol. 3. Brunswick: Vieweg, 1905. Pp. 109-282.
- LeGrand, Y. *Light, colour and vision* (Transl.). London: Chapman and Hall, 1957.
- Levinson, J. Fusion of complex flicker. *Science*, 1959, 130, 919-921.
- Lowry, E. M., & dePalma, J. J. Sine-wave response of the visual system I. The Mach phenomenon. *J opt Soc Amer*, 1961, 51, 740-746.
- Luce, R. D. *Individual choice behavior*. New York: Wiley, 1959.
- Luce, R. D. A choice theory analysis of similarity judgments. *Psychometrika*, 1961, 26, 151-163.

- Dartnall, H J A *The visual pigments* New York Wiley, 1957
- deLange Dzn, H Relationship between critical flicker-frequency and a set of low-frequency characteristics of the eye *J opt Soc Amer*, 1954, 44, 380-389
- deVries, H The quantum character of light and its bearing upon the threshold of vision, the differential sensitivity and visual acuity of the eye *Physica*, 9, 1943, 553-564
- Farnsworth, D Tritanomalous vision as a threshold function *Die Farbe*, 1955, 4, 185-196
- Graham, C H Color theory In S Koch (Ed), *Psychology a study of a science*, Vol I New York McGraw-Hill, 1959 Pp 145-287
- Granit, R The visual pathway In H Davson (Ed), *The eye*, Vol II New York Academic Press, 1962 Pp 537-763
- Granit, R, Munsterhjelm, A, & Zewi, M The relation between concentration of visual purple solutions and retinal sensitivity to light during dark adaptation *J Physiol*, 1939, 96, 31-44
- Hartline, H K, & Rathliff, F Spatial summation of inhibitory influences in the eye of limulus, and the mutual interaction of receptor units *J gen Physiol*, 1958, 41, 1049-1066
- Hartridge, H The change from trichromatic to dichromatic vision in the human retina *Nature*, 1945, 155, 657-662
- Hecht, S Vision II The nature of the photoreceptor process In C Murchison (Ed), *Handbook of general experimental psychology* Worcester, Mass Clark Univer Press, 1934 Pp 704-828
- Helmholtz, H L F von *Handbuch der physiologischen Optik* (2nd ed) Hamburg & Leipzig Voss, 1896
- Helson, H Fundamental problems in color vision I The principle governing changes in hue, saturation and lightness of nonselective samples in chromatic illumination *J exp Psychol*, 1938, 23, 439-476
- Hering, E Ueber Newton's Gesetz der Farbenmischung *Lotos, Jahrb Naturw NF* 1887, 7, 177-268
- Holway, A H, & Crozier, W J Differential sensitivity for somesthetic pressure *Psychol Rec*, 1937, 1, 170-176
- Holway, A H, & Hurvich, L M On the discrimination of minimal differences in weight I A theory of differential sensitivity *J Psychol*, 1937, 4, 309-332
- Holway, A H, & Hurvich, L M Visual differential sensitivity and retinal area *Amer J Psychol*, 1938, 51, 687-695 Also in R D Luce, R R Bush, & E Galanter (Eds), *Readings in mathematical psychology*, Vol II New York Wiley, 1964 Pp 332-339
- Hurvich, L M, & Jameson, D Some quantitative aspects of an opponent-colors theory II Brightness, saturation and hue in normal and dichromatic vision *J opt Soc Amer*, 1955, 45, 602-616
- Hurvich, L M, & Jameson, D Some quantitative aspects of an opponent-colors theory IV A psychological color specification system *J opt Soc Amer*, 1956, 46, 416-421
- Hurvich, L M, & Jameson, D An opponent process theory of color vision *Psychol Rev*, 1957, 64, 384-404
- Hurvich, L M, & Jameson, D Further development of a quantified opponent-colours theory In *Visual problems of colour*, II London, England Her Majesty's Stationery Office, 1958 Pp 691-723
- Hurvich, L M, & Jameson, D Perceived color, induction effects and opponent-response mechanism *J gen Physiol*, 1960, 43 (Suppl), 63-80

- Hurvich, L M, & Jameson, D Opponent chromatic induction and wavelength discrimination In R Jung & H Kornhuber (Eds), *The visual system neurophysiology and psychophysics* Berlin Springer, 1961 Pp 144-152
- Indow, T, & Kanazawa, K Multidimensional mapping of Munsell colors varying in hue, chroma and value *J exp Psychol*, 1960, 59, 330-336
- Jahn, T L Visual critical flicker frequency as a function of intensity *J opt Soc Amer*, 1946, 36, 76-82
- Jameson, D, & Hurvich, L M Some quantitative aspects of an opponent-colors theory I Chromatic responses and spectral saturation *J opt Soc Amer*, 1955, 45, 546-552
- Jameson, D, & Hurvich, L M Perceived color and its dependence on focal, surrounding and preceding stimulus variables *J opt Soc Amer*, 1959, 49, 890-898 Also in R D Luce, R R Bush, & E Galanter (Eds), *Readings in mathematical psychology* Vol II New York Wiley, 1964 Pp 340-348
- Jameson, D, & Hurvich, L M Opponent colors theory and physiological mechanisms In R Jung & H Kornhuber (Eds), *The visual system neurophysiology and psychophysics* Berlin Springer, 1961 Pp 152-161 (a)
- Jameson, D, & Hurvich, L M Opponent chromatic induction Experimental evaluation and theoretical account *J opt Soc Amer*, 1961, 51, 46-53 (b)
- Jameson, D, & Hurvich, L M Theory of brightness and color contrast in human vision *Vision Res*, 1964, 4, 135-154
- Judd, D B Radical changes in photometry and colorimetry foreshadowed by CIE actions in Zurich *J opt Soc Amer*, 1955, 45, 897-898
- Judd, D B, & Wyszecki, G *Color in business, science, and industry* (2nd ed) New York Wiley, 1963
- Kelly, D H Visual responses to time dependent stimuli I Amplitude sensitivity measurements *J opt Soc Amer*, 1961, 51, 422-429 (a)
- Kelly, D H Visual responses to time-dependent stimuli II Single-channel model of the photopic visual system *J opt Soc Amer*, 1961, 51, 747-754 (b) Also in R D Luce, R R Bush, & E Galanter (Eds), *Readings in mathematical psychology*, Vol II New York Wiley, 1964 Pp 364-371
- Kelly, D H Visual responses to time dependent stimuli III Individual variations *J opt Soc Amer*, 1962, 52, 89-95 (a)
- Kelly, D H Visual responses to time dependent stimuli IV Effects of chromatic adaptation *J opt Soc Amer*, 1962, 52, 940-947 (b)
- Kelly, D H Information capacity of a single retinal channel *IRE Trans*, 1962, 1, T-8, 221-226 (c)
- Kelly, K L, Gibson, K S, & Nickerson, D Tristimulus specification of the Munsell *Book of Color* from spectrophotometric measurements *J opt Soc Amer*, 1943, 33, 355-376
- König, A Ueber den menschlichen Sehpurpur und seine Bedeutung für das Sehen *Sitzber Akad Wiss Berlin*, 1894, 577-598
- Kries, J von Die Gesichtsempfindungen In W Nagel (Ed), *Handb Physiol d Menschen*, Vol 3 Brunswick Vieweg, 1905 Pp 109-282
- LeGrand, Y *Light, colour and vision* (Transl) London Chapman and Hall, 1957
- Levinson, J Fusion of complex flicker *Science*, 1959, 130, 919-921
- Lowry, E M, & dePalma, J J Sine-wave response of the visual system I The Mach phenomenon *J opt Soc Amer*, 1961, 51, 740-746
- Luce, R D *Individual choice behavior* New York Wiley, 1959
- Luce, R D A choice theory analysis of similarity judgments *Psychometrika*, 1961, 26, 151-163

- Luce, R D, & Galanter, E Discrimination In R D Luce, R R Bush, & E Galanter (Eds) *Handbook of mathematical psychology*, Vol I New York Wiley, 1963 Pp 191-243 (a)
- Luce, R D, & Galanter, E Psychophysical scaling In R D Luce, R R Bush, & E Galanter (Eds) *Handbook of mathematical psychology*, Vol I New York Wiley, 1963 Pp 245-307 (b)
- Ludvig, E *Perception of contour I Introduction II Effect of rate of change of retinal intensity gradient* Pensacola, Fla US Naval School of Aviation Med, 1953
- Lythgoe, R J The mechanism of dark adaptation A critical resume *Brit J Ophthal*, 1940, 24, 21-43
- MacAdam, D L Visual sensitivities to color differences in daylight *J opt Soc Amer*, 1942, 32, 247-274
- MacAdam, D L Colour discrimination and the influence of colour contrast on acuity *Documenta Ophthal*, 1949, 3, 214-233
- Mach, E Ueber die Wirkung der räumlichen Vertheilung des Lichtreizes auf die Netzhaut *Sitzber Akad Wiss (Wien), Math-Naturw Kl, Abt II*, 1865, 52, 303-322
- Mach, E Ueber den physiologischen Effect räumlich vertheilter Lichtreize *Sitzber Akad Wiss (Wien) Math Naturw Kl, Abt II*, 1866, 54, 131-144 (a)
- Mach, E Ueber den physiologischen Effect räumlich vertheilter Lichtreize *Sitzber Akad Wiss (Wien) Math Naturw Kl Abt II*, 1866, 54, 393-408 (b)
- Mach, E Ueber die physiologischen wirkung räumlich vertheilter Lichtreize *Sitzber Akad Wiss (Wien) Math Naturw Kl Abt II*, 1868, 57, 11-19 (a)
- Mach, E Ueber die Abhängigkeit der Netzhautstellen von einander *Vierteljahrsschrift Psychiatrie*, 1868, 2, 38-51 (b)
- Mach, E Ueber den Einfluss räumlich und zeitlich variirender Lichtreize auf die Gesichtswahrnehmung *Sitzber Akad Wiss (Wien) Math Naturw Kl, Abt IIa*, 1906, 115, 633-648
- Messick, S J An empirical evaluation of multidimensional successive intervals *Psychometrika*, 1956, 21, 367-375
- Moon, P, & Spencer, D E A modified photochemical theory of vision *J opt Soc Amer*, 1945, 35, 43-65
- Munsell, A H *A color notation* (1st ed) Boston Ellis, 1905
- Munsell, A H A pigment color system and notation *Amer J Psychol*, 1912, 23, 236-244
- Munsell, A H *A color notation* (9th ed) Baltimore Munsell Color Co, 1941
- Newhall, S M, Nickerson, D, & Judd, D B Final report of the O S A subcommittee on the spacing of the Munsell colors *J opt Soc Amer*, 1943, 33, 385-418
- Piéron, H *La vision en lumière intermittente* Paris Centre National de la Recherche Scientifique, 1961
- Pirenne, M H Absolute thresholds and quantum effects In H Davson (Ed), *The eye*, Vol II New York Academic Press, 1962 Pp 123-140
- Purdy, D M *Chroma as a function of retinal illumination* Unpublished doctoral dissertation, Harvard Univer, 1929
- Rushton, W A H, & Cohen, R D Visual purple level and the course of dark adaptation *Nature, London*, 1954, 173, 301-302
- Rushton, W A H, Campbell, F W, Hagns, W A, & Brindley, G S The bleaching and regeneration of rhodopsin in the living eye of the albino rabbit and of man *Optica Acta*, 1955, 1, 183-190
- Schouten, J F, & Ornstein, L S Measurements on direct and indirect adaptation by means of a binocular method *J opt Soc Amer*, 1939, 29, 168-182

- Schrödinger, E., *Grundlinien einer Theorie der Farbenmetrik im Tagesschen* *Ann Phys* 1920, 63, 481-520
- Shepard, R. The analysis of proximities. Multidimensional scaling with an unknown distance function. I *Psychometrika*, 1962, 27, 125-140 (a)
- Shepard, R. The analysis of proximities. Multidimensional scaling with an unknown distance function. II *Psychometrika*, 1962, 27, 219-246 (b)
- Silberstein, L., & MacAdam, D. L. The distribution of color matchings around a color center. *J opt Soc Amer*, 1945, 35, 32-39
- Sinden, R. H. A further search for the ideal color system. I. A new mechanico-graphical method. *J opt Soc Amer*, 1937, 27, 124-131
- Sinden, R. H. A further search for the ideal color system. II. A reconsideration of the Helmholtz line-element. *J opt Soc Amer*, 1938, 28, 339-347
- Stevens, S. S. On the psychophysical law. *Psychol Rev*, 1957, 64, 153-181
- Stiles, W. S. A modified Helmholtz line element in brightness colour space. *Proc phys Soc (London)*, 1946, 58, 41-65
- Stiles, W. S. Increment threshold and the mechanism of colour vision. *Documenta Ophthal*, 1949, 3, 138-163 (a)
- Stiles, W. S. Investigations of the scotopic and trichromatic mechanisms of vision by the two-colour threshold technique. *Rev d Optique*, 1949, 28, 215-237 (b)
- Stiles, W. S. Further studies of visual mechanisms by the two colour threshold method. *Coloquio sobre problemas opticos de la vision*. I Conferencias Generales. Madrid Union Int de Phys Pure et Appl, 1953. Pp 65-103
- Stiles, W. S. Interim report to the Commission Internationale de l'Éclairage Zurich, 1955 on the National Physical Laboratory's investigation of colour matching (1955). *Optica Acta*, 1955, 2, 168-175 (a)
- Stiles, W. S. The basic data of colour matching. *Phys Soc year book*, London. Phys Soc, 1955. Pp 44-65 (b). Also in R. D. Luce, R. R. Bush, & E. Galanter (Eds.), *Readings in mathematical psychology*, Vol II. New York: Wiley, 1964, Pp 382-405
- Stiles, W. S. Color vision. The approach through increment threshold sensitivity. *Proc Nat Acad Sci*, 1959, 45, 100-114
- Stiles, W. S., & Burch, J. M. N.P.L. colour-matching investigation (1955). Mean results for pilot group of ten subjects (Appendix to Interim report to CIE, Zurich, 1955). *Optica Acta*, 1955, 2, 176-181
- Thomson, L. C. The nervous system in visual adaptation. *Brit J Ophthal*, 1950, 34, 129-146
- Torgerson, W. S. Multidimensional scaling. I. Theory and method. *Psychometrika*, 1952, 17, 401-419
- Torgerson, W. S. *Theory and methods of scaling*. New York: Wiley, 1958
- Weale, R. A. Hue-discrimination in para-central parts of the human retina measured at different luminance levels. *J Physiol*, 1951, 113, 115-123
- Weale, R. A. The spectral reflectivity of the cat's tapetum measured *in situ*. *J Physiol*, 1953, 119, 30-42
- Wright, W. D. A redetermination of the trichromatic coefficients of the spectral colours. *Trans opt Soc*, 1928-1929, 30, 141-164
- Wright, W. D. *Photometry and the eye*. London: Hatton Press, Ltd, 1949
- Wright, W. D., & Pitt, F. H. G. Hue discrimination in normal colour vision. *Proc phys Soc (London)*, 1934, 46, 459-473

- Luce, R D, & Galanter, E Discrimination In R D Luce, R R Bush, & E Galanter (Eds) *Handbook of mathematical psychology*, Vol I New York Wiley, 1963 Pp 191-243 (a)
- Luce, R D, & Galanter, E Psychophysical scaling In R D Luce, R R Bush, & E Galanter (Eds) *Handbook of mathematical psychology*, Vol I New York Wiley, 1963 Pp 245-307 (b)
- Ludvig, E *Perception of contour I Introduction II Effect of rate of change of retinal intensity gradient* Pensacola, Fla US Naval School of Aviation Med, 1953
- Lythgoe, R J The mechanism of dark adaptation A critical resumé *Brit J Ophthalmol*, 1940, 24, 21-43
- MacAdam, D L Visual sensitivities to color differences in daylight *J opt Soc Amer*, 1942, 32, 247-274
- MacAdam, D L Colour discrimination and the influence of colour contrast on acuity *Documenta Ophthalmol*, 1949, 3, 214-233
- Mach, E Ueber die Wirkung der räumlichen Vertheilung des Lichtreizes auf die Netzhaut *Sitzber Akad Wiss (Wien), Math-Naturw Kl, Abt II*, 1865, 52, 303-322
- Mach, E Ueber den physiologischen Effect räumlich vertheilter Lichtreize *Sitzber Akad Wiss (Wien) Math Naturw Kl, Abt II*, 1866, 54, 131-144 (a)
- Mach, E Ueber den physiologischen Effect räumlich vertheilter Lichtreize *Sitzber Akad Wiss (Wien) Math Naturw Kl Abt II*, 1866, 54, 393-408 (b)
- Mach, E Ueber die physiologischen wirkung räumlich vertheilter Lichtreize *Sitzber Akad Wiss (Wien) Math-Naturw Kl Abt II*, 1868, 57, 11-19 (a)
- Mach, E Ueber die Abhängigkeit der Netzhautstellen von einander *Vierteljahrsschrift Psychiatrie*, 1868, 2, 38-51 (b)
- Mach, E Ueber den Einfluss räumlich und zeitlich variierender Lichtreize auf die Gesichtswahrnehmung *Sitzber Akad Wiss (Wien) Math-Naturw Kl, Abt IIa*, 1906, 115, 633-648
- Messick, S J An empirical evaluation of multidimensional successive intervals *Psychometrika*, 1956, 21, 367-375
- Moon, P, & Spencer, D E A modified photochemical theory of vision *J opt Soc Amer*, 1945, 35, 43-65
- Munsell, A H *A color notation* (1st ed) Boston Ellis, 1905
- Munsell, A H A pigment color system and notation *Amer J Psychol*, 1912, 23, 236-244
- Munsell, A H *A color notation* (9th ed) Baltimore Munsell Color Co, 1941
- Newhall, S M, Nickerson, D, & Judd, D B Final report of the O S A subcommittee on the spacing of the Munsell colors *J opt Soc Amer*, 1943, 33, 385-418
- Piéron, H *La vision en lumière intermittente* Paris Centre National de la Recherche Scientifique, 1961
- Pirenne, M H Absolute thresholds and quantum effects In H Davson (Ed), *The eye*, Vol II New York Academic Press, 1962 Pp 123-140
- Purdy, D M *Chroma as a function of retinal illumination* Unpublished doctoral dissertation, Harvard Univer, 1929
- Rushton, W A H, & Cohen, R D Visual purple level and the course of dark adaptation *Nature, London*, 1954, 173, 301-302
- Rushton, W A H, Campbell, F W, Hagins, W A, & Brindley, G S The bleaching and regeneration of rhodopsin in the living eye of the albino rabbit and of man *Optica Acta*, 1955, 1, 183-190
- Schouten, J F, & Ornstein, L S Measurements on direct and indirect adaptation by means of a binocular method *J opt Soc Amer*, 1939, 29, 168-182

*Identification Learning*¹

Robert R. Bush

University of Pennsylvania

1. *The preparation of this chapter was supported in part by grant NSF-G-14839 from the National Science Foundation to the University of Pennsylvania. I am grateful to E. W. Holman, D. H. Krantz, R. D. Luce, A. A. J. Marley, and R. M. Rose for many helpful discussions and for valuable assistance.*

*Identification Learning*¹

Robert R. Bush

University of Pennsylvania

1 The preparation of this chapter was supported in part by grant NSF G 14839 from the National Science Foundation to the University of Pennsylvania. I am grateful to E. W. Holman, D. H. Krantz, R. D. Luce, A. A. J. Marley, and R. M. Rose for many helpful discussions and for valuable assistance.

Contents

1	Dual-Process Models for Identification Learning	166
1 1	The Bush-Mosteller model,	167
1 2	The Restle model,	169
1 3	The Wyckoff model,	172
1 4	Stimulus-sampling approaches,	175
2	Single-Process Models	176
2 1	Path independence,	176
2 2	Outcome preference,	178
2 3	Generalization decrements,	179
2 4	Outcome consistency,	180
2 5	Symmetry,	181
2 6	Equal reward and nonreward effects,	183
3	Nongeneralization Single-Process Models	184
4	Single-Process Linear Models	185
4 1	Symmetric operators,	186
4 2	Experimenter controlled operators,	188
4 3	Position habits,	189
5	Single-Process Beta Models	193
5 1	Asymptotic properties,	194
5 2	Equal presentation probabilities,	199
5 3	Symmetric operators,	199
5 4	Experimenter-controlled operators,	200
6	Concluding Remarks	201
	References	202

Identification Learning

Following the strategy of the older sciences, mathematical psychology has been primarily concerned with the simplest possible abstractions of the real world. Mathematical learning theory, in particular, has concentrated on a particular kind of simple learning—two choice behavior in a single repeated stimulus environment. Except for Sec. 5 of Chapter 10, that chapter as well as Chapter 9 of this *Handbook* has been restricted to problems in this form of simple learning. Psychophysics, on the other hand, has long been concerned with two or more stimulus presentations by the experimenter, the questions being asked have no meaning when a subject believes that a single stimulus presentation is used on every trial of an experiment. In attempts to unify learning theory and psychophysics, therefore, we must deal with experiments containing two or more presentations.²

Experimental studies of animal learning include a large number of experiments with two distinct stimulus presentations. For some time the term "discrimination" has been applied to such studies. In jumping stand experiments, for example, one presentation includes a black card on the left and a white card on the right, and the other presentation has the cards reversed. Or in a Y-maze experiment, a steady light and a flickering light are used to differentiate the two presentations. Most informal theories of learning and a few published mathematical models have attempted to account for the results of such experiments.

The analogous human learning studies have used many stimulus presentations rather than just two. The most common design is paired-associates learning. With only two stimulus words, subjects learn too rapidly for the phenomenon to be interesting, and so twenty or so words are normally used in these experiments.

The psychophysical designs called "detection" and "recognition" can be given the same abstract characterization as the animal "discrimination" experiments, each such experiment uses two stimulus presentations. The so called *k*-alternative forced choice experiments in psychophysics, like the paired associates studies, use many presentations rather than only two, but otherwise they are similar.

² The term stimulus presentation pertains to the complete configuration of stimuli presented to a subject on a single trial and not to that portion which is "sampled" or perceived by the organism. See Chapter 2 for a discussion of terminology.

All these experimental designs, animal discrimination learning, paired-associates learning, detection, recognition, and forced-choice, have been placed into a single category called "complete identification" experiments (see Chapter 2). They are characterized by a one-to-one correspondence between the stimulus presentation set and the response set. This chapter is devoted to models for two alternative complete identification experiments, and so the prototypes are animal discrimination and human recognition designs. The stimulus presentation set is denoted by

$$S = \{s_1, s_0\},$$

and the response set by

$$R = \{r_1, r_0\}$$

The presentation probabilities are assumed constant, and so we let

$$P = \text{Pr}(s_1 \text{ on trial } n),$$

$$1 - P = \text{Pr}(s_0 \text{ on trial } n)$$

Two conditional response probabilities are needed

$$p_n = \text{Pr}(r_1 \text{ on trial } n \mid s_1 \text{ on trial } n),$$

$$q_n = \text{Pr}(r_1 \text{ on trial } n \mid s_0 \text{ on trial } n)$$

Note that in general, p_n and q_n do not add to unity

The only experiments to be considered are those in which r_1 is always "correct" when s_1 is presented, and r_0 is always "correct" when s_0 is presented, partial reinforcement procedures will not be discussed. The outcome set, then, has two elements

$$O = \{o_1, o_0\}$$

It is assumed that o_1 is preferred to o_0 and so the payoff matrix is as shown in the following display

$$\begin{array}{cc} & \begin{array}{c} r_1 \\ r_0 \end{array} \\ \begin{array}{c} s_1 \\ s_0 \end{array} & \begin{bmatrix} o_1 & o_0 \\ o_0 & o_1 \end{bmatrix} \end{array}$$

Two major factors beyond the species of subjects employed distinguish the animal discrimination and human recognition designs. First, the learning process is of primary interest in the animal studies, whereas only the asymptotic behavior usually is examined in the human experiments. Second, and more important, the stimulus presentations are intended to be very different for animals, but are made "similar" or "confusable" for the human subjects. Thus it is usually found that the animals "discriminate" perfectly after learning is complete. p_n tends to unity and q_n

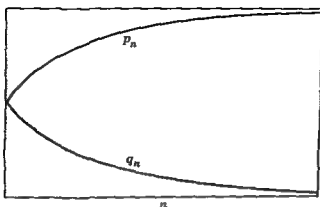


Fig 1 Typical learning curves obtained from animal experiments

tends to 0. The human subjects, on the other hand, stabilize at response probabilities other than one and zero. The animal data are usually plotted in the form of learning curves such as those sketched in Fig 1. (All too often these curves are combined into proportion of "correct" responses versus trials, this loses information which may be useful in model testing.) The human recognition data are often plotted, as shown in Fig 2, to give an "ROC curve" or "isosensitivity curve." Each point on such a curve is obtained from the asymptotic response probabilities that result from a fixed set of experimental conditions. Various points on the curve are obtained with fixed stimulus presentations and variable presentation probabilities, P and $1 - P$, or various values of the outcomes in the payoff matrix (cf Chapter 3 of this *Handbook*).

The traditional preoccupation with transient behavior in animals and asymptotic behavior in humans is probably an accident of history. In

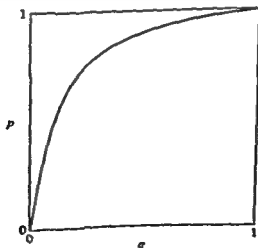


Fig 2 Typical isosensitivity curve obtained from asymptotic human data

most recognition experiments the subjects are explicitly instructed about which response is correct for each stimulus presentation, and so learning occurs very rapidly. This procedure, however, is not necessary, human learning can be examined easily if such instructions are omitted or modified. Confusable stimuli are as easy to present as distinct ones in animal experiments, and asymptotic behavior can then be examined if enough trials are run. The latter has seldom been done, perhaps because there is very little evidence that animals will, in fact, stabilize at response probabilities other than one and zero. As experimentalists well know, animals readily develop position habits in discrimination learning experiments. Often great care is taken to prevent them from developing by properly choosing the stimuli, modifying the early part of the presentation schedule, or using forced trials. Sometimes animals are discarded from an experiment because they never "learned the discrimination." It appears that discrimination, like woman, is a sometime thing. This observation presents a challenge to mathematical models—to specify the conditions under which an animal will or will not learn to respond asymptotically without error. The experimental facts seem to be these. Sometimes perfect learning occurs ($p_n \rightarrow 1$ and $q_n \rightarrow 0$), and sometimes position habits develop ($p_n \rightarrow 0$ and $q_n \rightarrow 0$, or $p_n \rightarrow 1$ and $q_n \rightarrow 1$). Whether or not intermediate asymptotes ever occur is not clear from the animal literature. The human data clearly show that intermediate asymptotes develop when the stimuli are confusable and that perfect identification occurs when the stimuli are sufficiently different from one another. Whether "position habits" (one response always occurring) can be produced with humans has not been explored systematically.

1 DUAL-PROCESS MODELS FOR IDENTIFICATION LEARNING

During the last fourteen years, several models for discrimination learning have appeared in the literature. None of these has turned out to be very satisfactory, either because it defied detailed mathematical analysis or because it led to a serious conceptual difficulty or both. Nevertheless, a brief summary of some of these models seems appropriate here if for no other reason than to warn the student about possible pitfalls in his own researches.

The various discrimination learning models have one thing in common: they postulate two distinct psychological processes. The first is a conditioning process of the type described by models for simple learning when a single stimulus is repeatedly presented, stimulus elements become

"conditioned" to a particular response or an operator is applied to a response probability to describe the effects of some experimental event. The other process is perceptual, by one mechanism or another the animal learns to pay attention to the appropriate parts of the stimulus configurations that are thought to differentiate them. This conception of discrimination learning has a good deal of intuitive appeal. Suppose that a rat is placed in the startbox of a T-maze on every trial but that on some trials a red light appears at the choice point and on other trials a green light is there. Let food be placed in the left goalbox when the red light is on and in the right-hand box when the green light is on. The light is clearly the "discriminative cue," but each stimulus presentation contains many other cues which appear on all trials. The rat must learn to pay attention to the light and to ignore everything else, so it is believed. Observed cues become conditioned or deconditioned to a performed response, depending on whether reward or nonreward follows that response. Thus the cues associated with the red light become conditioned to the left turning response only, those associated with the green light to the right-turning response only, and the common cues are conditioned in part to both responses. For perfect learning to occur, therefore, the common cues must become ineffective.

The two basic processes just described are embodied in each of the published models. Three examples will illustrate this general point and the difficulties to which it leads.

1.1 The Bush-Mosteller Model

One of the first formal models for discrimination learning was presented by Bush and Mosteller (1951). Stimulus presentation s_1 is represented by a set S_1 of stimulus elements and presentation s_0 by a set S_0 of elements. A measure function $m(X)$ is defined on every subset X of $S_1 \cup S_0$, and a similarity index is defined by

$$\eta = \frac{m(S_1 \cap S_0)}{m(S_1)}$$

Following an idea in an earlier paper by Estes (1950), a subset C of S_1 is assumed to be "conditioned" to response r_1 whose conditional probability, given s_1 , is

$$p = \frac{m(C)}{m(S_1)}$$

The conditioned subset C is partitioned into

$$I_c \subset S_1 \cap S_0 \quad \text{and} \quad T \subset S_1 - S_0$$

Thus

$$p = \frac{m(T_e) + m(I_e)}{m(S_1)}$$

Letting

$$\alpha = \frac{m(T_e)}{m(S_1 - S_0)}, \quad \beta = \frac{m(I_e)}{m(S_1 \cap S_0)},$$

the probability of response r_1 , given s_1 , becomes

$$p = \alpha(1 - \eta) + \beta\eta$$

The interpretation of this last equation is clear α is the proportion of measure on $S_1 - S_0$ that is conditioned to r_1 , β is the proportion on $S_1 \cap S_0$ that is conditioned to r_1 , and η is the relative measure of the intersection

Whenever s_1 is presented, elements of $S_1 - S_0$ become conditioned to r_1 when it occurs or deconditioned to r_0 when that response occurs. Linear operators were used, of course. Thus the ratio α tends to 1, which simply means that all elements of $S_1 - S_0$ are asymptotically conditioned to r_1 . A similar argument applies to the elements of $S_0 - S_1$. However, the elements in the intersection $S_1 \cap S_0$ are conditioned to r_1 or deconditioned to r_0 on s_1 trials, whereas they are conditioned to r_0 or deconditioned to r_1 on s_0 trials. Asymptotically, then, the ratio β tends to a value β_∞ less than 1 and greater than 0. We then see that

$$\begin{aligned} p_n &\rightarrow (1 - \eta) + \beta_\infty \eta \\ &= 1 - \eta(1 - \beta_\infty) \end{aligned}$$

The process just described which led to the last equation is a simple conditioning process. Alone it cannot lead to perfect learning ($p_n \rightarrow 1$). Therefore a second process is invoked, it is assumed that the measure of the intersection $S_1 \cap S_0$ (and thus the similarity index η) decreases toward 0 as learning progresses. This is intended to capture the notion that an animal pays less and less attention to the stimulus elements in the intersection. A "discrimination operator," D , is defined by

$$D\eta = k\eta \quad (0 < k < 1)$$

This formulation of the process raises the critical question. When should the operator D be applied to η ? Bush and Mosteller proposed the naive answer that D should be applied to η whenever an s_1 presentation follows an s_0 presentation or vice versa. It is the shift from one type of trial to the other type that decreases the relative importance of the common stimulus elements, regardless of the time interval between those adjacent trials. For this notion to be at all sensible, we must assume that the

animal (1) is "aware" of the shift and so "knows" how to distinguish s_1 from s_0 and (2) can "identify" the elements in the intersection in order to "know" what to pay less attention to in the future. Why should such an intelligent organism let the common elements influence him at all once he has seen both s_1 and s_0 just once? An elaborate sampling scheme might weaken the force of this criticism, but there are other problems.

Consider the limiting case when s_1 and s_0 are identical, that is, $\eta = 1$. The experiment reduces to one of simple learning with partial reinforcement; it is well known that learning does occur in such experiments. However, if the discrimination operator D is applied as suggested, the measures of S_1 and S_0 must both decrease to 0 and asymptotically the animal pays attention to nothing. Bush and Mosteller attempted to escape from this paradox by assuming that the operator D is an identity operator when $S_1 = S_0$, thus D depends on the initial value of η .

In addition to the conceptual difficulties just cited, there are mathematical problems also. The model leads to a complex stochastic process whose properties are virtually unknown. Even if we make the simplifying assumption that reward of one response and nonreward of the other response have effects of the same magnitude, there is trouble. In this experimenter-controlled case, we have

$$\alpha_{n+1} = \begin{cases} \alpha_n + \theta(1 - \alpha_n) & \text{if } s_1 \text{ on trial } n, \\ \alpha_n & \text{if } s_0 \text{ on trial } n, \end{cases}$$

and

$$\beta_{n+1} = \begin{cases} \beta_n + \theta(1 - \beta_n) & \text{if } s_1 \text{ on trial } n, \\ \beta_n - \theta\beta_n & \text{if } s_0 \text{ on trial } n. \end{cases}$$

These lead to

$$p_{n+1} = \begin{cases} p_n + \theta(1 - p_n) & \text{if } s_1 \text{ on trial } n, \\ p_n - \theta\eta_{n+1}\beta_n & \text{if } s_0 \text{ on trial } n. \end{cases}$$

From this transition rule, it can be seen that even if η_n were constant, the process is not path independent in the p_n 's, because p_{n+1} depends on β_n as well as on p_n . If this is not devastating enough, note that η_{n+1} depends not only on η_n but also on the stimulus presentation that occurred on trial $n - 1$. Such a path-dependent stochastic process has not been subjected to mathematical analysis, presumably because it is very difficult.

1.2 The Restle Model

A similar, but somewhat different, model was described by Restle (1955). Each stimulus element or "cue," as Restle called them, is either a

"relevant" cue or an "irrelevant" cue. In the notation of Sec 1.1, we can think of irrelevant cues as located in the intersection $S_1 \cap S_0$ and of relevant cues as in the remainder of $S_1 \cup S_0$. Restle, however, did not distinguish between the relevant cues in S_1 and those in S_0 . Similarly, he did not distinguish between a correct response, given s_1 , and a correct response, given s_0 . It is clear from Restle's later publications that his notion of a "cue" is somewhat different from Estes' notion of a "stimulus element," which was used in the Bush-Mosteller model previously described. In his book, Restle (1961) replaced the word "cue" with the term "situational variable" whose values are called "aspects." Examples are color, position, shape, etc. Thus situational variables present on s_1 trials are also present on s_0 trials, and vice versa, although the aspects are different on the two kinds of trials. Therefore it seems more consistent with the spirit of Restle's theorizing to think of a single set of cues, or aspects, as being present on each trial, this set is partitioned into relevant and irrelevant cues.

Also, conditioning in Restle's model is a concept that is somewhat different from that found in the Estes-type models. Restle says that "a conditioned cue is one which the subject knows how to use in getting reward." This is clearly a more cognitive notion than the idea of an S - R connection that is implicit in Estes' theory. Restle does not speak of a cue as being conditioned to a particular response but merely as being conditioned, that is, in a state of being useful in decision making. The specific conditioning axiom in the model concerns the probability $c(n)$ that any relevant cue has been conditioned by the beginning of trial n . Restle assumes that

$$c(n+1) = c(n) + \theta[1 - c(n)],$$

where $0 < 1$ is a rate parameter. Thus, if $c(1) = 0$, it follows that

$$c(n+1) = 1 - (1 - \theta)^n$$

Only relevant cues become conditioned in this model.

The second process introduced is the "adaptation" of irrelevant cues (called "suppression" in Restle's later writings). Once an irrelevant cue is adapted, or suppressed, it no longer has any influence on behavior. If $a(n)$ is the probability that any irrelevant cue has been suppressed by the start of trial n , Restle's perceptual axiom is

$$a(n+1) = a(n) + \theta[1 - a(n)]$$

Note that the same rate parameter θ is involved both in the conditioning axiom and in the perceptual axiom. From the latter it follows, if $a(1) = 0$, that

$$a(n+1) = 1 - (1 - \theta)^n.$$

The third major axiom relates the probability $p(n)$ of a correct response on trial n to the probabilities of conditioning and suppression. Restle assumed that unconditioned relevant cues and unsuppressed irrelevant cues lead to the correct and incorrect response with equal probability. If r is the number of relevant cues and i is the number of irrelevant ones, then the axiom is

$$p(n) = \frac{c(n) + \frac{1}{2}r[1 - c(n)] + \frac{1}{2}i[1 - a(n)]}{r + i[1 - a(n)]}$$

The denominator is the expected total number of unsuppressed cues, and the numerator is the expected number of conditioned relevant cues plus one-half the expected number of cues that have been neither conditioned nor suppressed.

The fourth axiom, introduced for reasons of simplification without a clear intuitive justification, is

$$\theta = \frac{r}{r + i}$$

The four axioms together yield, after algebraic manipulations, the final result, for $c(1) = a(1) = 0$

$$p(n) = 1 - \frac{1}{2} \left[\frac{(1 - \theta)^{n-1}}{\theta + (1 - \theta)^n} \right]$$

This result is very simple. It implies a nonbranching stochastic process that consists of a sequence of Bernoulli trials whose parameters $p(n)$ are related by a simple function with a single parameter θ . No one can complain about the complexity of this process. It is obtained, however, by making a number of strong simplifying assumptions. Why, for example, should the conditioning rate and the suppression rate be equal and why should they both equal the proportion of relevant cues? However, even if these specific assumptions are eliminated we still obtain an explicit formula for $p(n)$ in terms of n with three parameters now instead of only one. Although the equation would be a bit unsightly, the fact remains that the model predicts a unique value of $p(n)$ for each n . This is a consequence of the basic structure of the model, the probability $p(n)$ does not depend on which stimulus presentation occurred on trial $n - 1$ nor on which response was made. Such a powerful assumption should be testable directly, but no such experimental tests have yet been made. On the face of it, it seems unreasonable if not patently false. Consider, for example, a variety of presentation schedules, especially some extreme ones. First let $P = \frac{1}{2}$ and consider the sequences $\{s_1, s_0, s_1, s_0, \dots, s_1, s_0\}$ and $\{s_1, s_1, \dots, s_1, s_0, s_0, \dots, s_0\}$. With the alternating one, a monotonic increase in $p(n)$ is not too unreasonable, but with the second displayed

"relevant" cue or an "irrelevant" cue. In the notation of Sec 1.1, we can think of irrelevant cues as located in the intersection $S_1 \cap S_0$ and of relevant cues as in the remainder of $S_1 \cup S_0$. Restle, however, did not distinguish between the relevant cues in S_1 and those in S_0 . Similarly, he did not distinguish between a correct response, given s_1 , and a correct response, given s_0 . It is clear from Restle's later publications that his notion of a "cue" is somewhat different from Estes' notion of a "stimulus element," which was used in the Bush-Mosteller model previously described. In his book, Restle (1961) replaced the word "cue" with the term "situational variable" whose values are called "aspects." Examples are color, position, shape, etc. Thus situational variables present on s_1 trials are also present on s_0 trials, and vice versa, although the aspects are different on the two kinds of trials. Therefore it seems more consistent with the spirit of Restle's theorizing to think of a single set of cues, or aspects, as being present on each trial, this set is partitioned into relevant and irrelevant cues.

Also, conditioning in Restle's model is a concept that is somewhat different from that found in the Estes-type models. Restle says that "a conditioned cue is one which the subject knows how to use in getting reward." This is clearly a more cognitive notion than the idea of an S - R connection that is implicit in Estes' theory. Restle does not speak of a cue as being conditioned to a particular response but merely as being conditioned, that is, in a state of being useful in decision making. The specific conditioning axiom in the model concerns the probability $c(n)$ that any relevant cue has been conditioned by the beginning of trial n . Restle assumes that

$$c(n+1) = c(n) + \theta[1 - c(n)],$$

where $\theta < 1$ is a rate parameter. Thus, if $c(1) = 0$, it follows that

$$c(n+1) = 1 - (1 - \theta)^n$$

Only relevant cues become conditioned in this model.

The second process introduced is the "adaptation" of irrelevant cues (called "suppression" in Restle's later writings). Once an irrelevant cue is adapted, or suppressed, it no longer has any influence on behavior. If $a(n)$ is the probability that any irrelevant cue has been suppressed by the start of trial n , Restle's perceptual axiom is

$$a(n+1) = a(n) + \theta[1 - a(n)]$$

Note that the same rate parameter θ is involved both in the conditioning axiom and in the perceptual axiom. From the latter it follows, if $a(1) = 0$, that

$$a(n+1) = 1 - (1 - \theta)^n.$$

The third major axiom relates the probability $p(n)$ of a correct response on trial n to the probabilities of conditioning and suppression. Restle assumed that unconditioned relevant cues and unsuppressed irrelevant cues lead to the correct and incorrect response with equal probability. If r is the number of relevant cues and i is the number of irrelevant ones, then the axiom is

$$p(n) = \frac{c(n) + \frac{1}{2}r[1 - c(n)] + \frac{1}{2}i[1 - a(n)]}{r + i[1 - a(n)]}$$

The denominator is the expected total number of unsuppressed cues, and the numerator is the expected number of conditioned relevant cues plus one-half the expected number of cues that have been neither conditioned nor suppressed.

The fourth axiom, introduced for reasons of simplification without a clear intuitive justification, is

$$\theta = \frac{r}{r + i}$$

The four axioms together yield, after algebraic manipulations, the final result, for $c(1) = a(1) = 0$

$$p(n) = 1 - \frac{1}{2} \left[\frac{(1 - \theta)^{n-1}}{\theta + (1 - \theta)^n} \right]$$

This result is very simple. It implies a nonbranching stochastic process that consists of a sequence of Bernoulli trials whose parameters $p(n)$ are related by a simple function with a single parameter θ . No one can complain about the complexity of this process. It is obtained, however, by making a number of strong simplifying assumptions. Why, for example, should the conditioning rate and the suppression rate be equal and why should they both equal the proportion of relevant cues? However, even if these specific assumptions are eliminated we still obtain an explicit formula for $p(n)$ in terms of n with three parameters now instead of only one. Although the equation would be a bit unsightly, the fact remains that the model predicts a unique value of $p(n)$ for each n . This is a consequence of the basic structure of the model, the probability $p(n)$ does not depend on which stimulus presentation occurred on trial $n - 1$ nor on which response was made. Such a powerful assumption should be testable directly, but no such experimental tests have yet been made. On the face of it, it seems unreasonable if not patently false. Consider, for example, a variety of presentation schedules, especially some extreme ones. First let $P = \frac{1}{2}$ and consider the sequences $\{s_1, s_0, s_1, s_0, \dots, s_1, s_0\}$ and $\{s_1, s_1, \dots, s_1, s_0, s_0, \dots, s_0\}$. With the alternating one, a monotonic increase in $p(n)$ is not too unreasonable, but with the second displayed

sequence, it seems highly likely that $p(n)$ will increase monotonically during the s_1 presentations and then undergo a large decrease on the first s_0 trial. Second, let P be very large or very small. Should this lead to the same sequence $\{p(n)\}$ as schedules with $P = \frac{1}{2}$? The difficulty is abundantly clear: response probabilities cannot be independent of the presentation schedules.

1.3 The Wyckoff Model

Wyckoff (1952) was much influenced by the thinking of K. W. Spence and of C. J. Burke on the problem of discrimination learning. "Attending responses," or "orienting responses," or "observing responses" play a central role in their theories. Wyckoff built this concept into a model, or at least the formal structure of a model. He did essentially no mathematical analysis, however.

On each trial of an experiment, an animal is presumed first to make an attending or observing response. An overt motor response follows and is recorded. As Wyckoff suggested, if the discriminative stimulus is placed on the roof of the start alley, then the response of lifting the head is necessary for the animal to observe that stimulus. The recorded response is then, for example, a left or right turn in a T-maze. The abstract process is shown in Fig. 3. The rat begins in state s_1 or s_0 , depending on which stimulus is presented by the experimenter. Then with some probability u it makes attending response a_1 or a_0 , or, with probability $1 - u$, does not

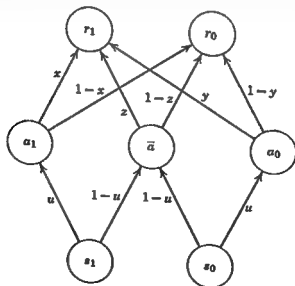


Fig. 3 State diagram of the Wyckoff model

make such a response. Depending on which of the three states the rat is then in, it makes response r_1 with probability x , y , or z , or response r_0 with the complementary probability. We thus have four basic conditional probabilities

$$u = Pr(a_1 | s_1) = Pr(a_0 | s_0),$$

$$x = Pr(r_1 | a_1),$$

$$y = Pr(r_1 | a_0),$$

$$z = Pr(r_1 | \bar{a})$$

It follows at once that

$$p = Pr(r_1 | s_1) = ux + (1 - u)z,$$

$$q = Pr(r_1 | s_0) = uy + (1 - u)z$$

Presumably, all the conditional probabilities change with learning. If reward always follows r_1 and never follows r_0 when s_1 is presented, and the

Table 1

Event	Probability	Δx	Δy	Δz	Δu
$s_1 a_1 r_1$	Pux	+	0	0	+
$s_1 a_1 r_0$	$Pu(1 - x)$	+	0	0	-
$s_1 a_0 r_1$	$P(1 - u)z$	0	0	+	+
$s_1 a_0 r_0$	$P(1 - u)(1 - z)$	0	-	0	-
$s_0 a_0 r_1$	$(1 - P)uy$	0	-	0	+
$s_0 a_0 r_0$	$(1 - P)u(1 - y)$	0	0	-	+
$s_0 a_1 r_1$	$(1 - P)(1 - u)z$	0	0	-	-
$s_0 a_1 r_0$	$(1 - P)(1 - u)(1 - z)$	0	0	-	-

reverse occurs when s_0 is presented, then x should increase whenever attending response a_1 is made and not change otherwise, and y should decrease when a_0 is made and not change otherwise. But z should either increase or decrease when a is made, depending on whether s_1 or s_0 was presented. The remaining question concerns changes in the attending probability u . Following Spence, Wyckoff argues that u should increase when an attending response is followed by reward later in the chain and decrease when it is not. Similarly, u should decrease when \bar{a} is rewarded and increase when \bar{a} is not rewarded. The signs of the increments in the four conditional probabilities for each of the eight possible events are shown in Table 1.

We first ask whether perfect learning is possible in this model. From Table 1, we see that Δx is either positive or zero and that Δy is either negative or zero. Thus, with reasonable transition rules, we expect that

$x \rightarrow 1$ and $y \rightarrow 0$ as learning progresses provided that $P \neq 0$ and $u \neq 0$. If this is true, then

$$\begin{aligned}p &\rightarrow u + (1 - u)z, \\q &\rightarrow (1 - u)z,\end{aligned}$$

and we see that $p \rightarrow 1$ and $q \rightarrow 0$ only if $u \rightarrow 1$, no matter what value z has. It may be helpful to note that

$$p - q = u(x - y),$$

and that when $x \rightarrow 1$ and $y \rightarrow 0$, then

$$p - q \rightarrow u$$

For perfect learning, $p - q \rightarrow 1$, which again requires that $u \rightarrow 1$. From Table 1 we see that Δu is either positive or negative, depending on the event, and so it is not clear whether it is possible for $u \rightarrow 1$.

To investigate further the asymptotic properties of the Wyckoff class of models, we must assume specific transition laws. Consider linear operators such that whenever some probability w_n increases,

$$w_{n+1} = w_n + \theta(1 - w_n),$$

and when it decreases,

$$w_{n+1} = w_n - \theta w_n$$

Like Restle, we assume that the same parameter θ is involved in all the transition equations. To simplify things further, consider only equal presentation probabilities, $P = 1 - P = 1/2$. The model then takes the form

$$\begin{aligned}x_{n+1} &= \begin{cases} x_n + \theta(1 - x_n) & \text{with prob } u_n/2, \\ x_n & \text{with prob } 1 - u_n/2, \end{cases} \\y_{n+1} &= \begin{cases} y_n - \theta y_n & \text{with prob } u_n/2, \\ y_n & \text{with prob } 1 - u_n/2, \end{cases} \\u_{n+1} &= \begin{cases} u_n + \theta(1 - u_n) & \text{with prob } \phi_n, \\ u_n - \theta u_n & \text{with prob } 1 - \phi_n, \end{cases}\end{aligned}$$

where

$$\phi_n = \frac{1}{2}[1 + u_n(x_n - y_n)]$$

Unless $u_n \rightarrow 0$, it seems evident that $x_n \rightarrow 1$ and $y_n \rightarrow 0$, although no rigorous proof of this is known. Now, if $x_n = 1$ and $y_n = 0$, then

$$\phi_n = \frac{1}{2}(1 + u_n),$$

and

$$\begin{aligned} E(u_{n+1} | u_n) &= (1 - \theta)u_n + \frac{\theta}{2}(1 + u_n) \\ &= u_n + \frac{\theta}{2}(1 - u_n) \end{aligned}$$

Under these conditions, the conditional expectation increases on each trial and so it is plausible that $u_n \rightarrow 1$

The plausibility arguments just presented in no sense constitute a proof, but they are consistent with Wyckoff's conjecture that the assumed process does lead to probability values of 1 for correct responding and attending. One would like to prove that this conjecture is true under more general conditions, but the mathematics is difficult.

The stochastic process implied by Wyckoff's model is very complex, even when linear operators and equal θ 's are assumed. The process is not path independent at the level of the p_n 's. This can be seen by writing down p_{n+1} for any of the eight possible events. For example, if $s_1a_1r_1$ occurs, then

$$\begin{aligned} p_{n+1} &= u_{n+1}x_{n+1} + (1 - u_{n+1})z_{n+1} \\ &= [u_n + \theta(1 - u_n)][x_n + \theta(1 - x_n)] + (1 - \theta)(1 - u_n)z_n \end{aligned}$$

Simplifications yield

$$p_{n+1} = p_n + \theta(1 - p_n) - \theta(1 - \theta)(1 - u_n)(1 - x_n),$$

and so p_{n+1} depends not only on p_n and θ , but also on u_n and x_n . Similar conclusions are reached for the other seven events. Detailed mathematical analyses of stochastic processes of this complexity are certain to be most difficult, if not impossible.

1.4 Stimulus-Sampling Approaches

The three models just described do not exhaust the published literature on dual process models for discrimination learning. Another major class of such models arose out of stimulus sampling theory. One begins with a finite population of stimulus elements that are sampled by a learning organism according to a specified set of axioms and become conditioned to his responses according to another set of axioms. Finite state Markov chains usually result. Chapter 10 of Volume II presents an extensive account of stimulus sampling theory in general, and Sec. 5 of that chapter is devoted to its application to "discrimination learning."

2 SINGLE-PROCESS MODELS

In the preceding section, three examples of dual-process models for identification learning were presented. The Restle model led to serious conceptual problems, the Wyckoff model led to major mathematical difficulties, and the Bush-Mosteller model exhibited some of each. It seems fair to conclude that the complications arise out of the attempt to include both conditioning and perceptual mechanisms in a theory having continuous operators. One direction of simplification, briefly mentioned in Sec. 1.4, is to replace continuous operators with discrete ones, thereby obtaining Markov models. Another direction is to eliminate the perceptual mechanism, regardless of its intuitive appeal. If "discrimination learning," as well as learning in recognition experiments, can be described by models that contain only conditioning mechanisms, then it would be hard to justify the complications that arise from the inclusion of a second process. This possibility has not been explored systematically.

Unlike the previous models, no intermediate hypothetical states are assumed, no use is made of stimulus elements or of observing states. The process is completely described by two sequences of random variables

$$Z_n = \begin{cases} 1 & \text{if } s_1 \text{ is presented on trial } n, \\ 0 & \text{if } s_0 \text{ is presented on trial } n, \end{cases}$$

$$X_n = \begin{cases} 1 & \text{if response } r_1 \text{ is made on trial } n, \\ 0 & \text{if response } r_0 \text{ is made on trial } n \end{cases}$$

Then, our previously defined probabilities are

$$P = \Pr(Z_n = 1),$$

$$p_n = \Pr(X_n = 1 \mid Z_n = 1),$$

$$q_n = \Pr(X_n = 1 \mid Z_n = 0)$$

The presentation probability P is taken to be constant, but the models must specify how the conditional response probabilities p_n and q_n vary from trial to trial. We shall begin by stating a general axiom and then introduce plausible conditions on the transformation rules. In the subsequent sections, specific models are examined.

2.1 Path Independence

Past experience with learning models has shown that they are rarely manageable unless an independence-of-path assumption is made. So this

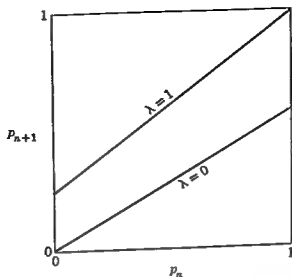


Fig 4 Probability transformations in the linear model

assumption is made here. Specifically, we assume that each conditional response probability on trial $n + 1$ depends only on the corresponding probability on trial n and on the stimulus presentation and response that occurred on trial n . Formally we have the

Basic Axiom

$$p_{n+1} = f(p_n, Z_n, X_n),$$

$$q_{n+1} = f^*(q_n, Z_n, X_n),$$

where f and f^* are continuous monotone increasing functions of p_n and q_n , respectively

Two examples, which will be explored in detail in later sections, are shown in Figs 4 and 5. The first is the well known linear model and the

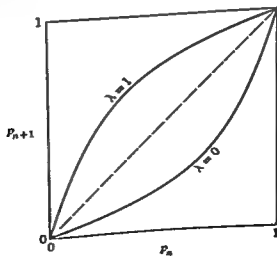


Fig 5 Probability transformations in the beta model

second is the beta model (see Chapter 9) For convenience, we use the operator notation,

$$p_{n+1} = T_{ij} p_n,$$

$$q_{n+1} = T_{ij}^* q_n,$$

where $i = 1, 0$ if $Z_n = 1, 0$, respectively, and $j = 1, 0$ if $X_n = 1, 0$, respectively Thus the first subscript denotes the presentation and the second subscript, the response on trial n In most cases of interest, the limits

$$\lambda_{ij} = \lim_{m \rightarrow \infty} T_{ij}^m p_n,$$

$$\lambda_{ij}^* = \lim_{m \rightarrow \infty} T_{ij}^{*m} q_n,$$

exist and have values of 1 or 0 (Identity operators sometimes arise)

The models to be considered are more than trivial only because we allow p_n , the probability of r_1 given s_1 , to change not only on s_1 trials but also on s_0 trials, similarly, q_n in general changes on both s_1 and s_0 trials At first glance, this may seem unreasonable, and on second thought, it may appear that two separate processes are being introduced in spite of publicity to the contrary But, as will be seen shortly, if one chooses to call them different processes, at least they are very similar

The change in the probability of r_1 given s_i during a trial on which $s_{i'}$ is presented is called "direct conditioning" when $i' = i$ and "generalized conditioning" when $i' \neq i$ These two kinds of operators are listed below

Presentation	Response	Direct Conditioning Operator	Generalized Conditioning Operator
s_1	r_1	T_{11}	T_{11}^*
s_1	r_0	T_{10}	T_{10}^*
s_0	r_1	T_{01}^*	T_{01}
s_0	r_0	T_{00}^*	T_{00}

The conditions presented next place restrictions on this set of eight operators

2.2 Outcome Preference

It is being assumed throughout this chapter that the presentation-response events $s_1 r_1$ and $s_0 r_0$ always lead to an outcome o_1 that is preferred to the other outcome o_0 which always follows the events $s_1 r_0$ and $s_0 r_1$

Thus we speak of r_i as "correct" when s_i is presented, $i = 1, 0$. In accordance with the law of effect, we assume that whenever the preferred outcome o_1 occurs, it produces an increase in the probability of the response that just preceded it. This assumption applies both to the direct conditioning effects and to the generalized conditioning effects. Similarly, it is assumed that the nonpreferred outcome o_0 decreases the probability of the response that just preceded it. This argument yields

Condition 1 For all p and q ,

$$\begin{array}{ll} T_{11}p \geq p, & T_{11}^*q \geq q, \\ T_{10}p \geq p, & T_{10}^*q \geq q, \\ T_{01}^*q \leq q, & T_{01}p \leq p, \\ T_{00}^*q \leq q, & T_{00}p \leq p \end{array}$$

The direct-conditioning portion of Condition 1 seems self-evident, and it stems directly from reinforcement theory. The generalized conditioning part, however, is a more substantive assumption. It says that if a response is reinforced in one stimulus situation, its strength is increased in some other situation (or at least not decreased). As long as the two situations are "similar" in some respects, this assumption seems plausible and is consistent with the known facts about stimulus generalization.

2.3 Generalization Decrements

It was just argued that the generalized effects of reinforcement should be positive and that the generalized effects of nonreinforcement should be negative. In addition, we expect that both generalized effects are smaller in magnitude than the corresponding direct conditioning effects. In all generalization studies, generalization decrements are observed. Thus we have

Condition 2 For all u ,

$$\begin{array}{l} |T_{11}u - u| \geq |T_{11}^*u - u|, \\ |T_{10}u - u| \geq |T_{10}^*u - u|, \\ |T_{01}^*u - u| \geq |T_{01}u - u|, \\ |T_{00}^*u - u| \geq |T_{00}u - u| \end{array}$$

This condition, along with the previous one about outcome preferences, leads to a small but important conclusion.

Lemma 1 If Conditions 1 and 2 are satisfied, then for all u and for all

$i, j = 1, 0$,

$$T_{ij}u \geq T_{ij}^*u$$

PROOF From the first two inequalities of Condition 2 and the first four inequalities of Condition 1 it follows immediately that $T_{11}u \geq T_{11}^*u$ and $T_{10}u \geq T_{10}^*u$. According to the last four inequalities of Condition 1, the quantities within the absolute value signs in the last two inequalities of Condition 2 are negative. Thus, when the absolute value signs are removed, the inequality signs are reversed, giving $T_{01}^*u \leq T_{01}u$ and $T_{00}^*u \leq T_{00}u$. QED

In Sec 5, this lemma leads to an important theorem about the asymptotic behavior of a version of the beta model

2.4 Outcome Consistency

So far nothing has been said about the relative effects of the two outcomes, reward and nonreward (o_1 and o_0 , respectively). We do not want a general condition that requires one of them to be more effective than the other, because there is convincing experimental evidence that either outcome can be more effective, depending on the type of experiment or particular experimental conditions. For example, without assuming a particular model, Sternberg has shown (see Sec 6.7 of Chapter 9) that avoidance is more effective than escape in a shuttle box experiment, but that reward is less effective than nonreward in a T-maze reversal experiment. On the other hand, within a particular experiment and with all experimental conditions held fixed, we expect reward to be *uniformly* more effective than nonreward or the reverse. In other words, if direct reward is more effective than direct nonreward with one of the stimulus presentations, then the same should be true concerning the direct effects with the other presentation and it should also be true about the generalized effects with both presentations. These assertions are embodied in Condition 3. *For all p, q, p', q' , either*

- (i) $T_{11}p \geq T_{10}p, T_{11}^*q \geq T_{10}^*q, T_{00}p \leq T_{01}p', T_{00}^*q' \leq T_{01}^*q'$,
- (ii) $T_{11}p \leq T_{10}p, T_{11}^*q \leq T_{10}^*q, T_{00}p' \geq T_{01}p', T_{00}^*q' \geq T_{01}^*q'$

With the aid of this condition and the previous ones we obtain
Lemma 2. *If Conditions 1, 2, and 3 are met, then one and only one of the following is true for all u*

- (a) $T_{11}u \geq T_{11}^*u \geq T_{10}u \geq T_{10}^*u \geq u$,
- (b) $T_{11}u \geq T_{10}u \geq T_{11}^*u \geq T_{10}^*u \geq u$,
- (c) $T_{10}u \geq T_{10}^*u \geq T_{11}u \geq T_{11}^*u \geq u$,
- (d) $T_{10}u \geq T_{11}u \geq T_{10}^*u \geq T_{11}^*u \geq u$

Also, one and only one of the following is true

$$(a') \quad T_{00}^*u \leq T_{00}u \leq T_{01}^*u \leq T_{01}u \leq u,$$

$$(b') \quad T_{00}^*u \leq T_{01}^*u \leq T_{00}u \leq T_{01}u \leq u,$$

$$(c') \quad T_{01}^*u \leq T_{01}u \leq T_{00}^*u \leq T_{00}u \leq u,$$

$$(d') \quad T_{01}^*u \leq T_{00}^*u \leq T_{01}u \leq T_{00}u \leq u$$

Furthermore, if and only if (a) or (b) is true, then (a') or (b') is true, if and only if (c) or (d) is true, then (c') or (d') is true

PROOF. Statements (a), (b), (a'), and (b') follow from Lemma 1 and part (i) of Condition 3. Statements (c), (d), (c'), and (d') follow from Lemma 1 and part (ii) of Condition 3. QED

It may appear from Lemma 2 that many possibilities are still open, and they are, but Lemma 2 reduces the 24 possible orderings of four quantities to only four orderings. As will now be argued, no further reduction should be made without reference to data or particular experimental designs.

Part (i) of Condition 3 represents the case for which reward is more effective than nonreward, and these restrictions are reflected in orderings (a), (b), (a'), and (b') of Lemma 2. The opposite case is described by (ii) of Condition 3 and orderings (c), (d), (c'), and (d'). Another distinction among the possible orderings has an equally simple interpretation: ordering (a), for example, implies that generalized reward is more effective than direct nonreward, whereas ordering (b) implies the converse. In this way, we see that orderings (a), (c), (a'), and (c') occur when the generalization is large and that orderings (b), (d), (b'), and (d') exist when the generalization is small. These interpretations lead to the following table.

	Amount of Generalization	
	Large	Small
Reward more effective	a, a'	b, b'
Nonreward more effective	c, c'	d, d'

Whether reward or nonreward is more effective and whether the generalization effects are large or small appear to be parametric questions, and so we do not want to restrict our models further without recourse to facts. One additional condition, however, may be appropriate for some experiments and so it is described next.

2.5 Symmetry

Many experimental designs possess an intrinsic symmetry. In a T-maze discrimination-learning apparatus, for example, the left-hand and the

right-hand alleys are of the same length and have similar goal boxes. The stimulus presentations are usually selected such that they are initially "neutral" to the animals. And the payoff matrix is symmetric by assumption. Although response biases may exist—the rat may be left-handed, for example—it seems more natural to describe these by initial response probabilities different from $\frac{1}{2}$ rather than by conditions on the event operators.

If the experimenter's attempts to produce a symmetric two-choice situation are successful, this should permit us to simplify a model. Reward of response r_1 with presentation s_1 should have the same effect on the strength of r_1 as reward of r_0 with s_0 has on the strength of r_0 . The same remark should apply to nonreward, and both remarks should apply to the generalized effects as well as to the direct effects. Another way of putting it is that the model should be invariant to a simultaneous relabeling of the presentations and responses. If s_1 and s_0 are interchanged, r_1 and r_0 interchanged, and all probabilities replaced by their respective complements, then no changes in predications should result. The following condition formulates this requirement.

Condition 4 For $i = 1, 0$, for $j = 1, 0$, and for all u ,

$$T_{ij}^*u = 1 - T_{ij}(1 - u)$$

One example of this condition may help to clarify its meaning. Suppose that at some point in learning we have $q = 1 - p$. Now if s_0 is presented and r_0 occurs, q decreases to T_{00}^*q . On the other hand, if s_1 is presented and r_1 occurs, p increases to $T_{11}p$. Condition 4 says that $T_{00}^*q = 1 - T_{11}p$ in this example.

Condition 4 reduces the system of eight operators to only four operators and this is a major simplification. The model builder would want strong evidence against this condition before he would be willing to give it up, four operators each with one or more parameters is quite enough to handle! This symmetry condition simplifies the statements in Lemma 2. We have

Lemma 3 If Condition 4 is met, orderings (a), (b), (c), and (d) of Lemma 2 imply and are implied by orderings (a'), (b'), (c'), and (d'), respectively, of that lemma.

PROOF If Condition 4 is imposed on each of the quantities in ordering (a'), we obtain

$$1 - T_{11}v \leq 1 - T_{11}^*v \leq 1 - T_{10}v \leq 1 - T_{10}^*v$$

where $v = 1 - u$. This leads at once to ordering (a). The other cases are shown in the same manner. Q E D

We may not always wish to impose Condition 4 on a model, but when we do the number of parameters is cut in half. Under such circumstances it is often convenient to deal with the four operators, T_{11} , T_{10} , T_{11}^* , T_{10}^* , all of which always increase their operands except when they are already 1. In principle, it is possible to analyze data to see if Condition 4 is approximately satisfied, but undoubtedly this would encounter serious technical difficulties.

2.6 Equal Reward and Nonreward Effects

Detailed mathematical analyses often demand simplifying assumptions. Models with experimenter controlled events have proved to be far simpler than other learning models, and so their investigation is worthwhile even though they seem to be inappropriate for many experiments. (See Bush & Mosteller, 1955, Chapter 3, or Sternberg, Chapter 9 of this *Handbook*, for a definition and analysis of experimenter controlled event models for simple learning.) The class of single process models implied by our basic axiom has the transition laws

$$p_{n+1} = \begin{cases} T_{11}p_n & \text{if } Z_n = 1, \quad X_n = 1, \\ T_{10}p_n & \text{if } Z_n = 1, \quad X_n = 0, \\ T_{01}p_n & \text{if } Z_n = 0, \quad X_n = 1, \\ T_{00}p_n & \text{if } Z_n = 0, \quad X_n = 0, \end{cases}$$

$$q_{n+1} = \begin{cases} T_{11}^*q_n & \text{if } Z_n = 1, \quad X_n = 1, \\ T_{10}^*q_n & \text{if } Z_n = 1, \quad X_n = 0, \\ T_{01}^*q_n & \text{if } Z_n = 0, \quad X_n = 1, \\ T_{00}^*q_n & \text{if } Z_n = 0, \quad X_n = 0 \end{cases}$$

To make the stochastic process independent of the response random variable X_n , we must impose

Condition 5 For all u ,

$$\begin{aligned} T_{11}u &= T_{10}u = T_1u \\ T_{01}u &= T_{00}u = T_0u, \\ T_{11}^*u &= T_{10}^*u = T_1^*u, \\ T_{01}^*u &= T_{00}^*u = T_0^*u \end{aligned}$$

This condition not only reduces the system of eight operators to four

operators (as did Condition 4), but it simplifies the transition laws to

$$p_{n+1} = \begin{cases} T_1 p_n & \text{if } Z_n = 1, \\ T_0 p_n & \text{if } Z_n = 0, \end{cases}$$

$$q_{n+1} = \begin{cases} T_1^* q_n & \text{if } Z_n = 1, \\ T_0^* q_n & \text{if } Z_n = 0 \end{cases}$$

Because Z_n is a Bernoulli random variable with fixed parameter P , the four operators are applied with constant probabilities. As a result, one can anticipate clean mathematical results when Condition 5 is imposed. They may not always be realistic, however.

3 NONGENERALIZATION SINGLE-PROCESS MODELS

A possible class of identification learning models arises from the preceding framework by assuming that no generalization occurs from s_1 to s_0 or from s_0 to s_1 . This is equivalent to assuming that the operators T_{11}^* , T_{10}^* , T_{01} , and T_{00} are identity operators. Condition 2 is thereby satisfied. By imposing Conditions 1 and 3, Lemma 2 holds and it reduces to

$$T_{11}u \geq T_{10}u \geq u \quad \text{and} \quad T_{00}^*u \leq T_{01}^*u \leq u,$$

or

$$T_{10}u \geq T_{11}u \geq u \quad \text{and} \quad T_{01}^*u \leq T_{00}^*u \leq u$$

The first possibility corresponds to reward being more effective than non-reward, whereas the second possibility corresponds to the converse.

We see at once that p_n either increases or remains constant on every trial and that q_n either decreases or remains constant. This leads at once to

Theorem 1 *If for all $u < 1$, $T_{11}u > u$, $T_{10}u > u$, and for all $u > 0$, $T_{00}^*u < u$, $T_{01}^*u < u$, then as the number of s_1 presentations and the number of s_0 presentations become large, $p_n \rightarrow 1$ and $q_n \rightarrow 0$.*

This result simply states that perfect learning occurs and so it is consistent with the available data from animal experiments. It is inconsistent, however, with the data from human recognition experiments.

A much more powerful (and probably wrong) prediction also follows from the assumption of no generalization. We see that p_n changes only on s_1 trials and that q_n changes only on s_0 trials. If we partition the trials according to which stimulus is presented, if we let p_k be the probability of

r_1 on the k th presentation of s_1 , and if we let q_i be the probability of r_1 on the i th presentation of s_0 , then we have

Theorem 2 *The two sequences $\{p_k\}$ and $\{q_i\}$ are independent*

This result says that the order of the s_1 and s_0 presentations is irrelevant. It is as if we were running two completely separate experiments without any interaction. This prediction, as unlikely as it seems, has never been tested, and so no data are available to condemn it. An obvious experimental design is to run one group of rats on a "random" 50-50 schedule and another group on a schedule in which all the s_1 presentations occur prior to all the s_0 presentations. The theorem predicts that the $\{p_k\}$ and $\{q_i\}$ sequences will be statistically identical for the two groups. If one finds this to be true, he will make a major discovery!

4 SINGLE-PROCESS LINEAR MODELS

Up to this point, the functional form of the learning operators T_{ij} has not been specified, only general conditions about them have been considered. As a result, few testable predictions about experimental data have been possible. Therefore in this section and the following one, two special types of learning operators are introduced.

The most extensively studied learning operators are linear in the response probabilities. The operators T_{ij} introduced in Sec. 2.1 are linear provided that

$$T_{ij}u = (1 - \theta_{ij})u + \theta_{ij}\lambda_{ij},$$

where $0 \leq \lambda_{ij} \leq 1$ and $0 \leq \theta_{ij} \leq 1$. Conditions 1, 2, and 3 of Sec. 2 are now imposed on these operators. Condition 1 requires that

$$\begin{aligned}\lambda_{11} &= \lambda_{10} = \lambda_{11}^* = \lambda_{10}^* = 1, \\ \lambda_{01} &= \lambda_{00} = \lambda_{01}^* = \lambda_{00}^* = 0,\end{aligned}$$

and so the eight operators are of the form

$$\begin{aligned}T_{11}p &= p + \theta_{11}(1 - p), \\ T_{10}p &= p + \theta_{10}(1 - p), \\ T_{01}p &= p - \theta_{01}p, \\ T_{00}p &= p - \theta_{00}p, \\ T_{11}^*q &= q + \theta_{11}^*(1 - q), \\ T_{10}^*q &= q + \theta_{10}^*(1 - q), \\ T_{01}^*q &= q - \theta_{01}^*q, \\ T_{00}^*q &= q - \theta_{00}^*q\end{aligned}$$

Condition 2, which implies Lemma 1, then requires that

$$\begin{aligned}\theta_{11} &\geq \theta_{11}^*, & \theta_{10} &\geq \theta_{10}^*, \\ \theta_{01} &\leq \theta_{01}^*, & \theta_{00} &\leq \theta_{00}^*\end{aligned}$$

Condition 3 requires either

or

$$\begin{aligned}\text{(i)} \quad & \theta_{11} \geq \theta_{10}, & \theta_{11}^* &\geq \theta_{10}^*, & \theta_{00} &\geq \theta_{01}, & \theta_{00}^* &\geq \theta_{01}^*, \\ \text{(ii)} \quad & \theta_{11} \leq \theta_{10}, & \theta_{11}^* &\leq \theta_{10}^*, & \theta_{00} &\leq \theta_{01}, & \theta_{00}^* &\leq \theta_{01}^*\end{aligned}$$

Restrictions (i) correspond to reward being more effective than nonreward, whereas restrictions (ii) correspond to the converse

Even with the restrictions imposed by Conditions 1, 2, and 3, the model contains eight parameters, a few too many for detailed data analysis. Therefore additional restrictions are desirable, two sets of which are considered

4.1 Symmetric Operators

For animal T-maze and Y-maze experiments, it seems natural to impose Condition 4 of Sec. 2.5, the symmetry condition. For our linear operators, this condition is equivalent to

$$\begin{aligned}\theta_{11}^* &= \theta_{00}, & \theta_{10}^* &= \theta_{01}, \\ \theta_{01}^* &= \theta_{10}, & \theta_{00}^* &= \theta_{11}\end{aligned}$$

Only four parameters remain and so we now examine the asymptotic behavior of the model. For given values of p_n and q_n , the conditional expectations on the next trial are

$$\begin{aligned}E(p_{n+1} | p_n, q_n) &= p_n + Z_n[p_n\theta_{11} + (1 - p_n)\theta_{10}](1 - p_n) \\ &\quad - (1 - Z_n)[q_n\theta_{01} + (1 - q_n)\theta_{00}]p_n \\ E(q_{n+1} | p_n, q_n) &= q_n + Z_n[p_n\theta_{00} + (1 - p_n)\theta_{01}](1 - q_n) \\ &\quad - (1 - Z_n)[q_n\theta_{10} + (1 - q_n)\theta_{11}]q_n\end{aligned}$$

From these equations it appears that perfect learning is not possible in general, because if $p_n = 1$ and $q_n = 0$, then

$$\begin{aligned}E(p_{n+1} | p_n = 1, q_n = 0) &= 1 - (1 - Z_n)\theta_{00}, \\ E(q_{n+1} | p_n = 1, q_n = 0) &= Z_n\theta_{00}\end{aligned}$$

These last equations say that if $p_n = 1$ and $q_n = 0$, then $p_{n+1} = 1$ and $q_{n+1} = \theta_{00}$ when $Z_n = 1$ or $p_{n+1} = 1 - \theta_{00}$ and $q_{n+1} = 0$ when $Z_n = 0$

In either case, the state $p = 1$ and $q = 0$ is not maintained. On the other hand, if a new restriction, $\theta_{00} = 0$, is introduced, it appears that the state of perfect learning can be maintained if it is ever reached. These ideas are now made precise.

Theorem 3 *If $\theta_{11}, \theta_{10}, \theta_{01} > 0$, a stable asymptotic distribution exists, but $p_n \rightarrow 1$ and $q_n \rightarrow 0$ with probability 1 if and only if $\theta_{00} = 0$.*

PROOF Using the method of Lamperti and Suppes (1959), David H. Krantz has shown that all the moments of the distribution of (p, q) converge if $\theta_{11}, \theta_{01} > 0$. Thus a stable asymptotic distribution exists. As a result, we can write

$$\begin{aligned} E(p_\infty) &= \lim_{n \rightarrow \infty} E(p_n) = \lim_{n \rightarrow \infty} E(p_{n+1}), \\ E(q_\infty) &= \lim_{n \rightarrow \infty} E(q_n) = \lim_{n \rightarrow \infty} E(q_{n+1}), \\ E(p_\infty^2) &= \lim_{n \rightarrow \infty} E(p_n^2), \\ E(q_\infty^2) &= \lim_{n \rightarrow \infty} E(q_n^2), \\ E(p_\infty q_\infty) &= \lim_{n \rightarrow \infty} E(p_n q_n). \end{aligned}$$

By taking the expectations over the (p, q) distribution of $E(p_{n+1} | p_n, q_n)$ and $E(q_{n+1} | p_n, q_n)$ and then taking limits, we obtain

$$\begin{aligned} 0 &= Z_n \{ \theta_{10} + (\theta_{11} - 2\theta_{10})E(p_\infty) + (\theta_{10} - \theta_{11})E(p_\infty^2) \} \\ &\quad - (1 - Z_n) \{ \theta_{00}E(p_\infty) + (\theta_{01} - \theta_{00})E(p_\infty q_\infty) \}, \\ 0 &= Z_n \{ \theta_{01} + (\theta_{00} - \theta_{01})E(p_\infty) - \theta_{01}E(q_\infty) + (\theta_{01} - \theta_{00})E(p_\infty q_\infty) \} \\ &\quad - (1 - Z_n) \{ \theta_{11}E(q_\infty) + (\theta_{10} - \theta_{11})E(q_\infty^2) \} \end{aligned}$$

(It is assumed that Z_n is independent of p_n and q_n .) If the asymptotic distribution has all its density at the point $(1, 0)$, then $E(p_\infty) = E(p_\infty^2) = 1$ and $E(q_\infty) = E(q_\infty^2) = E(p_\infty q_\infty) = 0$. These values satisfy the preceding two equations if and only if $\theta_{00} = 0$. The uniqueness of the asymptotic distribution is already established by Krantz's proof. Q.E.D.

Theorem 3 states that perfect learning occurs asymptotically only if the effects of reward do not generalize. The effects of nonreward may generalize, however, if they do not, the model reduces to a linear version of the nongeneralization models discussed in Sec. 3. The psychological interpretation of a learning process that permits nonreward to generalize from one stimulus presentation to another but does not permit reward to generalize is not evident.

4.2 Experimenter-Controlled Operators

Condition 5 of Sec. 2.6 sets the reward and nonreward effects equal, and as a result the models become experimenter-controlled. For the linear models considered here, the restrictions on the parameters are

$$\begin{aligned}\theta_{11} &= \theta_{10} \equiv \theta_1, & \theta_{01} &= \theta_{00} \equiv \theta_0, \\ \theta_{11}^* &= \theta_{10}^* \equiv \theta_1^*, & \theta_{01}^* &= \theta_{00}^* \equiv \theta_0^*,\end{aligned}$$

and the transition equations become

$$\begin{aligned}p_{n+1} &= \begin{cases} p_n + \theta_1(1 - p_n) & \text{if } Z_n = 1, \\ p_n - \theta_0 p_n & \text{if } Z_n = 0, \end{cases} \\ q_{n+1} &= \begin{cases} q_n + \theta_1^*(1 - q_n) & \text{if } Z_n = 1, \\ q_n - \theta_0^* q_n & \text{if } Z_n = 0 \end{cases}\end{aligned}$$

This is a model for recognition learning described by Bush, Luce, and Rose (1964). They proved the following theorem.

Theorem 4 *The marginal means of the asymptotic distribution are given by*

$$E(p_\infty) = \frac{1}{1 + \eta_2 b},$$

$$E(q_\infty) = \frac{\eta_1}{\eta_1 + b},$$

where

$$b = \left(\frac{1 - P}{P} \right) \frac{\theta_0^*}{\theta_1},$$

$$\eta_1 = \frac{\theta_1^*}{\theta_1}, \quad \eta_2 = \frac{\theta_0}{\theta_0^*},$$

$$P = \text{Pr}(Z_n = 1)$$

As was pointed out by Bush, Luce, and Rose (1964), these asymptotic means are identical in form to the asymptotes predicted by Luce's choice model (1959) which is based on entirely different considerations. Furthermore, it is easy to show that

$$\frac{1 - E(p_\infty)}{E(p_\infty)} \cdot \frac{E(q_\infty)}{1 - E(q_\infty)} = \eta_1 \eta_2,$$

which is the equation of the "isosensitivity" curve or "ROC" curve. The interpretation given was that η_1 and η_2 are similarity indices and thus properties of the stimulus presentations, whereas b is a bias parameter determined by the payoffs, the presentation probability P , and possibly by the stimulus presentations as well.

It follows immediately from Theorem 4 that perfect asymptotic learning can occur in the present model only if $\eta_1 = \eta_2 = 0$, but this implies that there are no generalization effects and so once again we have a model of the type described in Sec 3

4.3 Position Habits³

If we find as $n \rightarrow \infty$, either that $p_n \rightarrow 0$ and $q_n \rightarrow 0$ or that $p_n \rightarrow 1$ and $q_n \rightarrow 1$, we say that a position habit has developed. Conditions under which this behavior is predicted by the present linear model are now examined. For the position habit (0, 0) to develop, we must require first that the point (0, 0) be a *fixed point* of the process, by this we mean that $E(p_{n+1} | p_n = q_n = 0) = E(q_{n+1} | p_n = q_n = 0) = 0$. We must, however, also require that the point (0, 0) be an *absorbing point*, which means that there must be a nonzero probability that for all n , $p_{n+1} \leq p_n$ and $q_{n+1} \leq q_n$. Similar remarks apply to the position habit (1, 1).

In this section, only Conditions 1, 2, and 3 of Sec 2 are imposed. We see at once that

$$\begin{aligned} E(p_{n+1} | p_n = q_n = 0) &= \theta_{10}P, \\ E(q_{n+1} | p_n = q_n = 0) &= \theta_{10}^*P, \\ E(p_{n+1} | p_n = q_n = 1) &= 1 - \theta_{01}(1 - P), \\ E(q_{n+1} | p_n = q_n = 1) &= 1 - \theta_{01}^*(1 - P) \end{aligned}$$

Thus the points (0, 0) and (1, 1) are fixed points only if $\theta_{10} = \theta_{10}^* = \theta_{01} = \theta_{01}^* = 0$, and these restrictions imply that nonreward has no effect. Whether or not this seems reasonable, the resulting model is examined.

We have

$$\begin{aligned} p_{n+1} &= \begin{cases} p_n + \theta_{11}(1 - p_n) & \text{if } Z_n = 1, \quad X_n = 1, \\ p_n & \text{if } Z_n = 1, \quad X_n = 0, \\ p_n & \text{if } Z_n = 0, \quad X_n = 1, \\ p_n - \theta_{00}p_n & \text{if } Z_n = 0, \quad X_n = 0, \end{cases} \\ q_{n+1} &= \begin{cases} q_n + \theta_{11}^*(1 - q_n) & \text{if } Z_n = 1, \quad X_n = 1, \\ q_n & \text{if } Z_n = 1, \quad X_n = 0, \\ q_n & \text{if } Z_n = 0, \quad X_n = 1, \\ q_n - \theta_{00}^*q_n & \text{if } Z_n = 0, \quad X_n = 0 \end{cases} \end{aligned}$$

³ I am indebted to Ralph Roskies for assistance in developing the proofs given in this section.

Thus

$$E(p_{n+1} | p_n = q_n = 0) = E(q_{n+1} | p_n = q_n = 0) = 0,$$

and

$$E(p_{n+1} | p_n = q_n = 1) = E(q_{n+1} | p_n = q_n = 1) = 1,$$

which means that the points (0, 0) and (1, 1) are fixed points. It is also true, however, that

$$E(p_{n+1} | p_n = 0, q_n = 1) = 0,$$

$$E(q_{n+1} | p_n = 0, q_n = 1) = 1,$$

which means that the point (0, 1) is another fixed point. The point (1, 0), which corresponds to perfect learning, is not a fixed point, however, unless $\theta_{00} = \theta_{11}^* = 0$, this case, examined in Sec. 4.1, is not considered here.

The next task is to show that the points (0, 0) and (1, 1) are absorbing points. We have

Theorem 5 For the linear model with $\theta_{10} = \theta_{10}^* = \theta_{01} = \theta_{01}^* = 0$,

$$Pr(\lim_{n \rightarrow \infty} \phi_n = 0) = 0,$$

where

$$\phi_n = Pr(X_1 = X_2 = \dots = X_n = 0 | Z_1, Z_2, \dots, Z_n),$$

provided only that $p_1, q_1 \neq 1$, $\theta_{00}, \theta_{00}^* \neq 0$, and $P \neq 1$.

Similarly,

$$Pr(\lim_{n \rightarrow \infty} \psi_n = 0) = 0,$$

where

$$\psi_n = Pr(X_1 = X_2 = \dots = X_n = 1 | Z_1, Z_2, \dots, Z_n),$$

provided only that $p_1, q_1 \neq 0$, $\theta_{11}, \theta_{11}^* \neq 0$, and $P \neq 0$.

PROOF. Define random variables on the interval (0, 1) by

$$A_n = Z_n + (1 - Z_n)(1 - \theta_{00}),$$

$$A_n^* = Z_n + (1 - Z_n)(1 - \theta_{00}^*).$$

Then

$$\begin{aligned} \phi_n &= Pr(X_1 = 0 | Z_1) Pr(X_2 = 0 | X_1 = 0; Z_1, Z_2) \\ &\quad \dots Pr(X_n = 0 | X_1 = X_2 = \dots = X_{n-1} = 0; Z_1, Z_2, \dots, Z_n) \\ &= [1 - Z_1 p_1 - (1 - Z_1) q_1] [1 - Z_2 A_1 p_1 - (1 - Z_2) A_1^* q_1] \\ &\quad \dots [1 - Z_n A_{n-1} A_{n-2} \dots A_1 p_1 - (1 - Z_n) A_{n-1}^* A_{n-2}^* \dots A_1^* q_1]. \end{aligned}$$

Now let

$$s = \max(p_1, q_1),$$

$$\bar{\theta} = \min(\theta_{00}, \theta_{00}^*),$$

$$\bar{A}_n = Z_n + (1 - Z_n)(1 - \bar{\theta}).$$

Then

$$\phi_n \geq (1-s)(1-\tilde{A}_1s)(1-\tilde{A}_2\tilde{A}_1s)\dots(1-\tilde{A}_{n-1}\tilde{A}_{n-2}\dots\tilde{A}_1s)$$

Define

$$\pi_i = \tilde{A}_i\tilde{A}_{i-1}\dots\tilde{A}_1 \quad (i \neq 0),$$

$$\pi_0 = 1$$

Thus

$$\phi_n \geq \prod_{i=0}^{n-1} (1 - \pi_i s)$$

Taking logarithms,

$$-\log \phi_n \leq \sum_{i=0}^{n-1} -\log(1 - \pi_i s) = \sum_{i=0}^{n-1} \sum_{j=1}^{\infty} \frac{(\pi_i s)^j}{j}$$

Interchanging the order of summation,

$$-\log \phi_n \leq \sum_{j=1}^{\infty} \frac{s^j}{j} \sum_{i=0}^{n-1} \pi_i^j$$

The expectation then satisfies

$$E(-\log \phi_n) \leq \sum_{j=1}^{\infty} \frac{s^j}{j} \sum_{i=0}^{n-1} E(\pi_i^j)$$

Now the \tilde{A}_n are random variables defined on $(0, 1)$ and so the values of the π_i are in $(0, 1)$. Therefore

$$E(\pi_i^j) \leq E(\pi_i) \quad \text{for } j = 1, 2, \dots$$

However, the \tilde{A}_n are independent random variables because the Z_n are assumed independent. Thus

$$E(\pi_i) = E(\tilde{A}_i\tilde{A}_{i-1}\dots\tilde{A}_1) = E(\tilde{A}_i)E(\tilde{A}_{i-1})\dots E(\tilde{A}_1)$$

$$= [P + (1-P)(1-\theta)]^i,$$

and

$$\sum_{i=0}^{n-1} E(\pi_i^j) \leq \sum_{i=0}^{n-1} E(\pi_i) = \frac{1 - [P + (1-P)(1-\theta)]^n}{1 - [P + (1-P)(1-\theta)]}$$

$$= \frac{1 - [P + (1-P)(1-\theta)]^n}{(1-P)\theta}.$$

Thus we have

$$E(-\log \phi_n) \leq \sum_{j=1}^{\infty} \frac{s^j}{j} \left\{ \frac{1 - [P + (1-P)(1-\theta)]^n}{(1-P)\theta} \right\}$$

$$= [-\log(1-s)] \frac{1 - [P + (1-P)(1-\theta)]^n}{(1-P)\theta},$$

and taking limits,

$$\lim_{n \rightarrow \infty} E(-\log \phi_n) \leq \frac{-\log(1-s)}{(1-P)\theta}.$$

The requirement that $p_1, q_1 \neq 1$ implies that $s \neq 1$, the requirement that $\theta_{00}, \theta_{00}^* \neq 0$ implies $\bar{\theta} \neq 0$. We have also required that $P \neq 1$ and so the right-hand side of the last inequality is finite.

We conclude, therefore, that as $n \rightarrow \infty$, the expectation of $-\log \phi_n$ is finite and so the asymptotic distribution of ϕ_n can have no density at zero. This proves the first part of the theorem. The second part is proved by a parallel argument. Q E D

Theorem 5 says that, with probability 1, the probability of an infinite sequence of all r_0 responses and the probability of an infinite sequence of all r_1 responses are both positive, provided the parameters and starting values are appropriately bounded away from 0 and 1. Thus the points (0, 0) and (1, 1) are absorbing points as asserted.

Finally, it will be shown that the point (0, 1) is *not* an absorbing point. We define a random variable

$$Y_n = (1 - X_n)Z_n + X_n(1 - Z_n),$$

which has the value 1 if $X_n = 0$ when $Z_n = 1$ or if $X_n = 1$ when $Z_n = 0$, it has the value 0 otherwise. Thus

$$Pr(Y_n = 1 | Z_n) = (1 - p_n)Z_n + q_n(1 - Z_n),$$

which goes to 1 if (p_n, q_n) goes to (0, 1). For the point (0, 1) to be an absorbing point, we would need to show that the infinite sequence $Y_1 = Y_2 = \dots = 1$ had positive probability of occurring. This is not true because of

Theorem 6 *For the linear model with $\theta_{10} = \theta_{10}^* = \theta_{01} = \theta_{01}^* = 0$, if $p_1 \neq 0$ and $q_1 \neq 1$, then*

$$\lim_{n \rightarrow \infty} Pr(Y_1 = Y_2 = \dots = Y_n = 1 | Z_1, Z_2, \dots, Z_n) = 0$$

PROOF

$$\begin{aligned} Pr(Y_1 = Y_2 = \dots = Y_n = 1 | Z_1, Z_2, \dots, Z_n) \\ &= Pr(Y_1 = 1 | Z_1)Pr(Y_2 = 1 | Y_1 = 1, Z_1, Z_2) \\ &\quad Pr(Y_n = 1 | Y_1 = Y_2 = \dots = Y_{n-1} = 1, Z_1, Z_2, \dots, Z_n) \\ &= [(1 - p_1)Z_1 + q_1(1 - Z_1)][(1 - p_2)Z_2 + q_2(1 - Z_2)] \\ &\quad [(1 - p_n)Z_n + q_n(1 - Z_n)] \end{aligned}$$

Now let

$$t = \max(1 - p_1, q_1) \neq 1$$

Then

$$[(1 - p_i)Z_i + q_i(1 - Z_i)] \leq t,$$

and so

$$\Pr(Y_1 = Y_2 = \dots = Y_n = 1 \mid Z_1, Z_2, \dots, Z_n) \leq t^n.$$

In the limit as $n \rightarrow \infty$, $t^n \rightarrow 0$ and so this probability tends to zero.

Q.E.D.

It is not difficult to show that no other absorbing points besides (0, 0) and (1, 1) exist, and so we conclude that a position habit will develop with probability 1 if nonreward has no effect but both direct and generalized rewards have some effect.

5. SINGLE-PROCESS BETA MODELS

The beta model introduced by Luce (1959) has been applied to simple learning by Bush, Galanter, and Luce (1959), and the mathematical properties of the beta model for simple learning have been explored rather extensively by Luce (1959), Bush (1959), Lampert and Suppes (1960), and Kanal (1962a, 1962b). In this section, it is extended to identification learning within the framework of single-process models described in Sec. 2.

The learning operators that are applied to the response probabilities have the form

$$T_{ij}u = \frac{\beta_{ij}u}{\beta_{ij}u + (1 - u)},$$

where the β_{ij} are nonnegative parameters. If $\beta_{ij} > 1$, then the corresponding operator increases its operand, whereas if $\beta_{ij} < 1$, it decreases it. Thus Condition 1 of Sec. 2.2 requires that

$$\begin{aligned} \beta_{11} &\geq 1, & \beta_{11}^* &\geq 1. \\ \beta_{10} &\geq 1, & \beta_{10}^* &\geq 1. \\ \beta_{01}^* &\leq 1, & \beta_{01} &\leq 1. \\ \beta_{00}^* &\leq 1, & \beta_{00} &\leq 1. \end{aligned}$$

Furthermore, Condition 2 of Sec. 2.3 requires that

$$\begin{aligned} \beta_{11} &\geq \beta_{11}^*, & \beta_{10} &\geq \beta_{10}^*, \\ \beta_{01}^* &\leq \beta_{01}, & \beta_{00}^* &\leq \beta_{00}. \end{aligned}$$

or more briefly,

$$\beta_{ij} \geq \beta_{ij}^* \quad \text{for } i, j = 0, 1.$$

This follows also from Lemma 1 of Sec. 2.3.

5.1 Asymptotic Properties

Without imposing further restrictions on the beta model operators, an important asymptotic result can be obtained

Theorem 7 *If Conditions 1 and 2 are imposed on the beta model operators and if $\beta_{ij} > \beta_{ij}^*$ for $i, j = 0, 1$, then $\lim_{n \rightarrow \infty} p_n = 1$ or $\lim_{n \rightarrow \infty} q_n = 0$ or both*

PROOF If we let

$$v_n = \frac{p_n}{1 - p_n}, \quad u_n = \frac{q_n}{1 - q_n},$$

the beta model operators lead to

$$v_{n+1} = \beta_{ij} v_n, \quad u_{n+1} = \beta_{ij}^* u_n$$

Now let

$$t_n = \frac{u_n}{v_n} = \frac{q_n(1 - p_n)}{(1 - q_n)p_n}$$

Then we have

$$t_{n+1} = \frac{\beta_{ij}^*}{\beta_{ij}} t_n$$

Conditions 1 and 2 require that $\beta_{ij}^* \leq \beta_{ij}$ for each i and j , and by hypothesis, $\beta_{ij}^* < \beta_{ij}$. Thus $t_{n+1} < t_n$ for all n and so $\lim_{n \rightarrow \infty} t_n = 0$. From the definition of t_n , it follows that $p_n \rightarrow 1$ or $q_n \rightarrow 0$ or both. Q E D

The theorem just proved does not tell us whether $p_n \rightarrow 1$ or $q_n \rightarrow 0$ or both, but it does assure us that the asymptotic distribution lies entirely along the boundaries $q = 0$ and $p = 1$, no density appears in the interior of the (p, q) plane. In order to explore the asymptotic properties in more detail, we shall need two lemmas about stochastic processes

Lemma 4 *For a random walk defined by*

$$Y_{n+1} = \begin{cases} Y_n + a_1 & \text{with probability } P, \\ Y_n - a_2 & \text{with probability } 1 - P, \end{cases}$$

where $a_1, a_2 > 0$, as $n \rightarrow \infty$, $\Pr(Y_n \rightarrow \infty) \rightarrow 1$ provided that $P > 1/(1 + a_1/a_2)$ and $\Pr(Y_n \rightarrow -\infty) \rightarrow 1$ provided that $P < 1/(1 + a_1/a_2)$

PROOF Define a random variable Z_n such that $Z_n = 1$ if $Y_{n+1} = Y_n + a_1$ and $Z_n = 0$ if $Y_{n+1} = Y_n - a_2$. Then $P = \Pr(Z_n = 1) = E(Z_n)$ and

$$\begin{aligned} Y_{n+1} &= Y_1 + a_1 \sum_{i=1}^n Z_i - a_2 \sum_{i=1}^n (1 - Z_i) \\ &= Y_1 + \left[\frac{a_1}{n} \sum_{i=1}^n Z_i - \frac{a_2}{n} \sum_{i=1}^n (1 - Z_i) \right] n \end{aligned}$$

The law of large numbers assures us that with probability 1 as $n \rightarrow \infty$,

$$\frac{1}{n} \sum_{i=1}^n Z_i \rightarrow P$$

Thus, as $n \rightarrow \infty$, the coefficient of n in the preceding equation is positive and $Y_n \rightarrow \infty$ with probability 1 provided $a_1 P > a_2(1 - P)$. When this inequality is reversed, $Y_n \rightarrow -\infty$. Q E D

Lemma 5 Consider two sequences of random variables $\{Y_n\}$ and $\{L_n\}$ defined on the whole real line and let Y_1 and L_1 be finite. Then,

(i) if $Y_n \rightarrow \infty$ with probability 1 and if for all n ,

$$Y_{n+1} - L_{n+1} \leq Y_n - L_n,$$

then $L_n \rightarrow \infty$ with probability 1, and

(ii) if $Y_n \rightarrow -\infty$ with probability 1 and if for all n ,

$$Y_{n+1} - L_{n+1} \geq Y_n - L_n,$$

then $L_n \rightarrow -\infty$ with probability 1

PROOF If we sum both sides of the inequality

$$Y_{n+1} - L_{n+1} \leq Y_n - L_n$$

from $n = 1$ to $m - 1$, we obtain

$$Y_m - L_m \leq Y_1 - L_1,$$

or

$$L_m \geq Y_m - Y_1 + L_1$$

Thus, if Y_1 and L_1 are finite and if $Y_m \rightarrow \infty$ with probability 1, then $L_m \rightarrow \infty$ also. The proof of the second part of the lemma is similar. Q E D

The first of the preceding two lemmas was initially proved and used in an analysis of the beta model by Lamperti and Suppes (1960). The second is used in a "comparison" method, similar to the one used by Lamperti and Suppes, to prove the main theorem.

Theorem 8 For the beta models defined in this section and with Condition 1 imposed, we have with probability 1,

(i) $P_n \rightarrow 1$ if any one of the following sets of conditions holds

- | | | | |
|-----|----------------------------|----------------------------|---------------|
| (a) | $\beta_{11} < \beta_{10},$ | $\beta_{00} < \beta_{01},$ | $P > P_{10},$ |
| (b) | $\beta_{10} < \beta_{11},$ | $\beta_{01} < \beta_{00},$ | $P > P_{01},$ |
| (c) | $\beta_{11} < \beta_{10}$ | $\beta_{01} < \beta_{00}$ | $P > P_{11},$ |
| (d) | $\beta_{10} < \beta_{11},$ | $\beta_{00} < \beta_{01},$ | $P > P_{00},$ |

(ii) $p_n \rightarrow 0$ if any one of these four sets of conditions, with all inequalities reversed, holds,

(iii) $q_n \rightarrow 1$ if any one of the following sets of conditions holds

$$(a') \quad \beta_{11}^* < \beta_{10}^*, \quad \beta_{00}^* < \beta_{01}^*, \quad P > \rho_{10}^*,$$

$$(b') \quad \beta_{10}^* < \beta_{11}^*, \quad \beta_{10}^* < \beta_{00}^*, \quad P > \rho_{01}^*,$$

$$(c') \quad \beta_{11}^* < \beta_{10}^*, \quad \beta_{01}^* < \beta_{00}^*, \quad P > \rho_{11}^*,$$

$$(d') \quad \beta_{10}^* < \beta_{11}^*, \quad \beta_{00}^* < \beta_{01}^*, \quad P > \rho_{00}^*,$$

(iv) $q_n \rightarrow 0$ if any one of the sets of conditions in (iii), with all inequalities reversed, holds, where

$$\rho_{ij} = \frac{1}{1 + \frac{\log \beta_{1i}}{-\log \beta_{0j}}},$$

$$\rho_{ij}^* = \frac{1}{1 + \frac{\log \beta_{1i}^*}{-\log \beta_{0j}^*}}$$

PROOF Let $L_n = \log v_n = \log p_n$ and $b_{jk} = |\log \beta_{jk}|$. Condition 1 requires that $\beta_{11}, \beta_{10} > 1$ and $\beta_{01}, \beta_{00} < 1$. The process can then be described by the random walk

$$L_{n+1} = \begin{cases} L_n + b_{11} & \text{if } Z_n = 1, \quad X_n = 1, \\ L_n + b_{10} & \text{if } Z_n = 1, \quad X_n = 0, \\ L_n - b_{01} & \text{if } Z_n = 0, \quad X_n = 1, \\ L_n - b_{00} & \text{if } Z_n = 0, \quad X_n = 0 \end{cases}$$

Now define the following four comparison processes:

$$A_{n+1} = \begin{cases} A_n + b_{11} & \text{if } Z_n = 1, \\ A_n - b_{00} & \text{if } Z_n = 0, \end{cases}$$

$$B_{n+1} = \begin{cases} B_n + b_{10} & \text{if } Z_n = 1, \\ B_n - b_{01} & \text{if } Z_n = 0, \end{cases}$$

$$C_{n+1} = \begin{cases} C_n + b_{11} & \text{if } Z_n = 1, \\ C_n - b_{01} & \text{if } Z_n = 0, \end{cases}$$

$$D_{n+1} = \begin{cases} D_n + b_{10} & \text{if } Z_n = 1, \\ D_n - b_{00} & \text{if } Z_n = 0 \end{cases}$$

From Lemma 4 and the definition of p_{ij} , we see that as $n \rightarrow \infty$, with probability 1,

$$A_n \rightarrow \infty \quad \text{if } P > p_{10},$$

$$B_n \rightarrow \infty \quad \text{if } P > p_{01},$$

$$C_n \rightarrow \infty \quad \text{if } P > p_{11},$$

$$D_n \rightarrow \infty \quad \text{if } P > p_{00}$$

Furthermore, we see that

$$(A_{n+1} - L_{n+1}) \leq A_n - L_n \quad \text{if } b_{11} \leq b_{10} \quad \text{and } b_{01} \leq b_{00},$$

$$(B_{n+1} - L_{n+1}) \leq B_n - L_n \quad \text{if } b_{10} \leq b_{11} \quad \text{and } b_{00} \leq b_{01},$$

$$(C_{n+1} - L_{n+1}) \leq C_n - L_n \quad \text{if } b_{11} \leq b_{10} \quad \text{and } b_{00} \leq b_{01},$$

$$(D_{n+1} - L_{n+1}) \leq D_n - L_n \quad \text{if } b_{10} \leq b_{11} \quad \text{and } b_{01} \leq b_{00}$$

Thus, part (i) of Lemma 5 and the fact that $L_n \rightarrow \infty$ implies $p_n \rightarrow 1$ lead to part (i) of the theorem. Part (ii) of the theorem is proved in the same manner by reversing all inequalities. Parts (iii) and (iv) are similarly proved for the $\{q_n\}$ process. Q E D

Theorem 8 is valid when only Condition 1 of Sec 2 is imposed, but Conditions 2 and 3 of that section place some restrictions on the conclusions. First, Condition 3 implies that (a) and (b) of part (i) and the corresponding statements of parts (ii), (iii), and (iv) are impossible. Furthermore, it follows from Lemma 1, which is based on Conditions 1 and 2, that $p_{11} < p_{11}^*$ and $p_{00} < p_{00}^*$. Finally, Condition 3 implies that either

$$(i) \quad p_{11} < p_{00} \quad \text{and} \quad p_{11}^* < p_{00}^*,$$

or

$$(ii) \quad p_{11} > p_{00} \quad \text{and} \quad p_{11}^* > p_{00}^*$$

These conclusions lead to the following lemma

Lemma 6 *Conditions 1, 2, and 3 of Sec 2 require that one of the following orderings must hold*

$$(a) \quad p_{11} \leq p_{11}^* \leq p_{00} \leq p_{00}^*,$$

$$(b) \quad p_{11} \leq p_{00} \leq p_{11}^* \leq p_{00}^*,$$

$$(c) \quad p_{00} \leq p_{00}^* \leq p_{11} \leq p_{11}^*,$$

$$(d) \quad p_{00} \leq p_{11} \leq p_{00}^* \leq p_{11}^*$$

From this lemma and Theorem 8, the following corollary is readily proved

Corollary 1 If either ordering (a) or (b) of Lemma 6 holds, then with probability 1,

$$p_n \rightarrow 1 \quad \text{if } P > \rho_{00},$$

$$p_n \rightarrow 0 \quad \text{if } P < \rho_{11},$$

$$q_n \rightarrow 1 \quad \text{if } P > \rho_{00}^*,$$

$$q_n \rightarrow 0 \quad \text{if } P < \rho_{11}^*,$$

whereas if ordering (c) or (d) of Lemma 6 holds, then with probability 1,

$$p_n \rightarrow 1 \quad \text{if } P > \rho_{11},$$

$$p_n \rightarrow 0 \quad \text{if } P < \rho_{00},$$

$$q_n \rightarrow 1 \quad \text{if } P > \rho_{11}^*,$$

$$q_n \rightarrow 0 \quad \text{if } P < \rho_{00}^*.$$

These conclusions provide us with a more complete picture of what happens asymptotically. Figure 6 depicts the results for the four orderings listed in Lemma 6. As can be seen, the position habit (0, 0) develops when P is less than both ρ_{11} and ρ_{00} , and the position habit (1, 1) develops when P is larger than both ρ_{11}^* and ρ_{00}^* .

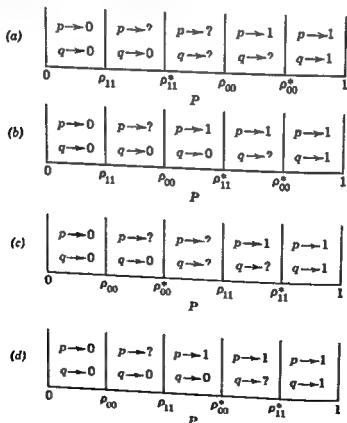


Fig. 6. Implications of Corollary 1 for each of the four orderings of Lemma 6

It should be stressed that, because of the method of proof used, the conditions that derive from Theorem 8 are only *sufficient* conditions. Thus the question marks in Fig 6 indicate our present state of ignorance, they do not mean that p and q tend to values other than 0 and 1. In the middle regions of Figs 6a and 6c, we know from Theorem 7 that $p \rightarrow 1$ or $q \rightarrow 0$ or both, but Theorem 8 is silent about which actually occurs.

5.2 Equal Presentation Probabilities

In most, if not all, published papers on animal discrimination learning, the two stimuli are presented with equal probabilities. Thus the special case of $P = \frac{1}{2}$ with the present model is of interest. In particular, we would like to find conditions, if they exist, for the development of position habits when $P = \frac{1}{2}$. Such conditions can be obtained directly from Corollary 1 and the definitions of ρ_{ij} and ρ_{ij}^* . We have

Corollary 2 *If ordering (a) or (b) of Lemma 6 applies, then when $P = \frac{1}{2}$ the position habit (0, 0) develops provided that*

$$\beta_{11}\beta_{01} < 1 \quad \text{and} \quad \beta_{11}^*\beta_{01}^* < 1,$$

and the position habit (1, 1) develops provided that

$$\beta_{10}\beta_{00} > 1 \quad \text{and} \quad \beta_{10}^*\beta_{00}^* > 1$$

If ordering (c) or (d) occurs and $P = \frac{1}{2}$, then the position habit (0, 0) develops provided that

$$\beta_{10}\beta_{00} < 1 \quad \text{and} \quad \beta_{10}^*\beta_{00}^* < 1,$$

whereas the position habit (1, 1) develops provided that

$$\beta_{11}\beta_{01} > 1 \quad \text{and} \quad \beta_{11}^*\beta_{01}^* > 1$$

Nothing we have required so far prevents the several conditions in this corollary from holding but as we will see in the next section, they imply an asymmetry in the experimental situation.

5.3 Symmetric Operators

When the experimental situation is completely symmetric, we wish to impose Condition 4 of Sec 2.5. This requires, for the beta model being considered here, that

$$\beta_{11}^*\beta_{00} = 1, \quad \beta_{10}^*\beta_{01} = 1, \quad \beta_{01}^*\beta_{10} = 1, \quad \beta_{00}^*\beta_{11} = 1,$$

and that

$$\rho_{11} + \rho_{00}^* = 1, \quad \rho_{00} + \rho_{11}^* = 1.$$

Furthermore, Lemma 6 and Condition 4 yield

Lemma 7 *Conditions 1, 2, 3, and 4 of Sec 2 require that one of the following pair of inequalities holds*

- (a) $\beta_{11}\beta_{01} \geq 1$ and $\beta_{10}\beta_{00} \leq 1$,
- (b) $\beta_{11}\beta_{01} \geq 1$ and $\beta_{10}\beta_{00} \geq 1$,
- (c) $\beta_{11}\beta_{01} \leq 1$ and $\beta_{10}\beta_{00} \geq 1$,
- (d) $\beta_{11}\beta_{01} \geq 1$ and $\beta_{10}\beta_{00} \geq 1$

Note that orderings (b) and (d) of Lemma 6 imply identical conditions on the β_{ij} , when Condition 4 is also assumed

From this lemma, we can see at once that none of the conditions of Corollary 2 can hold when Condition 4 holds. Thus, when $P = \frac{1}{2}$ and Condition 4 is met, we have found no conditions for which position habits develop. This does not prove that they cannot develop, however, because we are dealing only with sufficient conditions, not necessary ones. On the other hand, it is clear from Figs 6b and 6d that when $\rho_{11} + \rho_{00}^* = 1$ and $\rho_{00} + \rho_{11}^* = 1$, then with probability 1, $p_n \rightarrow 1$ and $q_n \rightarrow 0$. From Figs 6a and 6c, no such inference can be made, however, presumably because our sufficient conditions are not strong enough.

5.4 Experimenter-Controlled Operators

The single-process beta model is greatly simplified if we assume that reward and nonreward effects are equal as specified by Condition 5 of Sec 2.6, this implies that

$$\begin{aligned} \beta_{11} &= \beta_{10} = \beta_1, & \beta_{11}^* &= \beta_{10}^* = \beta_1^*, \\ \beta_{01} &= \beta_{00} = \beta_0, & \beta_{01}^* &= \beta_{00}^* = \beta_0^*, \end{aligned}$$

and

$$\rho_{11} = \rho_{00} = \rho, \quad \rho_{11}^* = \rho_{00}^* = \rho^*$$

Furthermore, Conditions 1 and 2 require that

$$\rho \leq \rho^*.$$

Corollary 1 simplifies to

Corollary 3 *If Conditions 1, 2, 3, and 5 are met, then with probability 1*

$$\begin{aligned} p_n &\rightarrow 1 \text{ if } P > \rho, & p_n &\rightarrow 0 \text{ if } P < \rho, \\ q_n &\rightarrow 1 \text{ if } P > \rho^*, & q_n &\rightarrow 0 \text{ if } P < \rho^* \end{aligned}$$

This result completely specifies the asymptotic distribution for all values of P except $P = \rho$ and $P = \rho^*$, values of P that are not likely to be of serious interest. Figure 7 depicts this result. If the symmetry condition is also imposed, then $\rho + \rho^* = 1$ and so if $\rho \neq \rho^*$, $p_n \rightarrow 1$ and $q_n \rightarrow 0$ when $P = \frac{1}{2}$. Only if the experimental situation is asymmetric can position habits develop when $P = \frac{1}{2}$. It is clear that when $P = \frac{1}{2}$, the position habit $(0, 0)$ can develop if $\rho > \frac{1}{2}$, which implies that $\beta_1\beta_0 < 1$, which, in turn, implies that the direct effects of s_1 presentations are weaker than the generalized effects of s_0 presentations. Similarly, when $P = \frac{1}{2}$, the position habit $(1, 1)$ can develop if $\rho^* < \frac{1}{2}$ that is, if $\beta_1^*\beta_0^* > 1$, which means that the direct effects of s_0 presentations are weaker than the generalized effects of s_1 presentations. Either situation suggests a perceptual asymmetry—a strong bias toward one of the stimulus presentations. On the other hand, the model predicts that position habits develop in symmetric experiments provided P is sufficiently small or sufficiently large.

$p \rightarrow 0$	$p \rightarrow 1$	$p \rightarrow 1$
$q \rightarrow 0$	$q \rightarrow 0$	$q \rightarrow 1$
0	ρ	ρ^*
p		
1		

Fig 7 Implications of Corollary 3 for the experimenter controlled beta model

6 CONCLUDING REMARKS

The analysis of single process models, presented in the preceding sections, has not led to a satisfactory, parsimonious theory of identification learning for the following reasons. The nongeneralization type of model defined in Sec 3 implies an independence of stimulus presentations that is very hard to believe in spite of the lack of relevant data. The linear models of Sec 4 provide a not unreasonable description of human recognition experiments that use confusable stimuli, but they cannot predict the development of either perfect learning or of position habits without rather implausible restrictions on the parameters. Perfect learning cannot occur unless the effects of reward do not generalize from one stimulus presentation to the other, position habits cannot develop unless nonreward has no direct or generalized effects. The beta models of Sec 5, on the other hand, predict quite plausible asymptotic behavior for experiments with nonconfusable stimuli, reasonable restrictions on the parameters and on the presentation probability P can lead to either perfect learning or position habits or both. However, the beta models cannot predict the sort of data obtained in human recognition experiments unless very reasonable restrictions (Conditions 1, 2, and 3) are eliminated.

It appears that if single-process models are to be taken seriously, something like the linear models must be used when the stimulus presentations are confusable, but beta-like models are more appropriate when they are not confusable. This lack of parsimony is hardly tolerable. The general conditions (Conditions 1, 2, and 3), which lead to the bulk of the conclusions reached, are not easily removed from any sensible theory, however. The alternatives are two (1) dual process models and (2) Markovian-type models such as those described in Sec. 5 of Chapter 10 or in Chapter 18 of this *Handbook*.

An experimental observation, not mentioned previously in this chapter, is the so-called facilitation effect of overlearning. It is often observed that overlearning *increases* the rate of relearning. In the numerous experiments on this phenomenon, various measures of learning rate have been used, but it seems evident that when facilitation occurs, the relearning curve of overlearned animals must cross the relearning curve of nonoverlearned animals. The overlearned ones start relearning at a lower probability of being correct, but must overtake their control group if facilitation, however defined, is said to occur. It has not been proved, but it appears evident, that no single-process model, of the type defined in Sec. 2, can predict such a facilitation effect. Perhaps this is reason enough to reject such single-process models, but, as usual, the experimental evidence is not overwhelming, facilitation sometimes occurs and sometimes does not. A single unified formal theory of identification learning is yet to come. It must specify the conditions under which imperfect asymptotic learning, perfect learning, position habits, and the facilitation effect occur in a variety of human and animal experiments.

References

- Bush, R. R., Galanter, E., & Luce, R. D. Tests of the "beta model." In R. R. Bush & W. K. Estes (Eds.), *Studies in mathematical learning theory*. Stanford: Stanford Univ. Press, 1959. Pp. 382-399.
- Bush, R. R., Luce, R. D., & Rose, R. M. Learning models for psychophysics. In R. C. Atkinson (Ed.), *Studies in mathematical psychology*. Stanford: Stanford Univ. Press, 1964. Pp. 201-217.
- Bush, R. R., & Mosteller, F. *Stochastic models for learning*. New York: Wiley, 1955.
- Bush, R. R., & Mosteller, F. A model for stimulus generalization and discrimination. *Psychol. Rev.*, 1951, 58, 413-423. Also in R. D. Luce, R. R. Bush, & E. Galanter (Eds.), *Readings in mathematical psychology*, Vol. I. New York: Wiley, 1963. Pp. 289-299.
- Estes, W. K. Toward a statistical theory of learning. *Psychol. Rev.*, 1950, 57, 94-107. Also in R. D. Luce, R. R. Bush, & E. Galanter (Eds.), *Readings in mathematical psychology*, Vol. I. New York: Wiley, 1963. Pp. 308-321.

- Kanal, L. A functional equation analysis of two learning models *Psychometrika*, 1962, 27, 89-104 (a) Also in R. D. Luce, R. R. Bush, & E. Galanter (Eds.), *Readings in mathematical psychology*, Vol. I New York: Wiley, 1963. Pp. 360-375
- Kanal, L. The asymptotic distribution for the two absorbing barrier beta model *Psychometrika*, 1962, 27, 105-109 (b) Also in R. D. Luce, R. R. Bush, & E. Galanter (Eds.), *Readings in mathematical psychology*, Vol. I New York: Wiley, 1963. Pp. 376-380
- Lamperti, J., & Suppes, P. Chains of infinite order and their application to learning theory *Pacific J. Math.*, 1959, 9, 739-754. Also in R. D. Luce, R. R. Bush, & E. Galanter (Eds.), *Readings in mathematical psychology*, Vol. I New York: Wiley, 1963. Pp. 413-428
- Lamperti, J., & Suppes, P. Some asymptotic properties of Luce's beta learning model *Psychometrika*, 1960, 25, 233-241. Also in R. D. Luce, R. R. Bush, & E. Galanter (Eds.), *Readings in mathematical psychology*, Vol. I New York: Wiley, 1963. Pp. 404-412
- Luce, R. D. *Individual choice behavior*. New York: Wiley, 1959
- Restle, F. A theory of discrimination learning *Psychol. Rev.* 1955, 62, 11-19. Also in R. D. Luce, R. R. Bush, & E. Galanter (Eds.), *Readings in mathematical psychology*, Vol. I New York: Wiley, 1963. Pp. 515-523
- Restle, F. *Psychology of judgment and choice*. New York: Wiley, 1961
- Wyckoff, L. B., Jr. The role of observing responses in discrimination learning *Psychol. Rev.*, 1952, 59, 431-442. Also in R. D. Luce, R. R. Bush, & E. Galanter (Eds.), *Readings in mathematical psychology*, Vol. I New York: Wiley, 1963. Pp. 524-535

18

Concept Utilization¹

Edwin Martin

University of Michigan

1 Preparation of this chapter was supported in part by the Advanced Research Projects Agency, Department of Defense, monitored by the Air Force Office of Scientific Research, under contract No AF 49(638)-1235 with the Human Performance Center, Department of Psychology. An earlier version was prepared as a contribution to a seminar in verbal processes while I was at State University of Iowa.

Contents

1	Component Formulations	208
2	Concept Utilization as Paired-Associate Learning	209
3	Concept Utilization as Cue Conditioning	214
3 1	Relevant redundancy,	220
3 2	Additivity of irrelevant dimensions,	222
4	Concept Utilization as Strategy Selection	223
4 1	Expected number of errors,	227
4 2	Additivity of cues,	230
4 3	A modification,	231
5	Concept Utilization as Selection and Conditioning	233
5 1	Additivity of categories,	238
5 2	Single-category solutions,	240
5 3	The all-or-none assumption,	242
5 4	The role of error trials,	243
6	Summary and Conclusions	245
	References	247

Concept Utilization

The purpose of this chapter is to present an overview of the mathematical treatments of concept utilization which are currently appearing in the psychological literature. There are four treatments and they can be sorted into four distinct categories: concept utilization as paired associate learning, as cue conditioning, as strategy selection, and as a combination of selection and conditioning.

In the typical concept-utilization experiment, a number of stimuli are presented to the subject, either singly or in certain combinations, who then makes one of a number of well defined responses. Such situations differ from ordinary paired associate learning in that (1) the number of allowable responses is less than the number of stimuli, and hence at least some of the responses must be given to more than one stimulus, and (2) stimuli associated with the same response are related in some way. For example, consider eight stimuli, each of which consists of a different combination of two levels each of form, color, and size, and let the response R_1 or R_2 be correct whenever the form circle or square, respectively, is presented, regardless of the form's size or color. The four stimuli associated with R_1 have circularity in common and the four associated with R_2 squareness. The subject's task is to learn to choose between R_1 and R_2 according to the form presented. Concept utilization experiments are thus studies of choice behavior (see Chapter 2) under conditions where the stimuli on which choices are based are categorizable in more than one way.

Consider in more detail the stimulus situation just described. Form is a dimension on the basis of which all the stimuli of the problem can be categorized, that is, each stimulus is clearly either a circle or a square, similarly for the color and size dimensions. Squareness and circularity are values on the form dimension, specific values on a dimension are often referred to as cues of the stimulus situation. Generally speaking in concept utilization experiments the dimensions are obvious and the values within a dimension are readily distinguishable: the subject is not required to engage in any stimulus differentiation learning.

If, as before, R_1 is correct whenever the stimulus is circular and R_2 is correct whenever it is square, regardless of color and size, the form dimension is said to be relevant and the other two irrelevant. When one binary dimension is relevant, regardless of how many other dimensions there may be, the subject has a two-choice problem: the subject must

choose between R_1 and R_2 . If form and color are both relevant so that four responses must be appropriately identified with the four form-color combinations, the subject has a four-choice problem. In certain concept-utilization problems, both form and color may be relevant but redundant, for example, squareness and redness may always appear together. Here the subject has a two-choice problem where response choice may be based on either form alone, color alone, or both simultaneously. In any situation, however, all that is required of the subject is that on presentation of a stimulus he choose one from a number of responses. The subject knows from the start what the possible responses are, furthermore, all responses are perfectly available, that is, the subject need not engage in any response learning. The final characteristic to note about responding is that all the responses specific to a given problem are utilized in criterial performance.

Thus, whatever concept utilization is (paired-associate learning, cue conditioning, strategy selection, or some combination), theorists and investigators do not conceive of it as including either stimulus-differentiation or response-learning processes.

1 COMPONENT FORMULATIONS

The seemingly standard division of psychological thinking into that concerning the stimulus situation as it is effective for a given organism, the processes that may be inferred of the organism, and the organism's actual overt behavior suggests that mathematical treatments of behavior might conveniently be studied in a similar fashion. In this chapter, an attempt is made to organize discussion of each of the treatments of concept utilization according to this tripartite schema, thereby isolating as much as possible the assumptions and implications peculiar to each of the three aspects of the total picture and facilitating comparison among the treatments.

The usual preliminary discussion of the type of situation to which the complete formulation is to apply plus the assumptions which specify how the subject is to perceive, or draw on, the various aspects of the presented stimulation is considered as the stimulus component of the formulation. The assumptions that describe hypothesized processes (conceptual or real) within the organism such as how connections are formed, how the effectiveness of certain aspects of the stimulus situation changes as a result of trials, and what is meant by reinforcement are, together, called the conditioning component. Finally, the response component is the set of assumptions, often a single statement, which tells

how the various conditioning states are translated into some response measure

2 CONCEPT UTILIZATION AS PAIRED-ASSOCIATE LEARNING

In the standard paired-associate (PA) task, where the pairs S_1-R_1, \dots, S_N-R_N are to be learned, three distinct processes may occur. First, in order successfully to form the pair S_i-R_i , S_i must be distinguishable from the other stimuli, thus, to the degree that the stimuli are similar among themselves, the subject is faced with a stimulus differentiation (discrimination) task. Second, in the event that the responses are either meaningless verbal units or never-before practiced actions, the subject, in order ever to score a correct response (CR), must acquire the ability to make the response. This process is the response learning phase of the familiar two-phase conceptualization of learning. The third process in the PA task is that of actually forming associative connections between the stimulus and response members of the pairs. It could be safely argued that the first and third processes, stimulus differentiation and response learning, are independent, however, there is substantial evidence which demonstrates that the second process, association formation, is dependent on both the other two. Mathematical formulations of learning treat only the association formation phase, the importance of stimulus differentiation and response learning in the over-all process is reflected in the general learning-rate parameter, if not specifically assumed to be absent.

Bower (1961, 1962), in an application of stimulus-sampling theory to PA learning, assumed that each S_i can be represented by a single element and that this element is either conditioned or not conditioned to R_i . He further assumed that the single element representing S_i is sampled on every trial without fail and, if it is not already in the conditioned state, becomes conditioned to R_i with probability θ . The probability of a CR, R_i to S_i , is unity if the element representing S_i is in the conditioned state, otherwise, when there are N responses, R_1, \dots, R_N , the probability is assumed to be $1/N$. These assumptions can be listed according to the component processes to which they apply as follows.

Stimulus axiom. Each stimulus item is represented by a single element which is sampled with probability 1 on every trial.

Conditioning axioms

- (1) *An element can be in either of two states C_0 , not conditioned, or C_1 , conditioned*

(ii) *On each reinforced² trial, the probability of a transition from C_0 to C_1 is a constant θ , the probability of a transition from C_1 to C_0 is 0*

(iii) *Initial condition all elements are in C_0 on the first trial*

Response axiom *If the element is in C_0 , then the probability of a CR is $1/N$, where N is the number of response alternatives, if the element is in C_1 , then the probability of a CR is 1.*

Several brief comments are in order. Regarding the stimulus axiom, note that the single element assumption is a determining factor for the conditioning axioms, for if two elements were assumed, each stimulus item would then be associated with one of three (instead of two) conditioning states C_0 , both elements not conditioned, C_1 , one element conditioned where the two elements are not distinguished, C_2 , both elements conditioned. Such a formulation yields some interestingly different statements

An interesting point is that even in data used to support Bower's model, the stimuli are clearly not unitary. In one study they were pairs of consonant letters, and it is known that subjects often use only the first letter of a nonsense syllable (when it is a stimulus) in forming associations. This makes one suspect that the value of the one-element assumption is primarily its role in determining the number of states in the conditioning model and not its verity as a statement about the stimulus situation.

The assumption that the single element is sampled with probability 1 on every trial is probably reasonable in PA learning, but the subject's occasional failure to observe the stimulus is certainly possible, and should be recognized as a source of error in a poor fit.

With respect to the conditioning axioms, to say that θ is constant from trial to trial is the same as saying that prior to the critical event, the shift from C_0 to C_1 , there is no accumulation of response tendency and that following the critical event there is no further modification. This means that the trials preceding the critical event are independent events and that the trial number of the critical event is independent of both the number and the outcomes of preceding trials. Note that specifying such a probability at all is a departure from the strict contiguity principle utilized in the original stimulus-sampling formulations. There, in the Guthrie tradition, any element sampled became conditioned with probability 1.

Of special interest is the matrix of transition probabilities which restates the conditioning model. The entries give the probability of moving from

² Reinforcement refers to the contiguous presentation of stimulus and response members

the state listed at the left to the state listed above

$$P = \begin{matrix} & \begin{matrix} C_0 & C_1 \end{matrix} \\ \begin{matrix} C_0 \\ C_1 \end{matrix} & \begin{bmatrix} 1 - \theta & \theta \\ 0 & 1 \end{bmatrix} \end{matrix}$$

Each row sums to unity, since if a subject is in, say, C_0 , then staying in and moving to C_1 exhaust the possible outcomes of the trial. The matrix P gives the transition probabilities for trial n to trial $n + 1$ for any n , P^2 gives the transition probabilities for trial n to trial $n + 2$, the probability of moving from one state to another in exactly two trials, and P^n gives the probabilities of having moved from one state to another in exactly n trials.

$$P^n = \begin{matrix} & \begin{matrix} C_0 & C_1 \end{matrix} \\ \begin{matrix} C_0 \\ C_1 \end{matrix} & \begin{bmatrix} (1 - \theta)^n & 1 - (1 - \theta)^n \\ 0 & 1 \end{bmatrix} \end{matrix}$$

Thus with probability $(1 - \theta)^n$, the subject will still be in C_0 after n trials. In all transition matrices P^n , $n = 1, 2, \dots$, the transition probability of C_1 to C_0 is zero, as stated in the conditioning model. In the language of Markov processes, C_1 is called an absorbing state.

Regarding the response model, to say that $Pr(CR | C_0) = 1/N$ is to say that (1) the subject must respond on every trial with one of the responses from the list, a requirement which in turn requires that (a) no response learning is necessary and (b) all the responses can easily be made in the time allotted, and (2) when an element is in C_0 each of the N response alternatives is equally likely to be chosen. With regard to the latter implication, in some studies integers are used as responses, but it is known that of the integers 2 through 9, the odd integers are more likely to be guessed than the even ones, hence a possible cause of a poor fit.

The foregoing model of PA learning is referred to as the one-element model and is discussed by Atkinson and Estes in Chapter 10 of this *Handbook* where they present derivations within the model of several statements about the PA situation. A statement not derived by them, namely, the number of trials before the critical event occurs, will be derived here as an illustrative example.

Let \bar{n}_0 be the expected number of trials the element remains in C_0 . Then to calculate \bar{n}_0 from the model, simply sum the weighted trial numbers 1, 2, 3, ..., where the weight for a given trial number is the probability that the critical event occurs on that trial. For example, the weight for trial 2 is $(1 - \theta)\theta$, the probability that the element did not shift as a result

of the first trial but did as a result of the second, thus making the second term in the sum $2(1 - \theta)\theta$

$$\begin{aligned}\bar{n}_0 &= 1\theta + 2(1 - \theta)\theta + 3(1 - \theta)^2\theta + \\ &= \sum_{n=1}^{\infty} n(1 - \theta)^{n-1}\theta \\ &= \theta \sum_{n=1}^{\infty} n(1 - \theta)^{n-1} \\ &= \theta \left[\frac{1}{1 - (1 - \theta)} \right]^2 \\ &= \theta \left(\frac{1}{\theta^2} \right) \\ &= \frac{1}{\theta}\end{aligned}$$

It remains to show that

$$\sum_{n=1}^{\infty} n(1 - \theta)^{n-1} = \left[\frac{1}{1 - (1 - \theta)} \right]^2 = \frac{1}{\theta^2}$$

To do this, first note that

$$\sum_{n=0}^{\infty} (1 - \theta)^n = \frac{1}{\theta},$$

the sum of a geometric sequence, then by taking the derivative with respect to θ of both sides, we obtain

$$-\sum_{n=0}^{\infty} n(1 - \theta)^{n-1} = -\frac{1}{\theta^2}$$

or

$$\sum_{n=1}^{\infty} n(1 - \theta)^{n-1} = \frac{1}{\theta^2}$$

since $n(1 - \theta)^{n-1} = 0$ for $n = 0$. Thus the one element model predicts that $1/\theta$ trials will be required before the element moves from C_0 to C_1 . This does not mean, of course, that $1/\theta$ errors are expected. The probability of an error while the element is in C_0 is $1 - (1/N)$, hence of the $1/\theta$ trials prior to shifting to C_1 , only $[1 - (1/N)](1/\theta)$ of them are predicted to result in an error.

In considering the application of Bower's one-element model to the stimulus situation of a concept-utilization task, an interesting choice must be made. Consider three binary dimensions, D_1, D_2, D_3 , where only one of them, say D_1 , is relevant. What are the S_i in this situation? If D_{11} and

D_{12} are the two values of D_1 , it is possible to take $S_1 = D_{11}$ and $S_2 = D_{12}$ and hence construe the concept-utilization task as that of forming associations between the two responses R_1 and R_2 and the two values of the relevant dimension D_{11} and D_{12} . According to this view, the PA task would consist of learning the pairs D_{11} - R_1 and D_{12} - R_2 . On the other hand, since there are eight actual stimuli in the problem, it is possible to designate each as a distinct stimulus and therefore construe the concept-utilization task as learning the following eight pairs S_i - R_i , for $i = 1, 2, 3, 4$, and S_i - R_2 for $i = 5, 6, 7, 8$, where S_1, \dots, S_4 are instances of D_{11} and S_5, \dots, S_8 are instances of D_{12} .

Since Bower's one-element PA model is already known to work quite well when the specifications of the response model are met (see Fig 1, Chapter 10 of this *Handbook*), and since it can be applied twice to any concept-utilization situation, once using D_{11} and D_{12} as the stimuli and once using the eight instances as stimuli, it seems reasonable that the model provides a technique for distinguishing the contributions of straight PA learning and concept utilization to over-all performance on the task. If the model fits, for example, when D_{11} and D_{12} are taken as the stimuli but does not fit when the eight instances are taken as the stimuli, then perhaps it is safe to conclude that the subjects were conditioning R_1 and R_2 to conceptual aspects of the stimulus situation and not to specific instances of those concepts. The task of the rest of this section is to argue that such a conclusion is not warranted.

Suppes and Ginsberg (1962) conducted just such an experiment and applied Bower's one-element PA model twice in the manner outlined. Using the specific instances as stimuli, they found Bower's model to fit excellently, however, using D_{11} and D_{12} as stimuli, the fit was very poor. In view of this, they suggest that perhaps learning was primarily PA learning of individual instances and that the concepts (D_{11} and D_{12}) were of only limited value.

In order to fit Bower's model using D_{11} and D_{12} as the stimuli, satisfaction of the stimulus axiom requires that all the instances of D_{11} be, for the subject, indistinguishable among themselves and identified with the single cue denoted by D_{11} . However, the stimuli of a concept utilization problem are deliberately selected so as to be unambiguously categorizable on all the dimensions involved. This means that if the specifications of the experimental situation are met, satisfaction of the stimulus axiom is doubtful. Thus an interpreter of the data must consider the additional alternative that the model is not appropriate. This does not constitute an argument to the effect that concept utilization cannot be usefully viewed as concept-wise PA learning, rather it constitutes an argument to the effect that Bower's one-element model cannot be used to decide

which process best represents the facts, the learning of responses to D_{11} and D_{12} or to instances of D_{11} and D_{12}

A final point of interest will be considered regarding the two-way application of the one-element model to a concept-utilization task. On an intuitive basis, it would be expected that if the model fits using individual instances as stimuli, it will also fit using the D_{1i} as stimuli. [The latter goodness-of-fit also depends on homogeneity (with respect to ease of learning, not distinctiveness of stimuli) of the pairs associated with D_{1i} .] This can be seen by considering how the model is applied when using D_{1i} as stimuli. Since all the instances of D_{1i} are taken as identical, the error score for D_{1i} is the combined number of errors for the instances of D_{1i} . Thus if there are four instances of D_{1i} , a single trial for D_{1i} involves four stimulus presentations, one for each instance, and hence the expected number of errors per trial for D_{1i} is the combined number of errors per trial for each instance of D_{1i} . This means the learning rate when the D_{1i} are taken as the stimuli should be one-fourth of that when the individual instances are taken as the stimuli. Suppes and Ginsberg, in fact, report such a relationship. The actual mechanics of applying Bower's model directly to the concepts (the D_{1i} 's), then, is seen to be a matter of selectively blocking the trials and then calculating a new rate parameter for the blocked data.

Suppes and Ginsberg's application of Bower's one-element model appears to be the only mathematical attempt to treat concept utilization as a form of PA learning. It has been argued that the model is inadequate for this purpose, this is because it involves an oversimplified representation of the stimulus situation. Works by Bower and Trabasso (1963), using a combination of concept selection and PA learning, will be considered later.

3 CONCEPT UTILIZATION AS CUE CONDITIONING

In Bower's one element model for PA learning, the stimulus member of a given pair was assumed to be represented by a single element. In concept utilization experiments, however, the stimuli are deliberately constructed so as to be unambiguously classifiable onto each of two or more dimensions. For example, if form, color, and size are the dimensions involved in a given experiment, every stimulus is jointly classifiable according to form, color, and size. Suppose each dimension has two values: square and circle on the form dimension, red and green on the color, and large and small on the size dimension. A particular stimulus,

say a large green circle, then has at least the three attributes imposed by the experimenter—largeness, greenness, and circularity. However, note that other stimuli share some of these attributes, for example, a large green circle and a small red circle both possess circularity. A shared attribute, such as circularity, is called a cue. Thus in an ideal situation where the only cues are those arising from the three dimensions under discussion, cues and stimuli may be distinguished by a complete listing of each, as in Table 1.

Table 1 Complete Listing of Cues and Stimuli Associated with the Three Binary Dimensions Form, Color, and Size

Cues	Stimuli
Squareness	Large red square
Circularity	Large red circle
Redness	Large green square
Greenness	Large green circle
Largeness	Small red square
Smallness	Small red circle
	Small green square
	Small green circle

Briefly, a stimulus is whatever is presented to the subject (for example, a large green circle), and a cue (for example, greenness) is a value on a dimension and hence something shared by more than one stimulus. Some writers call such cues concepts, thus referring to all stimuli possessing the attribute greenness, say, as instances of the concept green. A theory about cue conditioning, then, is a theory about the effectiveness of dimensional values.

Bourne and Restle's (1959) cue-conditioning model has as its basic elements the cues of the stimulus situation. They treat an abstract set K of cues which includes not only those cues built in by the experimenter (squareness, redness, etc.) but any other cues, either environmental or internal, to which the subject might be sensitive. As will become apparent, it is not necessary to know what the elements of K are or even to know whether K has very many or only a few elements. The utility of the model arises from the fact that whatever the contents of K may be, they can be operationally divided into two kinds. The operation is the experimenter's reinforcing procedure, and the result is a partitioning³ of K into two

³ K is said to be partitioned into the two subsets K_1 and K_2 if and only if (1) $K = K_1 \cup K_2$, and (2) $K_1 \cap K_2 = \phi$. In other words, K_1 and K_2 must be (1) exhaustive and (2) mutually exclusive.

subsets relevant and irrelevant cues. A relevant cue is one to which a particular response is either always correct or never correct, that is, it is reinforced with probability 1, an irrelevant cue is one which is reinforced only randomly (probability $\frac{1}{2}$ in the two choice situation). The conditioning model of Bourne and Restle's formulation then treats the two types of cues differentially: irrelevant cues are assumed to become adapted and unadapted cues are assumed to become conditioned.

The component axioms of Bourne's and Restle's cue conditioning formulation may be listed as follows:

Stimulus axioms

- (i) *The stimulus situation may be represented as a set K of cues*
- (ii) *The set K is partitioned by the experimenter's procedures into two subsets: relevant and irrelevant cues*

Conditioning axioms *Two processes are assumed to operate simultaneously*

- (i) *Conditioning: on each reinforced^{*} trial, a constant proportion θ of the unadapted, unconditioned cues become conditioned; cues already conditioned remain so; nonreinforcement produces no change*
- (ii) *Adaptation: on each reinforced trial, a constant proportion θ of the unadapted irrelevant cues become adapted; irrelevant cues already adapted remain so; nonreinforcement produces no change*

(Note: the conditioning and adaptation rates are assumed to be equal, that is, θ is the rate parameter for both processes.)

Response axiom *The probability of a response equals the proportion of unadapted cues which are conditioned to that response*

The statements of both the conditioning and the response axioms may be written as equations, thus permitting the use of the probability calculus for deriving new statements. Turning first to the conditioning axiom, let k be any cue in K and let $F(k, n)$ be the probability that k is conditioned on trial n . Then

$$F(k, n+1) = F(k, n) + \theta[1 - F(k, n)] \quad (1)$$

This equation follows immediately from the conditioning statement, in order to have element k conditioned on trial $n+1$, it must have been either (1) already conditioned on trial n and remained so or (2) not conditioned on trial n but conditioned as a result of reinforcement on that trial. Since these two alternatives exhaust the possibilities and since they are mutually exclusive events, their separate probabilities add to give the

* Reinforcement refers to knowledge of whether the response was correct or incorrect. Being informed that a wrong response is wrong and being informed that a correct response is correct are equally reinforcing.

probability of k being conditioned on trial $n + 1$. The former alternative has probability $F(k, n) + \theta$, the joint probability of k being conditioned on trial n and remaining so, the latter has probability $\theta[1 - F(k, n)]$ the joint probability of k not being conditioned on trial n , $1 - F(k, n)$ and of its becoming conditioned, θ . Equation 1 is a first order difference equation, and can be solved for $F(k, n)$ by use of the calculus of finite differences to give

$$F(k, n) = (1 - \theta)^{n-1}F(k, 1) + [1 - (1 - \theta)^{n-1}] \quad (2)$$

Note that although $F(k, n + 1)$ is a linear function of $F(k, n)$ in Eq. 1, $F(k, n)$ is an exponential function of n in Eq. 2. The conditioning model is thus said to be a linear model since its transition law is a linear equation but the trial by-trial solution for $F(k, n)$ is necessarily exponential.

If reinforcement is not given on trial n , then the equation representing the conditioning model's assumption that nonreinforcement produces no change is simply

$$F(k, n + 1) = F(k, n) \quad (3)$$

The question now arises as to what form Eq. 2 assumes when some trials are reinforced and some are not, that is if reinforcement is given only on π percent of the trials, how does $F(k, n)$ increase as a function of trials? This question can be answered as follows. A response is either reinforced or not reinforced, and because these two events are exhaustive and mutually exclusive, their probabilities are combined by simple addition. Therefore, since the former event occurs with probability π and the latter with probability $1 - \pi$, the probability of an element k being conditioned on trial $n + 1$ is

$$F(k, n + 1) = \pi\{F(k, n) + \theta[1 - F(k, n)]\} + (1 - \pi)F(k, n), \quad (4)$$

or, after rearranging,

$$F(k, n + 1) = F(k, n) + \pi\theta[1 - F(k, n)], \quad (5)$$

which intuitively is satisfactory. The first term on the right hand side represents the case where k was already conditioned, hence it makes no difference whether or not reinforcement occurred, and the second term represents the case where k was not conditioned but becomes so with probability $\pi\theta$, the probability that reinforcement is given, π , times the probability that conditioning occurs θ .

Equation 5 is a first order difference equation, similar to Eq. 1, and has the solution

$$F(k, n) = (1 - \pi\theta)^{n-1}F(k, 1) + [1 - (1 - \pi\theta)^{n-1}] \quad (6)$$

Equation 6 then, is a more general form of Eq. 2 that takes into account

the possibility of partial reinforcing schedules. For 100% reinforcement, $\pi = 1$ and Eq 6 is the same as Eq 2.

The restatement of the adaptation portion of the conditioning model as an equation is identical with the foregoing treatment of the conditioning portion. Let k' be any irrelevant cue in K and let $A(k', n)$ be the probability that k' is adapted on trial n . Then

$$A(k', n+1) = A(k', n) + \theta[1 - A(k', n)], \quad (7)$$

which has the solution

$$A(k', n) = (1 - \theta)^{n-1}A(k', 1) + [1 - (1 - \theta)^{n-1}] \quad (8)$$

In the event that reinforcement occurs with probability π , Eq 8 becomes

$$A(k', n) = (1 - \pi\theta)^{n-1}A(k', 1) + [1 - (1 - \pi\theta)^{n-1}] \quad (9)$$

The adapting rate parameter θ is the same θ that appears in the conditioning equations, hence adaptation is assumed to progress at the same rate as conditioning.

A careful reading of the assumptions of the conditioning model reveals that an irrelevant cue may become adapted whether conditioned or not, thus it is possible in this model for a cue (albeit an irrelevant one) to become unconditioned once conditioned. That a once conditioned, now adapted irrelevant cue is now unconditioned follows also from the assumption of the response model which says that adapted cues are not effective in determining response probability.

Turning now to the response model, let p_n be the probability of a CR on trial n , then, translation of the statement of the response model directly into a ratio gives

$$p_n = \frac{N[(n\text{-adapted}) \wedge (\text{conditioned})]}{N(n \text{ adapted})}, \quad (10)$$

where $N(x)$ and $N(n x)$ denote the number of cues which are " x " and "not x ", respectively. Equation 10 is equivalent to

$$p_n = 1 - \frac{\frac{1}{2}(1 - \theta)^{n-1}}{r + (1 - r)(1 - \theta)^{n-1}}, \quad (11)$$

where r is the proportion of cues in K which are relevant. Derivation of Eq 11 from Eq 10 proceeds as follows.

The numerator and denominator of Eq 10 can be rewritten in proportions by dividing both by the number of cues in K , whatever that number may be

$$p_n = \frac{P[(n \text{ adapted}) \wedge (\text{conditioned})]}{P(n\text{-adapted})}$$

Since there are two kinds of nonadapted cues, relevant cues which cannot become adapted, and irrelevant cues which have not become adapted, $P(n\text{-adapted})$ can be rewritten as

$$P(\text{relevant}) + P[(\text{irrelevant}) \wedge (n \text{ adapted})]$$

Let r equal $P(\text{relevant})$, then $P(\text{irrelevant}) = 1 - r$, and $P(n \text{ adapted})$ becomes $r + (1 - r)[1 - A(k', n)]$. This last expression is the denominator of p_n . The procedure for obtaining the numerator is similar

$$\begin{aligned} P[(n \text{ adapted}) \wedge (\text{conditioned})] \\ &= P[(\text{relevant}) \wedge (\text{conditioned})] \\ &\quad + P[(\text{irrelevant}) \wedge (n\text{-adapted}) \wedge (\text{conditioned})] \\ &= rF(k, n) + (1 - r)[1 - A(k', n)]F(k', n) \end{aligned}$$

Note that in the two choice problem, the probability $F(k', n)$ that the irrelevant cue k' is conditioned on trial n is $\frac{1}{2}$. Substituting this into the foregoing expression and then substituting the new versions of the numerator and denominator into Eq 11 gives

$$p_n = \frac{rF(k, n) + (1 - r)[1 - A(k', n)]\frac{1}{2}}{r + (1 - r)[1 - A(k', n)]} \quad (11a)$$

At this point two initial conditions are introduced (1) there are no response biases, that is, $F(k, 1) = \frac{1}{2}$, and (2) initially none of the irrelevant cues is adapted, that is, $A(k', 1) = 0$. If these two substitutions are now made in Eqs 2 and 8, respectively, and the results substituted into Eq 11a, p_n becomes

$$p_n = \frac{r[\frac{1}{2}(1 - \theta)^{n-1} + 1 - (1 - \theta)^{n-1}] + (1 - r)(1 - \theta)^{n-1}\frac{1}{2}}{r + (1 - r)(1 - \theta)^{n-1}},$$

which, by adding and subtracting $(1 - \theta)^{n-1}$ and rearranging can be rewritten as

$$p_n = \frac{r + (1 - r)(1 - \theta)^{n-1} - \frac{1}{2}(1 - \theta)^{n-1}}{r + (1 - r)(1 - \theta)^{n-1}}$$

This final form is now seen to be the same as Eq 11

The curve described by Eq 11, which is sigmoidal, begins at $p_1 = \frac{1}{2}$ and approaches the asymptote $p_\infty = 1$. This means that the concept-utilization problem of concern is assumed to be capable of complete solution with perfect performance

Equation 11 involves two parameters, the proportion r of relevant cues in K and the learning rate parameter θ . Bourne and Restle make the

assumption that $r = \theta$, thus reducing Eq 11 to

$$p_n = 1 - \frac{\frac{1}{2}(1 - \theta)^{n-1}}{\theta + (1 - \theta)^n} \quad (12)$$

This simplification leaves one parameter, which is usually estimated by means of a formula involving the mean number of observed errors \bar{E} . The formula is straightforwardly developed on pp 282 to 283 in Bourne and Restle's (1959) article, the final result being

$$\bar{E} = \frac{\frac{1}{2} \log \theta}{(1 - \theta) \log (1 - \theta)} \quad (13)$$

Applications of this model to two concept-utilization problems will now be considered

3.1 Relevant redundancy

An immediate implication of the assumption that $\theta = r$ is that an increase in the proportion of relevant cues should speed up learning. Such an increase can be effected by adding relevant but redundant dimensions to the problem. Let R be the number of redundant relevant dimensions and assume that each contributes the same number of cues, then the number of relevant cues in K should be proportional to R , that is, $N(\text{relevant cues}) = cR$, where c is a proportionality constant. Let $a = N(\text{cues from irrelevant dimensions})$ and $bR = N(\text{relevant and irrelevant cues from the redundant relevant dimensions})$, then the proportion of relevant cues in K is given by

$$r = \frac{cR}{a + bR} \quad (14)$$

Now R , the number of redundant relevant dimensions, is known, hence it remains to estimate a , b , and c . If b is arbitrarily set equal to 1, this reduces the number of constants to be estimated and makes the number of cues associated with each relevant dimension the unit of measurement. Without further ado, Bourne and Restle state that "The data indicate that $c = \frac{1}{2}$ and that $a = 3.4 + I$, where I is the number of irrelevant dimensions in the problem" (p 284) for their experimental setup. Taking this as true, under penalty of not being able to go further with the application, Eq 14 becomes

$$r = \frac{\frac{1}{2}R}{3.4 + I + R} \quad (15)$$

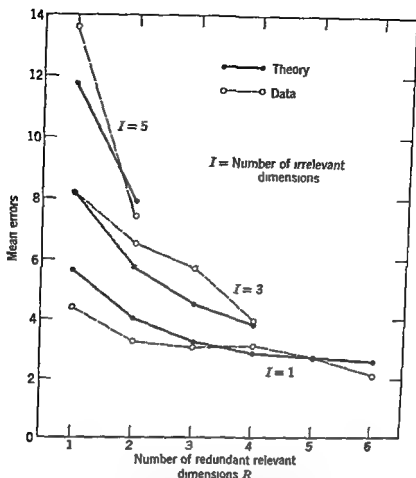


Fig 1 Predicted and observed errors in a concept utilization experiment analyzed by a cue conditioning model. Adapted with permission from Bourne & Restle (1959, p. 285)

We then use this r , which is taken as equal to θ , in Eq. 13 to predict the number of errors for various numbers of redundant relevant (R) and irrelevant (I) dimensions. Figure 1 shows the predicted and the observed errors from the concept-utilization experiment described in the article. In lieu of rigorous techniques for examining theory-data discrepancies, a fair amount of appropriateness for the model might be admitted, recalling, however, that (1) in developing Eq. 15 it was assumed that each dimension contributes an equal number of cues and (2) the estimation procedures for c and a were not presented by Bourne and Restle and hence whether or not a further assumption was introduced is not known.

Up to this point, only the two-choice situation has been considered. Bourne and Restle extend their model by treating the four-choice problem as if it were composed of two independent two-choice problems. This means that if p_{1n} and p_{2n} are the probabilities of CR's on the two relevant dimensions taken singly as two-choice problems, then the probability of a

CR in the four choice problem is given by

$$p_{cn} = p_{1n}p_{2n} \quad (16)$$

In the event that the separate two choice problems are equally difficult, p_{cn} is given by

$$p_{cn} = p_{1n}^2 = \left[1 - \frac{\frac{1}{2}(1 - \theta)^{n-1}}{r + (1 - r)(1 - \theta)^{n-1}} \right]^2 \quad (17)$$

In the two choice situation, it was assumed that $\theta = r$, in the present context, they assume that $\theta = r/2$. As in the two choice situation, an estimate of θ may be obtained and predictions made about the expected number of errors for different numbers of irrelevant dimensions. The only additional assumption involved in the extension is what might be called the independence assumption, stated formally in Eq. 16

3.2 Additivity of Irrelevant Dimensions

On the assumption that all the dimensions of a problem are equally effective and using the previous estimate of c , namely, $c = \frac{1}{2}$, r can be written

$$r = \frac{\frac{1}{2}R}{B + I + R}, \quad (18)$$

where $B = N(\text{residual or background or internal, irrelevant cues})$. Now B can reasonably be supposed to be constant in a given experimental situation, and R can be held fixed by the experimenter. Thus, after appropriate algebraic manipulation, $1/r$ is found to be a linear function of I

$$\frac{1}{r} = \frac{2(R + B)}{R} + \frac{2}{R}I \quad (19)$$

A value for B for this experimental situation can be estimated by first estimating θ , which yields an estimate of r , and then entering the values for R and I into Eq. 19. This estimate of B is then used in predicting the value of $1/r$ for experiments with different values of R and I , the numbers of redundant relevant and nonredundant irrelevant dimensions, respectively. Bourne and Restle report experimental results involving both two- and four-choice problems that are in excellent agreement with predictions from the model.

Bourne and Restle also develop two more applications of their cue conditioning model which will not be presented here: delayed reinforcement and partial reinforcement. Again, the agreement between the model and the data is very good and highly encouraging. This implies that the

model as a whole is appropriate for those variations of concept-utilization problems so far investigated

Although Bourne and Restle do not make reference to reversal- and nonreversal-shift tasks, it is tempting to assert that the model straightforwardly predicts that the latter should be more difficult than the former. This would seem to be deducible as follows. In the reversal situation, the same cues are irrelevant as before and hence are already adapted, leaving only reconditioning, whereas in the nonreversal situation, the now relevant cues are adapted from the preshift task and the now irrelevant cues are inappropriately conditioned with probability $\frac{1}{2}$. Such an argument cannot be made, however, because there is no provision in the model for the conditioning of previously adapted cues and no provision for the extinction of previously conditioned relevant cues. The model has two explicit initial conditions, namely, that there are no response biases and that initially none of the irrelevant cues is adapted. Thus Bourne and Restle's model cannot be applied to reversal- and nonreversal shift problems in its present state.

Two final comments seem worthwhile. The first has to do with the model's inability to handle the variance observed in the data: once θ has been estimated, the probability of a CR on trial n , p_n , is determined. Thus every subject is required to have p_n as his response probability, and deviations from p_n can only be random deviations with variance as prescribed by the model. The fact that the mean number of errors, averaged over a group of subjects, was used in estimating θ does not mean that p_n is an average. In a subsequent model proposed by Restle (1962), treated in the next section, this defect is eliminated by making learning itself a random event, a feature that allows the model to generate variability comparable with observed variability.

The second comment is a common sense observation that only prescient subjects can adapt θ per cent of the irrelevant cues and condition θ per cent of the unadapted cues on the first trial, or even on the second or third trials. It seems unrealistic to assume, as is done implicitly in the conditioning model, that subjects can "know" which cues are relevant and irrelevant with sufficient certainty prior to complete solution to ensure that a full θ per cent of each type are appropriately conditioned or adapted.

4 CONCEPT UTILIZATION AS STRATEGY SELECTION

A strategy is a rule according to which decisions are made. In a two-choice concept utilization task, a subject may adopt the strategy of making response R_1 or R_2 according to whether the stimulus is red or green.

respectively. If color is the relevant dimension, this strategy is either correct or wrong—correct if R_1 and R_2 have been designated by the experimenter as appropriate to red and green, respectively, wrong if they have been designated the other way around. If some other dimension is relevant and color is irrelevant, then this strategy is irrelevant.

A strategy need not be related to the stimuli presented, for example, a subject may decide to double-alternate the two possible responses, R_1, R_1, R_2, R_2 . Such a strategy is irrelevant. In a concept-utilization task with one relevant binary dimension, a rule for responding (a strategy) can be labeled formally as follows: (1) *relevant* if it is either correct or wrong, where it is (a) *correct* if it leads to a CR with probability 1 and (b) *wrong* if it leads to a CR with probability 0, or (2) *irrelevant* if it leads to a CR with some probability other than 1 and 0.

Strategies must specify under what conditions (or according to what plan) the alternative responses are to be made, that is, a strategy cannot be a rule for only a segment of the over-all task. For example, the strategy "make R_1 whenever the stimulus is red" is not permissible because the subject then has no plan of action for occasions when the stimulus is not red. (Of course, if all stimuli are red, then it is implicit that $R_j, j \neq 1$, is never to be made, thus making the strategy acceptable, and irrelevant.) A permissible strategy would be "make R_1 whenever the stimulus is red, otherwise make R_2 ." This simple strategy thus specifies a course of action for all possible stimulus situations.

At the risk of belaboring the issue, a final example is presented. Consider a concept utilization task with two binary dimensions—form, with values circle and square, and color, with values green and red. Let color be the relevant dimension such that "green $\rightarrow R_1$ and red $\rightarrow R_2$ " is the correct strategy. Suppose that the subject selects the strategy "green square $\rightarrow R_1$ and everything else $\rightarrow R_2$." Because this strategy leads to a CR with probability $\frac{1}{2}$, it is labeled irrelevant.

Restle (1962) has constructed a mathematical model based on strategy selection. His conjecture was that subjects in a concept-utilization task select at random a single strategy from a set of strategies. The subject retains this strategy as long as it leads to CR's. If it leads to an error, the subject is assumed to return it to the set and to select again at random. If the subject selects a wrong strategy, he makes an error with probability 1, if he selects a correct strategy, he makes a CR with probability 1, if he selects an irrelevant strategy he makes a CR with some probability different from 0 and 1. Of course, if he selects a correct strategy, he retains it and never makes another error, since reselection only occurs when an error is made.

The statements which together constitute Restle's strategy-selection model are not readily classifiable into component axioms. The fundamental

entities about which the model is concerned are strategies, not stimuli or cues, and hence there is no stimulus axiom, and since strategies are selected, not conditioned, there is no conditioning component to his model. In other words, the fundamental entities, strategies, do not pass from an initial state to some sort of conditioned state. Finally, because the selection of a strategy immediately implies a specific response (a response associated with a strategy is part of the definition of that strategy), a distinct response model is not apparent. In view of these difficulties, another organizational approach must be used.

STIMULUS SITUATION *Restle's strategy selection model applies to stimulus situations about which a subject can form testable hypotheses regarding how to respond correctly, each response being designated correct or wrong by the experimenter.*

FUNDAMENTAL TERM *The fundamental term is strategy. A strategy is a rule according to which the subject decides how to respond given any stimulus. A strategy yields a consistent pattern of responses to the stimuli of the task situation.*

CLASSIFICATION OF STRATEGIES *Let H be the set of strategies involving the permissible responses in the task. Then, according to which responses the experimenter labels correct or wrong, H is partitioned into three subsets*

$C = \{\text{strategies which always lead to a CR}\},$

$W = \{\text{strategies which always lead to a wrong response}\},$

$I = \{\text{strategies which can lead to either type response}\}$

The strategies falling into C , W , and I are uniquely determined by the experimenter's labeling rules, a change in his rules (for example, in reversal- and nonreversal-shift problems) induces a new partition on H .

ASSUMPTION *The subject selects at random a strategy from H , if the response is labeled an error by the experimenter, the subject again selects at random from H , sampling with replacement, if the response is labeled as correct, he retains the strategy.*

This assumption has two essential parts. (1) strategy selection is random with replacement, and (2) selection is made only after an error, except on the first trial. The first part, random selection with replacement, means that the proportions of C , W , and I strategies in H remain constant over trials, in other words, every time an error is made, the subject is conceived of as returned to an initial state—the concept-identification task is reset, so to speak, to zero. No provision is made for a change in the sampling probability of the strategy just rejected. The subject has the same probability of redrawing that strategy as he had of drawing it initially on the first trial. Thus, although certain mathematical complexities (which would follow were the sampling probabilities of previously sampled strategies

allowed to vary over trials) are avoided, the subject is theoretically conceived of as not remembering previously selected and rejected strategies. The second part, selection only after an error, means, by definition of C , that once the subject selects a strategy from C , he makes no more errors. The occurrence of an error therefore necessarily requires that the strategy selected not be a C strategy. On the other hand, the occurrence of a CR does not necessarily mean that the strategy selected is a C strategy since utilization of an I strategy may also lead to a CR . For this reason, derivations within the model are generally in terms of errors.

The model can be expressed as two equations. Let c , w , and i represent the proportions of C , W , and I strategies, respectively, in H . Because C , W , and I partition H , $c + w + i = 1$. Let p_j be the probability that an error occurs for the first time on the j th trial. Note that given an error on trial n for any n , p_j is the probability that the next error will occur on trial $n + j$. This is because following an error the strategy that led to the error is returned to H and the subject selects a new strategy at random, which is precisely the situation the subject was in on trial 1. For the two-choice concept-utilization task, the two equations are

$$\begin{aligned} p_1 &= w + \frac{1}{2}i, \\ \text{and} \quad p_j &= i\left(\frac{1}{2}\right)^j, \quad \text{for } j \geq 2 \end{aligned} \quad (20)$$

The first equation follows immediately from the experimenter's labeling procedures and from the definition of irrelevancy, for the probability of an error on the first trial (or on the first trial after an error) is the probability of drawing a W strategy plus the probability of drawing an I strategy and its not being labeled a CR , which is assumed by Restle to be $\frac{1}{2}$ in the two-choice situation. (The implications of this last assumption will be discussed later.) The second equation can be evolved recursively as follows. Let $Pr(CR_i | CR_{i-1} \text{ and } I_{i-1})$ read "the probability of a CR on trial i given a CR on trial $i - 1$ and an irrelevant strategy on trial $i - 1$ ". Then

$$\begin{aligned} p_2 &= Pr(CR_1 | I_1)Pr(I_1)Pr(error_2 | I) = \frac{1}{2}i\left(\frac{1}{2}\right) = i\left(\frac{1}{2}\right)^2 \\ p_3 &= Pr(CR_2 | CR_1 \text{ and } I_1)Pr(CR_1 | I_1)Pr(I_1)Pr(error_3 | I) \\ &= \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)i\left(\frac{1}{2}\right) = i\left(\frac{1}{2}\right)^3 \\ p_4 &= Pr(CR_3 | CR_{1,2} \text{ and } I_1)Pr(CR_{1,2} | I_1)Pr(I_1)Pr(error_4 | I) \\ &= \frac{1}{2}\left[\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\right]i\left(\frac{1}{2}\right) = i\left(\frac{1}{2}\right)^4 \\ &\dots \\ p_j &= Pr(CR_{j-1} | CR_1, \dots, CR_{j-2} \text{ and } I_1)Pr(CR_1, \dots, CR_{j-2} | I_1)Pr(I_1) \\ &\quad \times Pr(error_j | I) \\ &= \frac{1}{2}\left[\left(\frac{1}{2}\right)^{j-2}\right]i\left(\frac{1}{2}\right) = i\left(\frac{1}{2}\right)^j. \end{aligned}$$

Thus Eq 20 expresses the assumptions of the model plus the assumption that $Pr(CR | I) = \frac{1}{2}$ in the two-choice situation, and hence describes mathematically the concept-utilization process hypothesized by Restle

A comment concerning the assumption that $Pr(CR | I) = \frac{1}{2}$ in the two-choice situation is in order. According to this assumption, either (1) exactly $\frac{1}{2}$ of the I strategies are wrong and all I strategies have the same sampling probability, or (2) some fraction other than $\frac{1}{2}$ are wrong and the distribution of sampling probabilities is not uniform. A similar statement can be made for the situation involving r response alternatives where the assumption that $Pr(CR | I) = 1/r$ would presumably be made by Restle. It seems difficult to make psychological sense out of this restriction in view of the definition of a strategy. Because strategies are rules according to which decisions are made, it seems reasonable that the value of $Pr(CR | I)$ should depend on the stimulus situation as well as the number of response alternatives. The equality $Pr(CR | I) = \frac{1}{2}$ for the two-choice problem is used in a number of derivations of new statements within the model and hence must be considered as one of the axioms of the model.

4.1 Expected Number of Errors

As an illustration of how new statements may be derived from the assumptions, consider the model's prediction of the number of errors (selection of W or I strategies) a subject will make before selecting a C strategy. Since c is the proportion of C strategies in H , c is the probability of no further errors following an error. The probability of exactly one more error, then, is $(1-c)c$, the probability $1-c = w + i$ of first drawing either a W or I strategy times the probability c of then drawing a C strategy. In general, the probability of exactly k errors is $(1-c)^k c$. Therefore the expected value of k is the expected number of errors

$$\begin{aligned}
 E(k) &= \sum_{k=1}^{\infty} k(1-c)^k c \\
 &= c(1-c) \sum_{k=1}^{\infty} k(1-c)^{k-1} \\
 &= c(1-c) \frac{d}{dc} \left[- \sum_{k=1}^{\infty} (1-c)^k \right] \\
 &= c(1-c) \frac{d}{dc} \left(1 - \frac{1}{c} \right) \\
 &= c(1-c) \frac{1}{c^2} \\
 &= \frac{1-c}{c}
 \end{aligned} \tag{21}$$

Thus the model predicts that a subject will make on the average $(1 - c)/c$ errors during the course of learning to utilize the correct concept

Note that in order to predict the number of errors in an actual experimental situation, the parameter c must be known. Since other statements involving c can also be derived within the model, the procedure for obtaining a numerical value for c is to expend one of them in estimation. Suppose that of a number of derived statements, Eq. 21 is chosen to estimate c . Let \bar{T} denote the mean (over a number of subjects from a homogeneous population) number of errors actually made in the experiment, now substitute \bar{T} for $E(k)$ in Eq. 21 and solve for c . Thus the estimate of c is given by

$$c = \frac{1}{\bar{T} + 1} \quad (22)$$

It should be noted that \hat{c} slightly overestimates c . This is because $\bar{T} \leq E(k)$, which is due to the fact that in an actual experiment only a finite number of trials is given. In other words, $E(k)$ is influenced by extreme outcomes like prolonged correct responding under an I strategy, whereas \bar{T} is not.

The estimate of c given in Eq. 22 is also the maximum likelihood estimator for c . Suppose a group of N subjects makes X errors during the concept-utilization task. The probability, or likelihood, of such an event when c is the proportion of C strategies in H is

$$L(X \text{ errors}, c) = (1 - c)^X c^N,$$

the probability of X errors during the presolution period. The question is, what value of c maximizes this likelihood? Since the function $\log L$ has its maximum at the same value of c as does L , the foregoing likelihood is first rewritten as

$$\log L = \log [(1 - c)^X c^N] = X \log (1 - c) + N \log c$$

Proceeding with maximization as usual,

$$\frac{d}{dc} \log L = -\frac{x}{1 - c} + \frac{N}{c},$$

which, when set equal to zero and solved for c , gives

$$\hat{c} = \frac{N}{\lambda + N} = \frac{1}{(X/N) + 1}.$$

However, since λ is the number of errors made by N subjects, X/N is the

mean number of errors made, or T . Therefore

$$\hat{c} = \frac{1}{T+1},$$

the maximum likelihood estimator for c

It is of interest that the model, as formalized in Eq. 20, also permits a maximum likelihood estimate of w , the proportion of W strategies in H . To obtain the likelihood function, begin by noting that before a C strategy is selected and no more errors are made, the following conditional probabilities are immediate

$$Pr(\text{error}_{n+1} | \text{error}_n) = w + \frac{1}{2}c = \frac{1-c+w}{2},$$

$$Pr(\text{CR}_{n+1} | \text{error}_n) = \frac{1}{2}c = \frac{1-c-w}{2}$$

The former is simply the first equation of Eq. 20. Regarding the latter, in order to make a CR in the presolution period, that is, prior to selecting a C strategy, the subject must draw an I strategy, which then leads to a CR with probability $\frac{1}{2}$. Now if $T+1$ is the total number of errors, counting "trial 0" as an error, then $T+1 = M_1 + M_0$, where M_1 is the number of errors followed by another error and M_0 is the number followed by a CR. For a given subject, then, one can speak of $T+1$ errors, M_1 of which are followed by another error and M_0 of which are followed by a CR. Clearly, $T+1$, M_1 , and M_0 can be counted in the subject's protocol. By using the preceding conditional probabilities, the likelihood of such an outcome for the given subject can be written as

$$\begin{aligned} L(T+1 \text{ errors}, M_1 \text{ followed by another error}, M_0 \text{ by a CR}) \\ &= [Pr(\text{error} | \text{error})]^{M_1} [Pr(\text{CR} | \text{error})]^{M_0-1} c \\ &= \left(\frac{1-c+w}{2} \right)^{M_1} \left(\frac{1-c-w}{2} \right)^{M_0-1} c, \end{aligned}$$

where c is the probability that no more errors are made. The reason for making the exponent of the second factor $M_0 - 1$ is that a CR on the first trial, that is, a CR following the trial 0 error, is not to be counted since the number of errors T counted from the subject's protocol does not include trial 0, thus $M_1 + (M_0 - 1) = T$, as required. The likelihood of a whole set of such data for N subjects is

$$L_N = \prod L_i = \left(\frac{1-c+w}{2} \right)^{\sum_{i=1}^N M_{1i}} \left(\frac{1-c-w}{2} \right)^{\sum_{i=1}^N (M_{0i}-1)} c^N$$

If this likelihood function is maximized simultaneously with respect to w and c , it turns out that

$$\hat{w} = \frac{\bar{M}_1 - \bar{M}_0 - 1}{\bar{T} + 1}$$

Thus both c and w are capable of being estimated. This means that because $i = 1 - c - w$, and hence $\hat{i} = 1 - \hat{c} - \hat{w}$, the structure of H can be estimated in concept-utilization problems for which the model is appropriate.

4.2 Additivity of Cues

Consider a concept-utilization task involving at least two binary dimensions D_1 and D_2 . Suppose there are three experimental groups such that for Group 1 only D_1 is relevant, for Group 2 only D_2 , and for Group 3 D_1 and D_2 are both relevant but redundant, in each case all the remaining dimensions are irrelevant. Thus all three groups have a two choice problem. If c_1 and c_2 are the proportions of C strategies in H for Groups 1 and 2, respectively, then Restle asserts that $c_3 = c_1 + c_2$ should be the proportion of C strategies in H for Group 3. This equality is not presented as a theorem within the model, but merely as what Restle (1962) calls "the simplest interpretation of the experiment" (p. 340). It is of importance to notice, however, that the assertion $c_3 = c_1 + c_2$ involves a shift from speaking about cues (values on dimensions) to speaking about strategies. The model provides no formal bridge by means of which manipulations of the stimulus situation (making D_1 and D_2 redundant) can be translated into manipulations of subsets of H (making $c_3 = c_1 + c_2$).

The plausibility of adding subsets of H in a manner corresponding to cue addition arises from the intuition that if strategies involving hypotheses about irrelevant cues are irrelevant, then if the cues become relevant so should the associated strategies. What is intended could, of course, be stated completely in terms of strategies and their combinations. This would not be convenient, however, because such a formulation would make no direct reference to the stimulus situation and thus would make the description of a cue-additivity problem very difficult within the model. Actually, it is not until we attempt to apply the model to a variant stimulus situation that we realize there is very little contact between the model and the real world, this is evident in the absence of a stimulus model, the only translation of the hypothetical (the set H and its structural properties) into the real being the error equations.

Nevertheless, Restle presents data from several studies in which c_3 is closely approximated by $c_1 + c_2$. In one of these, a response-versus-place

learning study by Scharlock (1955), rats were run on an elevated T-maze. Group 1 had to learn to make a particular turn regardless of which goalbox had a light over it, Group 2 had to learn to run to the goalbox with the light over it regardless of whether it was to the right or left, and Group 3 had to learn to run to a lighted goalbox which was always on the same side. Group 1 subjects were thus required to utilize turning behavior in order to solve the problem, Group 2 subjects, light-selecting behavior, and Group 3 subjects, either or both since for them place and response cues were redundant. In 28 trials, the mean number of errors made by the subjects of Groups 1 and 2 were 6.7 and 9.7 respectively. If these values are substituted into Eq. 22, the estimates of c for the two groups are $\hat{c}_1 = 1/7.7 = 0.13$ and $\hat{c}_2 = 1/10.7 = 0.09$. Then $\hat{c}_3 = 0.13 + 0.09 = 0.22$, and hence for Group 3 the expected number of errors as given by Eq. 21 is $E(k) = 0.78/0.22 = 3.54$. The mean number of errors actually observed for Group 3 was 4.01, "which is adequately close" (Restle, 1962, p. 340).

In a discrimination problem for cats, Warren (1959) reports 11.13 errors for position relevant, 28.63 for object relevant, and 8.00 for object and position relevant and redundant. Estimates of c_1 and c_2 are calculated from the first two values: $c_1 = 1/11.13 = 0.082$ and $\hat{c}_2 = 1/28.63 = 0.034$. Then $\hat{c}_3 = 0.082 + 0.034 = 0.116$, and hence by Eq. 21 it must be that $E(k) = 0.884/0.116 = 7.62$, which is to be compared with the observed 8.00. As will be seen in Sec. 5, where the Bower and Trabasso (1963) model is discussed, prediction of the cue-additivity data of the foregoing experiments can be considerably improved.

4.3 A Modification

The seeming anachronism notwithstanding, an interesting modification of the present model (Restle, 1961, 1962) was proposed in 1960 by Restle which involves a specific probability r that reselecting will follow an error. The new assumption is: the subject selects at random a strategy from H , if the response is labeled an error by the experimenter, with probability r the subject again selects (sampling with replacement) at random from H , and with probability $1 - r$ he retains the non- C strategy, if the response is labeled correct, he retains the strategy with probability 1, as before. The original model is obtained by setting $r = 1$.

This modification was intended to handle the possibility that the subject might persevere, that is, that he might stick with a strategy even though it led to an error. The probability that perseveration occurs on an error trial is $1 - r$, perseveration on nonerror trials is assumed to occur with

probability 1 both here and in the original model since a strategy that leads to a CR is always retained. The term "perseveration" is therefore used only in reference to retaining non-*C* strategies. If no perseveration exists, an estimate of r will be approximately 1.

In this modified model, the probability that a given error is the last is rc , the probability r that the subject chooses a new hypothesis times the probability c that it is correct. Proceeding in a fashion similar to the original development, let $f(k)$ be the probability of exactly k errors. Then

$$\begin{aligned} f(0) &= c \\ f(1) &= (1 - c)rc \\ f(2) &= (1 - c)(1 - rc)rc \\ f(3) &= (1 - c)(1 - rc)^2rc \\ f(k) &= (1 - c)(1 - rc)^{k-1}rc, \quad \text{for } k > 0 \end{aligned} \quad (23)$$

This distribution is similar to the geometric distribution, which was the distribution of the expected number of errors in the original model, and a similar argument shows it to have the mean

$$E(k) = \frac{1 - c}{rc} \quad (24)$$

The model permits maximum likelihood estimation of r and c . Consider the experimental outcome where N subjects together make X errors, N_0 of whom make zero errors. The likelihood of such an event is given by

$$L = c^{N_0} [(1 - c)(1 - rc)^{X-1} (rc)]^{N - N_0}$$

where the first factor c^{N_0} is the probability of N_0 subjects selecting a *C* strategy on the first trial, and the second factor is the probability of $N - N_0$ or the remaining subjects making k errors. Now if $N - N_0$ subjects make all X of the errors, then k must be approximately $X/(N - N_0)$. If this substitution is made, the foregoing likelihood function becomes

$$\begin{aligned} L &= c^{N_0} [(1 - c)(1 - rc)^{X/(N - N_0) - 1} (rc)]^{N - N_0} \\ &= c^{N_0} (1 - c)^{N - N_0} (1 - rc)^{X - N + N_0} (rc)^{N - N_0} \end{aligned}$$

Maximizing L with respect to c and r simultaneously gives

$$\hat{c} = \frac{N_0}{N}$$

and

$$\hat{r} = \frac{N(N - N_0)}{N_0 X}$$

Thus since r and c can be estimated and because a number of statements, such as Eq 24, can be derived within the model, an evaluation of the perseveration hypothesis is possible in concept-utilization situations for which the strategy-selection model is appropriate. Restle (1960) cites several studies for which this modified model seems appropriate. For example, in an application of the model to a discrimination experiment with infant monkeys reported by Harlow (1959), Restle estimated r to be 0.50 and c was found to increase with age. This analysis suggests that with an increase in age there is a concomitant increase in the proportion of C strategies in H but not a change in the willingness to give up a disconfirmed strategy—an interpretation entirely consonant with Harlow's (1960) own views.

In summary, the strategy-selection model is concerned with entities, strategies, which have no referents in the stimulus situation. This leads to difficulties when attempting to make applications involving variations in the stimulus field, as in cue-additivity experiments. When such applications are attempted, the arguments leading to predictions can no longer be made completely within the model, and hence the resulting comparisons between theory and data are not strictly legitimate.

5 CONCEPT UTILIZATION AS SELECTION AND CONDITIONING

Consider a concept-utilization task where there are m independent binary dimensions under the experimenter's control, then there are $2m$ distinct cues, 2 from each of the m dimensions, and 2^m distinct categories to which the stimuli of the task can be assigned. Thus, in the ordinary situation, each stimulus has m attributes, or, more precisely, carries m cues. For example, let $m = 3$, where the three dimensions are color (red, green), form (circle, square), and size (large, small), then one of the stimuli is a large red circle and thus carries the $m = 3$ cues of largeness, redness, and circularity. If the relevant dimension is color and if the two response alternatives are R_1 and R_2 such that $\text{red} \rightarrow R_1$ and $\text{green} \rightarrow R_2$ are the associations to be acquired for correct utilization of the concept color, then when the large red circle is presented, the subject must learn to select redness from among largeness, redness, and circularity and to associate R_1 with it, that is, he must categorize the stimulus as red (as opposed to other possible categorizations) and associate R_1 with that category.

Hypothesizing that this is the way in which subjects actually proceed when faced with a concept-utilization task, Bower and Trabasso (1964) constructed a model that formalizes the selection and association processes.

In their model, categories are sampled according to their "attention-getting" properties and the subject shifts from an unconditioned state, where the response alternatives are chosen at random, to a conditioned state, where the correct response alternative is completely conditioned to the appropriate category. Thus, in the preceding example, the set of all stimuli that are red is the category with which R_1 must be associated. The shift from not associated to associated is probabilistic and contingent on the selection of the appropriate category. The component axioms may be stated as follows:

Stimulus axioms

- (i) *The stimuli can be classified into distinct categories such that each stimulus belongs to more than one category but not to all categories*
- (ii) *The set of categories is partitioned according to the experimenter's response labeling procedures into relevant and irrelevant categories*
- (iii) *The probability of selecting a relevant category from among all the categories to which the presented stimulus belongs is*

$$r = \frac{\sum_{i \in R} w_i}{\sum_{i \in R} w_i + \sum_{i \in I} w_i}, \quad (25)$$

where the w_i 's are the weights or measures ("attention-getting" values) of the relevant (R) and irrelevant (I) categories

Conditioning axioms Only the case of one binary relevant dimension will be treated at this point

- (i) *The subject can be in exactly one of two states: U, an initial unconditioned state, or C, a terminal conditioned state*
- (ii) *For each reinforced trial on which the subject has selected the correct category, there is a constant probability θ that the correct response is conditioned to that category; once the correct response is conditioned, it remains so*
- (iii) *Initial condition: all subjects begin in U*

Response axioms

- (i) *There is exactly one, unique, completely available response which is correct for each relevant category*
- (ii) *If the subject is in U, the probability of a correct response is some chance probability p ; if the subject is in C, the probability of a correct response is 1*

From these assumptions, what might be called a critical event may be identified, namely, the conditioning of a response to its appropriate

category This is a compound event with probability $c = r\theta$, the joint probability that the relevant aspect of the presented stimulus is selected and that the correct response becomes conditioned to it Since the parameter c is estimated from the data, and because it is the product of r and θ , it should be pointed out that there are at least two major sources of estimation error. First, r is variable across subjects to the extent that perceptual identification skills play a role in category selection, and second, θ varies to the extent that subjects differ in responding ability The model does not allow for such differences, in fact, it applies to a single subject and is evoked in connection with group data only under the assumption of identical subjects Therefore the appropriateness of the model to a given concept identification situation is directly related to the extent to which perceiving and responding skills are at the same asymptote for all subjects A subject is seen by the model as making two responses only (1) selecting a category and (2) selecting a response

Each subject on each trial has probability $c = r\theta$ of moving from U , the initial unconditioned state, to C , the terminal conditioned state Stated this way, the model is formally identical with Bower's (1961) one element model, in other words, both are two state Markov chains characterized by initial and terminal probabilities A very important property of two-state Markovian models is that as long as a subject is in U , that is, over the trials prior to shifting into C , the occurrence of correct responses should follow the binomial distribution with parameter p , where p is the probability that R_1 will be selected over R_2 (In the two-choice problem when there are no response biases, $p = \frac{1}{2}$) Furthermore, each trial prior to shifting into C is an independent event Trials for which (1) only two response alternatives are possible, (2) the chance probabilities of the two responses are fixed over trials, and (3) the individual trials are mutually independent are called *independent Bernoulli trials* Bower and Trabasso report very impressive data demonstrating that, in the concept utilization task they use, the trials prior to conditioning are independent Bernoulli trials This condition is an important "go ahead" sign regarding the appropriateness of the model for the particular situation An example of such data is shown in Fig 2

Over 200 subjects were trained on a number of two-choice single classification problems Different subjects had differing combinations of relevant attributes, this affects the learning rate on a problem but is immaterial to deciding whether performance is constant prior to a subject's final error... Each point plotted involves only those subjects who made their final error on some later block As subjects make their final error (and learn), they are dropped from consideration, hence the number of subjects involved decreases over successive trials The number of observations is quite large averaging around 550 responses per block (Bower & Trabasso, 1964, pp 36-37)

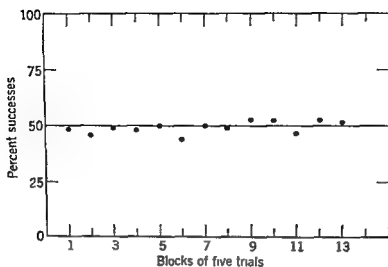


Fig 2 Percent successes prior to the last error plotted in blocks of five trials Adapted with permission from Bower & Trabasso (1963 p 36)

It is relevant at this point to mention the findings of Suppes and Ginsberg (1963). They report evidence to support the possibility that the kind of stationarity of response probability prior to the last error shown in Fig 2 may be an artifact of the method of organizing the data. It may be that response probability increases incrementally, so that on any trial the mean response probability is actually a composite of high and low response probabilities, high for subjects about to shift from U to C , low for subjects who will not be shifting from U to C until somewhat later in the problem. If such is, in fact, so, then Vincentizing the presolution data should show an increase in response probability instead of stationarity. Suppes and Ginsberg performed this analysis on a number of sets of data that have the property of stationarity when reported in the manner of Fig 2. They found systematic increases in response probability as the critical event (solution) was approached. Thus caution is in order when model evaluation is in terms of presolution stationarity of response probability.

The state to state transition probabilities of the present model can be written in matrix form

$$P = \begin{matrix} & \begin{matrix} U & C \end{matrix} \\ \begin{matrix} U \\ C \end{matrix} & \begin{bmatrix} 1-c & c \\ 0 & 1 \end{bmatrix} \end{matrix}$$

Thus, for example, on any trial the probability of remaining in state U is $1 - c$, whereas the probability of shifting to C is c . Note that C is absorbing. The n th power of P gives the probabilities of moving from one

state to another in exactly n trials

$$P^n = \begin{matrix} & U & C \\ \begin{matrix} U \\ C \end{matrix} & \begin{bmatrix} (1-c)^n & 1 - (1-c)^n \\ 0 & 1 \end{bmatrix} \end{matrix}$$

To apply the model to a particular concept utilization situation requires an estimate of the parameter c . Although parameters r and θ are used in deriving a number of statements from the assumptions, they seldom appear separately in the final form of the statement, r and θ nearly always occur in statements jointly as $r\theta = c$. Estimation of c is rather straightforward and does not involve either r or θ . In Sec. 2, where Suppes and Ginsberg's application of Bower's one-element model was treated, it was shown that if the probability of moving from U to C were a constant c for each trial, then the expected number of trials before the shift occurs is $1/c$. In other words, $1/c$ trials should be required before the critical event occurs. However, for each of these trials, there is a constant probability p that the subject makes a correct response simply by guessing. Therefore, of the $1/c$ trials prior to conditioning, p/c of them should be correct responses and $(1-p)/c$ should be errors. Now p can easily be estimated since learning is assumed to be all or-none, performance prior to the critical event should be at a chance level, hence p can be taken as the observed proportion of successes prior to the last error, denoted \hat{p} . From within the model, then, the expected number of errors is

$$E(\text{errors}) = \frac{1 - \hat{p}}{c} \quad (26)$$

From the data, we estimate p and find, over subjects, the average number of errors \bar{T} . Substituting \bar{T} for $E(\text{errors})$, we have as the estimate of c ,

$$\hat{c} = \frac{1 - \hat{p}}{\bar{T}} \quad (27)$$

Equation 27 can also be derived by means of the maximum likelihood technique. In a given concept utilization experiment, the protocol of the i th subject consists of a sequence of errors and CR's. Prior to the critical event, suppose subject i makes T_i errors and Z_i CR's. The likelihood of this outcome is

$$L_i = p^{Z_i}(1-p)^{T_i}(1-c)^{Z_i+T_i-1}c,$$

where p^{Z_i} is the probability of the Z_i CR's, $(1-p)^{T_i}$ the probability of the T_i errors, $(1-c)^{Z_i+T_i-1}$ the probability of being in U for all $Z_i + T_i$ trials except the last, and c the probability of shifting into C on the last

The likelihood of N subjects producing protocols with T errors and Z presolution CR's is

$$L = \prod_{i=1}^N L_i = p^Z (1-p)^T (1-c)^{Z+T-N} c^N,$$

where

$$Z = \sum_{i=1}^N Z_i \quad \text{and} \quad T = \sum_{i=1}^N T_i$$

Converting L to $\log L$ gives

$$\log L = Z \log p + T \log (1-p) + (Z+T-N) \log (1-c) + N \log c$$

Maximizing with respect to p and c simultaneously,

$$\begin{aligned} \frac{\partial \log L}{\partial p} &= \frac{Z}{p} - \frac{T}{1-p} = 0, \\ \frac{\partial \log L}{\partial c} &= \frac{N}{c} - \frac{Z+T-N}{1-c} = 0 \end{aligned}$$

The former yields $p = Z/(Z+T)$ the proportion of presolution trials which are CR's, whereas the latter yields $c = N/(Z+T)$, which can be rewritten as

$$\begin{aligned} c &= \frac{T + (Z - Z)}{(T/N)(Z+T)} \\ &= \frac{1 - Z/(Z+T)}{T} \\ &= \frac{1 - p}{T}, \end{aligned}$$

as given in Eq. 27

5.1 Additivity of Categories

From Eq. 25, it is clear that r , and therefore c , must vary directly with the number and weights of relevant categories (cues) and inversely with the number and weights of irrelevant categories (cues). Specifically, c is written

$$c = \theta \frac{\sum_{i \in R} n_i}{\sum_{i \in R} n_i + \sum_{i \in I} n_i} \quad (28)$$

By using Eq 28,⁵ statements regarding the effects on performance of certain variations in the stimulus situation can be derived within the model

Consider the ordinary two choice cue-additivity situation as described in Sec 4.2 Let T_1 , T_2 , and T_3 be the total number of errors made in the conditions where D_1 , D_2 , and D_1 plus D_2 are relevant, respectively. The model predicts that

$$T_3 = \frac{T_1 T_2}{T_1 + T_2}$$

The proof of this equation is facilitated by first proving a lemma

Lemma If c_1 , c_2 , and c_3 are the learning rates for the conditions where

D_1 , D_2 , and D_1 plus D_2 are relevant, respectively, then $c_3 = c_1 + c_2$

PROOF From the definition of r as given in Eq 25 or 28,

$$c_1 = \theta \frac{w_1}{w_1 + w_2 + w_3},$$

$$c_2 = \theta \frac{w_2}{w_1 + w_2 + w_3},$$

$$c_3 = \theta \frac{w_1 + w_2}{w_1 + w_2 + w_3} = \theta \frac{w_1}{w_1 + w_2 + w_3} + \theta \frac{w_2}{w_1 + w_2 + w_3} = c_1 + c_2,$$

as required

Note that although the weights were used in the foregoing proof, they do not appear in the lemma. Thus, although there are (complicated) ways of determining category weights, it is seen that their formal properties may be utilized in derivations within the model without the need of determining their actual values arising. Using the lemma, we may now prove the following theorem

Theorem If T_1 , T_2 , and T_3 are the total number of errors for the conditions where D_1 , D_2 and D_1 plus D_2 are relevant, respectively, then

$$T_3 = \frac{T_1 T_2}{T_1 + T_2} \quad (29)$$

PROOF From Eq 27, $c = (1 - p)/T$, or, for a particular condition, $c_i = (1 - p)/T_i$. Therefore $T_i = (1 - p)/c_i$, and in particular $T_3 = (1 - p)/c_3$. However, from the lemma, $c_3 = c_1 + c_2$. Therefore, by

⁵ Equation 28 as well as Eq 25, is properly a definition. The assumption is that category selection is probabilistic, as given in the stimulus axioms. Equation 28 is then offered as one of the possible ways for interpreting the assumed probability

The likelihood of N subjects producing protocols with T errors and Z presolution CR's is

$$L = \prod_{i=1}^N L_i = p^Z (1-p)^T (1-c)^{Z+T-N} c^N,$$

where

$$Z = \sum_{i=1}^N Z_i \quad \text{and} \quad T = \sum_{i=1}^N T_i$$

Converting L to $\log L$ gives

$$\log L = Z \log p + T \log (1-p) + (Z+T-N) \log (1-c) + N \log c$$

Maximizing with respect to p and c simultaneously,

$$\begin{aligned} \frac{\partial \log L}{\partial p} &= \frac{Z}{p} - \frac{T}{1-p} = 0, \\ \frac{\partial \log L}{\partial c} &= \frac{N}{c} - \frac{Z+T-N}{1-c} = 0 \end{aligned}$$

The former yields $\hat{p} = Z/(Z+T)$, the proportion of presolution trials which are CR's, whereas the latter yields $\hat{c} = N/(Z+T)$, which can be rewritten as

$$\begin{aligned} \hat{c} &= \frac{T + (Z - Z)}{(T/N)(Z+T)} \\ &= \frac{1 - Z/(Z+T)}{T} \\ &= \frac{1 - \hat{p}}{T}, \end{aligned}$$

as given in Eq. 27

5.1 Additivity of Categories

From Eq. 25, it is clear that r , and therefore c , must vary directly with the number and weights of relevant categories (cues) and inversely with the number and weights of irrelevant categories (cues). Specifically, c is written

$$c = \theta \frac{\sum_{i \in R} n_i}{\sum_{i \in R} n_i + \sum_{i \in I} n_i} \quad (28)$$

By using Eq 28,⁵ statements regarding the effects on performance of certain variations in the stimulus situation can be derived within the model

Consider the ordinary two-choice cue-additivity situation as described in Sec 4.2. Let T_1 , T_2 , and T_3 be the total number of errors made in the conditions where D_1 , D_2 , and D_1 plus D_2 are relevant, respectively. The model predicts that

$$T_3 = \frac{T_1 T_2}{T_1 + T_2}$$

The proof of this equation is facilitated by first proving a lemma

Lemma *If c_1 , c_2 , and c_3 are the learning rates for the conditions where*

D_1 , D_2 , and D_1 plus D_2 are relevant, respectively, then $c_3 = c_1 + c_2$

PROOF From the definition of r as given in Eq 25 or 28,

$$c_1 = \theta \frac{w_1}{w_1 + w_2 + w_3},$$

$$c_2 = \theta \frac{w_2}{w_1 + w_2 + w_3},$$

$$c_3 = \theta \frac{w_1 + w_2}{w_1 + w_2 + w_3} = \theta \frac{w_1}{w_1 + w_2 + w_3} + \theta \frac{w_2}{w_1 + w_2 + w_3} = c_1 + c_2,$$

as required

Note that although the weights were used in the foregoing proof, they do not appear in the lemma. Thus, although there are (complicated) ways of determining category weights, it is seen that their formal properties may be utilized in derivations within the model without the need of determining their actual values arising. Using the lemma, we may now prove the following theorem

Theorem *If T_1 , T_2 , and T_3 are the total number of errors for the conditions where D_1 , D_2 , and D_1 plus D_2 are relevant, respectively, then*

$$T_3 = \frac{T_1 T_2}{T_1 + T_2} \quad (29)$$

PROOF From Eq 27, $c = (1-p)/T$, or, for a particular condition, $c_i = (1-p)/T_i$. Therefore $T_i = (1-p)/c_i$, and in particular $T_3 = (1-p)/c_3$. However, from the lemma, $c_3 = c_1 + c_2$. Therefore, by

⁵ Equation 28, as well as Eq 25 is properly a definition. The assumption is that category selection is probabilistic, as given in the stimulus axioms. Equation 28 is then offered as one of the possible ways for interpreting the assumed probability

substitution,

$$\begin{aligned}
 T_3 &= \frac{1-p}{c_1 + c_2} \\
 &= \frac{1-p}{(1-p)/T_1 + (1-p)/T_2} \\
 &= \frac{1-p}{[(1-p)(T_2 + T_1)]/T_1 T_2} \\
 &= \frac{T_1 T_2}{T_1 + T_2},
 \end{aligned}$$

as required

Note that Eq 29 does not involve any of the parameters of the model nor any of the category weights used in the definition of r . All these are used in the proof, but the final statement involves only the total number of errors made under the three conditions

Bower and Trabasso cite data which give outstanding support to their model. In a T-maze task for rats, Scharlock (1955) reports $T_1 = 9.7$ for place learning, $T_2 = 6.7$ for response learning, and $T_3 = 4.0$ for place plus response learning. The last value is to be compared with 3.97, the value predicted by the model using Eq 29. In a discrimination problem for cats, Warren (1959) reports $T_1 = 11.13$ for position relevant, $T_2 = 28.63$ for object relevant, and $T_3 = 8.00$ for object plus position relevant. Substituting the T_1 and T_2 values into Eq 29 yields the model's prediction of 8.03 for T_3 .

In summary, Eq 29 is a statement, derived from the assumptions of the model, which is true in several category additivity situations. This should mean that the assumptions are cogent for such concept-utilization situations, that is, that the stimulus axioms are true of the stimulus situation, etc. Some caution should be exercised, however, since the possibility exists that the excellent agreement between model and data is because of a fortuitous combination of axioms, not all of which are true taken individually. This is not a comment on this model in particular, but rather on all models utilizing more than one axiom. It seems especially relevant at this point, however, since a careful examination of the foregoing lemma and theorem reveals that all the assumptions, plus the definition of r given by Eq 25, were used in their proofs.

5.2 Single-Category Solutions

According to the stimulus axioms, the subject selects a category on each trial and following the critical event, he always makes the correct response

for that category. What if, as in the preceding section, there are two or more relevant but redundant categories for each stimulus? Since on any given trial only one of them is selected, it follows that on the critical event trial the correct response becomes conditioned to only one category. Furthermore, because the category to which the correct response is conditioned is the only category selected thereafter, the correct response cannot become conditioned to any of the additional redundant categories. Thus in a situation where there are two relevant redundant categories for each stimulus, the subject is seen as solving the problem using only one of them.

Suppose D_1 and D_2 are relevant and redundant, and suppose that on solving the concept-utilization problem a new problem is introduced where D_1 is relevant and D_2 does not appear. Then, if the model is correct, the subjects should be neatly divided into two groups: those who used D_1 in solving the first problem should show perfect transfer to the second problem, whereas those who used D_2 should show no transfer. One of the conditions included in Trabasso's doctoral dissertation (Trabasso, 1963) was designed to test this prediction, and may be introduced as follows:

If D_1 and D_2 are the two relevant and redundant dimensions, then a proportion $w_{D_1}/(w_{D_1} + w_{D_2})$ of the subjects will solve the concept-utilization task using only D_1 , whereas a proportion $w_{D_2}/(w_{D_1} + w_{D_2})$ will solve using only D_2 . This is because these two proportions are the probabilities that, for any given stimulus presentation prior to solution, D_1 and D_2 , respectively, will be selected. Now w_{D_1} and w_{D_2} can be estimated from data based on situations in which only D_1 and D_2 , respectively, are relevant (see Trabasso, 1963, for procedural details). Thus if D_1 and D_2 are relevant and redundant in one task, and if in a second task D_1 is relevant and D_2 is absent, it follows that proportion $w_{D_1}/(w_{D_1} + w_{D_2})$ of the subjects will make no errors in the second task and that proportion $w_{D_2}/(w_{D_1} + w_{D_2})$ will learn the second task with learning rate c_1 . Using this argument, Trabasso predicted that 4.72 out of 20 subjects solved the first task using D_1 and hence would make no errors in the second task. Five such subjects were observed. Using the parameter c_1 as estimated from the appropriate control situation (a single problem with D_1 relevant and D_2 absent), Trabasso predicted that the remaining 15 subjects would make, on the average, 16.07 errors while learning the second task. The observed mean was 16.45 errors.

A second aspect of the single-category solution implication is that once a subject attains the concept, that is, once the critical event has occurred, responding should be (1) unaffected by manipulations of the irrelevant categories and (2) at the chance level to irrelevant cues presented in

isolation Bower and Trabasso report experimental results which suggest that the latter might be true. They used trigrams as stimuli which could be categorized two ways (R_1 or R_2) according to whether the middle letter was an F or a Y . After reaching a criterion on this task, the subjects were then tested on the various letters separately, on the pairs which could be formed from the various letters, and on the trigrams themselves. The probability of a CR when a trigram was tested was 0.92, when either F or Y was tested alone, 0.89, and when any pair which contained F or Y was tested, 0.88. The two response alternatives R_1 and R_2 occurred about equally often when irrelevant letters were presented, either singly or in pairs, thus confirming the prediction that responding should be at the chance level to irrelevant cues presented in isolation.

5.3 The All-or-None Assumption

The prediction of Bernoulli trials prior to the critical event and the absorption property of the terminal state C together constitute an assumption that learning is all or-none: a subject is either in a learned or an unlearned state regarding any particular category-response pair. This assumption characterizes the conditioning model and is open to direct experimental test without any formal derivations within the model. Because an error is direct evidence that the subject is still in state U , that is, that the critical event has not occurred, it must be that a change in the experimenter's response labeling procedures immediately following an error can have no effect on the subject's performance.

To test this implication, Bower and Trabasso proceeded as follows. Three groups of subjects were given the same concept-utilization task where D_1 and D_2 were independent binary dimensions which were, respectively, relevant and irrelevant. On the first error after trial 9, the subjects of Group R were given a reversal shift task, those of Group NR were given a nonreversal shift task, and those of Group C were continued on the original task. In other words, if $D_{11} \rightarrow R_1$ and $D_{12} \rightarrow R_2$ are the associations to be acquired in the first task, then those to be acquired in the second task would be $D_{11} \rightarrow R_2$ and $D_{12} \rightarrow R_1$ for Group R and $D_{21} \rightarrow R_1$ and $D_{22} \rightarrow R_2$ for Group NR . (Subjects beginning a criterion run of successes prior to trial 9 were dropped from the experiment.) If, in fact, nothing has been learned up to the trial after which the shift was made, then, in terms of performance following that error, that is, following the shift, the three groups should perform identically. Counting from the error trial that initiated the final task, the mean number of errors for Groups C , R , and NR were 19.11, 19.11, and 18.28, respectively, and the mean trial of the last error was 38.33, 39.56, and 36.94, respectively. Since

these values do not differ among themselves significantly, Bower and Trabasso conclude that the all-or-none implication of their model is acceptable.

5.4 The Role of Error Trials

Section 5.3 dealt with a particular composite assumption, namely, that learning is all-or-none, this section deals with a more limited aspect of the conditioning model, namely, that there is a constant probability of moving from U to C on *each* trial. As the model now stands, a subject in state U who selects an irrelevant category and then by chance gives a correct response is assumed again to select a category at random on the next trial. Intuitively, one might suspect that a subject would actually continue to select that same category until he made an error, and then either shift to another category or reselect at random. The question is, can the model be improved by altering the conditioning model in such a way that the subject is assumed to shift (probabilistically) from U to C only on error trials and never on correct-response trials? The comparison of such a new model with the original one then permits at least some evaluation of the conditioning assumption of the original model.

To assume that the subject shifts from U to C only on error trials is to construct a new model. The new model is identical with the old one except that the second conditioning axiom is replaced by the following new statement: For each reinforced trial on which the subject has selected the correct category and has made an error, there is a constant probability θ' that the correct response is conditioned to that category, if a correct response is made, conditioning does not occur. If we now take $e = r\theta'$, e is the probability of the critical event on an error trial. In this new model, U is subdivided into two states, E and S , and hence there are three states in all:

- C the terminal state, as before,
- E the unconditioned state just prior to an error,
- S the unconditioned state just prior to a chance success

The probability of a correct response, then, in the three states is 1, 0, and 1, respectively, and the probability of being in each of the three states on the first trial is 0, $1 - p$, and p , respectively, where p is the probability of selecting R_1 at random. The matrix of transition probabilities for this new model is

$$P = \begin{matrix} & \begin{matrix} C & E & S \end{matrix} \\ \begin{matrix} C \\ E \\ S \end{matrix} & \begin{bmatrix} 1 & 0 & 0 \\ e & (1-p)(1-e) & p(1-e) \\ 0 & 1-p & p \end{bmatrix} \end{matrix}$$

Thus, for example, the probability of shifting from E to S is the probability of the joint event of conditioning not occurring, $1 - e$, and of the next trial being a chance correct response, p . As before, C is absorbing.

As in the original model, statements can be derived that then can be checked against data. Most of these derived statements, however, look very much like those derived in the original model and hence comparison of the two forms of the conditioning axiom becomes problematic. The difficulty arises because most of the test statistics (derived statements) are in terms of errors and therefore are in some way dependent on the prediction, made in both models, that the trials prior to the critical event are Bernoulli trials. One way to circumvent this is to design two experimental situations that yield considerably different p values (probability of randomly selecting a given response alternative) and base the comparison of the two forms of the axiom on test statistics that involve p .

To see this more clearly, consider the expected number of errors predicted from the two models. The prediction of the original model is given by Eq. 26

$$E(\text{errors}) = \frac{1 - p}{c} \quad (30)$$

In the modified model, however, the prediction is

$$E(\text{errors}) = \frac{1}{e} \quad (31)$$

Recall that $1/c$ trials were expected prior to the critical event in the original model, counting both error and correct-response trials, and hence $(1 - p)/c$ of them should be errors. In the modified model, only error trials are taken into consideration, hence the expected number of errors is simply $1/e$. The basis for comparing the two conditioning assumptions arises from the simple fact that the expression giving the expected number of errors for the one model involves p , whereas for the other it does not. Therefore one model predicts variations in the expected number of errors with variations in the number of response alternatives, whereas the other does not.

Another test statistic which is highly similar for the two forms of the conditioning axiom but differentially sensitive to variations in p is the expected trial number of the last error. For the original model, where the critical event may occur on any type trial, the expectation of this statistic is

$$\frac{1/c}{1 + cp/(1 - p)}, \quad (32)$$

whereas for the modified model, where the critical event can occur only on error trials, it is

$$\frac{1}{(1 - p)e} \quad (33)$$

Equation 32, which is only slightly sensitive to variations in p , becomes smaller as p increases, whereas Eq 33, which is very sensitive to variations in p , becomes larger as p increases. Thus the original model predicts that the trial number of the last error will be slightly lower in a two choice than in a four-choice concept-identification task, a prediction to be compared with that of the modified model which says that the trial number of the last error will be appreciably higher in the two choice than in the four-choice task.

Bower and Trabasso report results of original research that give rather convincing support to the modified model. In essence, the experimental situation consisted of two- and four-choice tasks where, for example, one group learned to give R_1 , R_2 , R_3 , and R_4 to red, green, blue, and brown stimuli, respectively, whereas another group learned to give R_1 to both red and green and R_2 to both blue and brown. For the former subjects p turned out to be approximately $\frac{1}{3}$, whereas for the latter it was close to $\frac{1}{2}$, thus the two conditions, the two- and four choice tasks, afforded the desired manipulation of p . The critical result, however, is the average trial of the last error for the two tasks. The mean for the two choice task was 1.44 times the mean for the four-choice task ($26.36/18.32 = 1.44$). This finding is almost exactly what is predicted by the modified model. If the estimated p value for the four-choice task is substituted in Eq 33 and the result divided into the corresponding value obtained when the estimated p value for the two-choice task is substituted in Eq 33, the result is 1.5. The original model, on the other hand, predicts a ratio less than 1 by an amount which is a function of the size of c . For $c = 0.037$, the ratio of the trial numbers for the last error for the two- and four choice tasks is predicted to be $26.04/26.52 = 0.98$. It is clear that something is wrong with the original model and that the modification is an important one. Bower and Trabasso conclude that learning takes place only on error trials, which is the hypothesis underlying the modification.

6 SUMMARY AND CONCLUSIONS

Four speculations as to what subjects do in a concept utilization task have been considered. None of these speculations is testable by direct observation. In each case, however, the speculation was formulated with sufficient

precision to allow the deduction of new statements which are testable by direct observation. Usually, the deductions rested on the probability calculus, a convenience arising from the fact that the conjectures were expressible as equations. The several theorists involved hypothesized variously that concept utilization could be viewed as paired-associate learning, as cue conditioning, as strategy selection, and as a combination of selection and conditioning.

Bower's one-element model for paired associate learning was seen to be inadequate as a descriptive device for the concept-utilization situation. In making this application, Suppes and Ginsberg fitted the model twice, once using the relevant categories as the stimuli and once using the separate instances of the relevant categories as stimuli. It was argued that the former procedure involved a greatly oversimplified representation of the stimulus situation and hence that the stimulus axiom of the model was not satisfied.

The derived statements following from Bourne and Restle's cue-conditioning conjecture turned out to be impressively appropriate for the experimental situation studied, especially regarding variations in relevant redundancy and irrelevant additivity. To date, this model has generated and handled more research data than any of the others.

Restle's strategy selection model was seen to be inadequately tied to the stimulus situation, a shortcoming which produces difficulties when statements about variations in the stimulus situation are to be derived. The difficulty is that the arguments leading to predictions cannot be made completely within the model. This makes comparisons between theory and data of doubtful legitimacy.

The final model considered was Bower and Trabasso's selection-conditioning model, a conjecture which represents concept utilization as a two-stage process: selection of a category and selection of a response alternative. Conditioning of a response alternative was hypothesized to be an all-or-none affair contingent on selection of a relevant category. The logical consequences were examined, and a modified conjecture about the role of error trials was shown to improve the model.

Making comparisons among the four models, we cannot help but detect a special emphasis on the utility of models which (1) make a firm contact with both the stimulus situation and response classes and (2) allow for the operation of more than one process at a time. Restle's strategy selection is poor in contact and is a single-process model, and consequently poor in verification. The two most successful models, Bourne and Restle's cue-conditioning model and Bower and Trabasso's selection-conditioning model, both make good contact and postulate multiple processes.

References

- Bourne, L E, & Restle, F Mathematical theory of concept identification *Psychol Rev*, 1959, 66, 278-296
- Bower, G Application of a model to paired-associate learning *Psychometrika*, 1961, 26, 255-280
- Bower, G An association model for response and training variables in paired associate learning *Psychol Rev*, 1962, 69, 34-53
- Bower, G, & Trabasso, T Concept identification In R C Atkinson (Ed) *Studies in mathematical psychology* Stanford Stanford Univer Press, 1964 Pp 32-94
- Harlow, H F Learning set and error factor theory In S Koch (Ed), *Psychology a study of a science*, Vol 2 New York McGraw-Hill, 1959 Pp 492-537
- Harlow, H F, Harlow, M K, Reuping, R R & Mason, W A Performance of infant rhesus monkeys on discrimination learning, delayed response, and discrimination learning set *J comp physiol Psychol*, 1960, 53, 113-121
- Restle, F A note on the 'hypothesis' theory of discrimination learning *Psychol Rep*, 1960, 7, 194
- Restle, F Statistical methods for a theory of cue learning *Psychometrika* 1961, 26, 291-306
- Restle, F The selection of strategies for cue learning *Psychol Rev*, 1962, 69, 329-343
- Scharlock, D P The role of extramaze cues in place and response learning *J exp Psychol*, 1955, 50, 249-254
- Suppes, P, & Ginsberg, R Application of a stimulus-response model to children's concept formation of binary numbers, with and without a response *J exp Psychol*, 1962, 63, 330-336
- Suppes, P, & Ginsberg, R A fundamental probability distribution of responses prior to conditioning: its application to children's concept formation *Psychol Rev*, 1963, 70, 139
- Trabasso, T R Stimulus emphasis and discrimination learning by children *J exp Psychol*, 1963, 65, 398-406
- Warren, J M Stimulus perseveration *J exp Psychol*, 1959, 52, 99-101

19

Preference, Utility, and Subjective Probability¹²

R. Duncan Luce

University of Pennsylvania

Patrick Suppes

Stanford University

1. The preparation of this chapter was supported in part by National Science Foundation grants G17637 and GB1462 to the University of Pennsylvania and by the Group Psychology Branch of the Office of Naval Research under contract NR 171-034 with Stanford University

2. We very much appreciate the extensive comments and criticisms of drafts of various sections that were given by Ward Edwards, F. W. Irwin, Jacob Marschak, and Dana Scott.

Contents

1	General Remarks on the Study of Preference	252
1 1	Origins of the mathematical theories of preference, 252	
1 2	Preference, discrimination, and bias, 254	
1 3	A classification of theories of preference, 255	
1 4	Previous surveys of the literature, 258	
2	General Algebraic Choice Theories	258
2 1	Ordinal utility functions, 259	
2 2	Topological assumptions, 264	
2 3	Additivity assumptions, 267	
2 4	Higher ordered metrics, 272	
2 5	JND assumptions, 279	
3	Algebraic Choice Theories for Uncertain Outcomes	281
3 1	The expected utility hypothesis, 281	
3 2	Axiom systems that assume numerical probability, 284	
3 3	Axiom systems for subjective probability, 291	
3 4	Other decision principles, 299	
4	Experimental Tests of Algebraic Models	306
4 1	Higher ordered metrics, 307	
4 2	Measurement of utility, 310	
4 3	Measurement of subjective probability, 321	
4 4	Experimental studies of variance-preference, utility-of-gambling, and other models, 327	
5	General Probabilistic Choice Theories	331
5 1	Response probabilities, 331	
5 2	Constant utility models, 332	
5 3	Random utility models, 337	
5 4	Observable properties, 339	
5 5	Relations among the observable properties, 342	
5 6	Relations between the models and observable properties, 346	
6	General Probabilistic Ranking Theories	351
6 1	A function relating choice to ranking probabilities, 351	

6.2	A function relating ranking to choice probabilities, 353	
6.3	An "impossibility" theorem, 356	
7.	Probabilistic Choice Theories for Uncertain Outcomes	359
7.1	Expected utility models, 359	
7.2	Decomposition assumption, 362	
7.3	Several theories applied to a special case, 367	
8.	Experimental Tests of Probabilistic Models	377
8.1	Some experimental issues, 377	
8.2	Stochastic transitivity, 380	
8.3	Plots of p_∞ versus π , 390	
8.4	Probabilistic expected utility models, 397	
	References	402

Preference, Utility, and Subjective Probability

Of the major areas into which experimental psychology has been traditionally partitioned, motivation is the least well understood and systematized. This is true whether we consider theory, experimental paradigms, or experimental results.

The psychology of motivation, compared with that of learning or of sensory and perceptual processes, is peculiarly retarded and confused. Much of what passes for discussion of motivation is pieced together out of fragments from physiological psychology, learning, personality, social psychology, and psychopathology. These patches often look violently out of place, worse still, they conceal the underlying cloth so well that one may doubt whether it exists at all (Irwin, 1958, p. 152).

Nevertheless, as Irwin goes on to point out, at least one motivational concept, *preference*, is just as basic to psychological theory as, for example, are the concepts of discrimination and instrumental conditioning that have arisen in the other more developed branches of psychology. Moreover, of the various notions usually considered to be primarily motivational, preference is the only one that mathematical psychologists have attempted to analyze with any care. There are almost no satisfactory formal theories concerning, for example, drive and incentive, and those that exist are best discussed as aspects of learning. So this chapter on mathematical theories of motivation is limited to a study of preference and to the closely related constructs of utility and subjective probability.

1. GENERAL REMARKS ON THE STUDY OF PREFERENCE

1.1 Origins of the Mathematical Theories of Preference

Although we are correct in viewing formal theories of preference as a part of mathematical psychology, it would badly wrench history to suggest that the research was mostly carried out by people who classify themselves as psychologists. Much of the theory, and certainly the best of it, was worked out by economists and statisticians who needed psychological

underpinnings for their decision theories. Only in the last half dozen years have psychologists begun to isolate for separate study these inherently psychological theories of preference. While being elaborated as distinct and testable psychological theories, the theories of preference have begun to acquire a richness and complexity—hopefully reflecting a true richness and complexity of behavior—that renders them largely useless as bases for economic and statistical theories. Perhaps we may ultimately find simple, yet reasonably accurate, approximations to the more exact descriptions of behavior that can serve as psychological foundations for further theoretical developments, but at the moment this is not the main trend. The psychological reader should be warned that statisticians and economists have long employed a technique of nonempirical, rationalistic argument which is totally foreign to many psychologists, who tend to the other extreme of automatically rejecting plausible hypotheses unless they have powerful experimental support. Psychologists sometimes seem mightily naive in their reverence for what are held to be pure empirical facts, when actually most experimental inferences depend on some more or less implicit theoretical position, often partially embodied in a statistical model. Be that as it may, many theories of preference were formulated and evaluated initially in terms of what a “rational” person having limited computational abilities and resources ought to do. Frequently these debates have a somewhat tenuous quality for psychologists, especially since there does not exist any really satisfactory comprehensive definition of rationality and only a few of its properties are generally agreed on. Some psychologists have simply rejected work of this genre as sterile, but those who have taken an active interest in it have found two research paths open. First, can these so called normative theories also be adequate descriptions of behavior? Attempts to answer this question have led to laboratory tests of the normative theories—with, however, mostly ambiguous results. Second, can theories be devised that are frankly descriptive in intent, that try to encompass explicitly some of the phenomena that seem to be found in the laboratory? Several attempts are in vogue.

Our attitude in this chapter is primarily psychological. We ask whether a theory seems to describe behavior, not whether it characterizes a rational person. We report experiments, although admittedly not enough experimental work has yet been done to provide us with either completely satisfactory designs or highly reliable phenomena, and we explore some of the relations between theories of preference and other psychological theories. At the same time, we try to recount the normative considerations that led originally to the theories and to cite the more important rationalistic criticisms that have been leveled against them.

1.2 Preference, Discrimination, and Bias

Irwin (1958, p. 152) developed in detail the widely accepted thesis that, "preference is exactly as fundamental as discrimination and that, indeed, the two are so intimately related in behavior that if the organism exhibits a discrimination, it must also exhibit a preference, and conversely." Without reproducing the supporting argument in detail, we may suggest the main point. Suppose that outcome x_1 results either when response r_1 is made to stimulus presentation s_1 or when r_2 is made to s_2 , whereas x_2 results when r_2 is made to s_1 or when r_1 is made to s_2 . Before we can expect a subject to respond differentially and thus to show his ability to discriminate between s_1 and s_2 , it must matter to him which outcome occurs—he must prefer one outcome to the other. Equally well, he can evidence a preference between x_1 and x_2 only if he is capable of discriminating the discriminative stimuli, for only then can he know (or learn) which response is appropriate on each trial to achieve the preferred outcome.

A careful examination of Irwin's discussion suggests that this view is correct: the notions of discrimination and preference are both fundamental to an understanding of behavior and they are profoundly intertwined in parallel roles. If so, it appears perverse, if not worse, to divide psychology—in particular, that portion based upon choice experiments—into a part concerned primarily with the discrimination of stimuli and a part concerned primarily with preferences among outcomes. We are saved from total chaos, however, by the experiments that in principle we should perform, but rarely do because we are certain of the results on the basis either of the informal experimentation of experience or of studies deeply buried in the past of psychology. In a psychophysical or learning experiment we assume that we know what a subject's preferences are among the outcomes. We are confident that hungry animals prefer receiving food pellets to not receiving them, that a student prefers winning five cents to losing three, etc. Equally well, in (human) preference experiments, we always attempt to select discriminative stimuli, for example, labels to identify the outcomes, that we are sure from our knowledge of people are perfectly discriminable. We could perform the necessary auxiliary experiments to prove these assumptions, but usually we do not because we are so certain of the results. This means, for example, that when a subject exhibits some inconsistency in his choices in a preference experiment, we automatically attribute it to an ambivalence about the worth of the outcomes, not to his inability to tell which outcomes are associated with which stimulus response pairs nor to his inability to discriminate among the relevant stimuli.

Although the accuracy of these assumptions can always be questioned, there is little doubt that serious attempts have been made to realize them in the preference experiments we shall discuss here. Therefore, throughout the chapter, we take it for granted that the subject has no difficulty in discriminating the alternatives presented to him, our focus is entirely on his preferences. This is not to deny that any ultimately satisfactory theory of choice behavior will have to encompass discrimination, preference, and learning. The current partitioning is a convenient way to subdivide a complex problem into somewhat more manageable chunks.

A second point emphasized by Irwin (1958) is more deeply confounding in practice than the interlocking of preference and discrimination. Subjects often—almost always, in fact—exhibit “preferences” among the responses as well as among the outcomes. To keep the terminology straight, we follow Irwin by speaking of these as *biases* among the responses. Perhaps the most striking example of response biases is the frequently noticed position habits of animals that seem to occur whenever the discrimination is at all difficult or whenever the differences among the outcomes are slight. Furthermore, it should be recalled that response biases are ubiquitous in psychophysics (see Chapters 3, 4, and 5 of Vol. I) and that the more recent psychophysical theories explicitly provide for them.

Unfortunately, the same cannot be said for the mathematical theories of preference. None of them acknowledges response biases. To some extent, theoreticians either have not been particularly sensitive to the phenomenon or they have felt that the biases arise from relatively trivial experimental errors that can be overcome by appropriate techniques, such as randomization. Actually, however, response biases seem to be a fairly deep problem, and no one knows how to eliminate them experimentally. Randomization is no solution, despite its widespread acceptance among experimentalists, because it only buries the biases in increased variability, it does not eliminate them. We badly need theories of preference that explicitly provide for response biases.

1.3 A Classification of Theories of Preference

As in psychophysical research, preference studies have been largely restricted to steady state (asymptotic) behavior. Behavioral transients, which are the province of learning, are sufficiently complicated to analyze that only the simplest learning experiments have been dealt with in any detail. These learning experiments are too simple to provide answers to many of the questions that we wish to ask about preferences, and so

we are forced to confine our attention to asymptotic behavior. Unfortunately, some recent studies (Edwards, 1961a, Lindman & Edwards, 1961) suggest that true experimental asymptotes may be harder to come by than one would wish.

Because the focus is on equilibrium behavior, many of the theories of preference are simply static in character, including no explicit mechanism for temporal changes. A few recent theories, however, derive asymptotic mean predictions from well-specified stochastic learning processes. As we shall see (Sec 7.3), the predictions of these two types of theories are not very compatible, but the problem does not seem to be simply to choose between them, rather, some sort of fusion is needed. The static theories embody, admittedly roughly, something of a subject's cognitive or rational analysis of the choice problem, whereas the asymptotic learning theories encompass the fine-grain adjustments made by subjects to their recent experiences. One can hardly doubt that both mechanisms exist in people.

Several distinctions that are important both substantively and mathematically can be made among these equilibrium theories of preference. We make three and use them to organize the chapter.

The first is between what may be called *algebraic* and *probabilistic theories*. Let us suppose that the responses that a subject makes to stimulus presentations are governed by probability mechanisms. Then the theory can be viewed as algebraic if all such probabilities are either 0, $\frac{1}{2}$, or 1. Although this definition is strictly correct, it misleadingly suggests that the algebraic theories are simply special cases of the probabilistic ones. They are not. The algebraic ones employ mathematical tools, namely, algebraic tools, that are different from those used in the probabilistic theories, and so it is appropriate to develop them separately. Historically, the algebraic theories were studied first, and they have been used in economics and statistics almost exclusively. The probabilistic ones are largely the product of psychological thought, forced upon us by the data we collect in the laboratory.

Some authors have chosen to refer to the probabilistic theories of preference as stochastic theories, and this usage is gaining in popularity. In our opinion, it is unfortunate because it blurs a valuable distinction. A probability process that includes a time parameter, continuous or discrete, has come to be called a *stochastic process* (see Chapter 20). One that does not involve transitions over time should be called something else. Thus, for example, an asymptotic learning theory for a preference experiment is properly described as a stochastic theory of preference, but theories that simply postulate a probabilistic choice mechanism without a means of changing the probabilities in time (for example, with experience) we shall speak of as probabilistic theories.

The second distinction, an experimental one, is between certain and uncertain outcomes. It is simply a question of whether the outcome is prescribed by the stimulus presentation plus the response or whether these only determine a probability distribution over the outcomes. In terms of the notation of Chapter 2 (p. 86), if s is the presentation and r the response, the conditional outcome schedule $\omega = \phi(r, s)$ is a random variable with the trials as its domain and the set A of outcomes as its range. If the outcome schedules have the special property that for all $x \in A$ and all $\omega \in \Omega$, where Ω is the set of conditional outcome schedules, the conditional probability $\pi(x | \omega)$ is either 0 or 1, then we shall say that the experiment has *certain outcomes*. Otherwise, the outcomes are *uncertain*.

Theories for certain outcomes are usually stated in such a way that, in principle, they can be applied to uncertain outcomes, however, in that context they usually seem to be very weak theories because they fail to take into account the complex structure of the uncertain outcomes. On the other hand, theories for uncertain outcomes that explicitly include the probability distribution over the set of outcomes can always be specialized to the certain case, but again the effect is to produce weak theories. Presumably, this is a temporary problem and ultimately the specialization of a theory for uncertain outcomes will yield a satisfactory theory for certain ones.

Statisticians sometimes attempt to make further distinctions among what we are calling uncertain outcomes. When the subject knows the probability distribution over the outcomes, they speak of risky choices, and they reserve the word "uncertain" either for all cases that are neither risky nor certain or for those in which the subject has no information at all about the distribution. As these distinctions are difficult to make precise and they do not seem particularly useful for our purposes, we shall not bother with them.

To add further to the terminological confusion, information theorists use the word "uncertain" to describe situations in which the probability distributions are known, that is, to refer to situations that decision theorists describe as pure risk. Their use is not inconsistent with ours, but it is less broad.

The third and final distinction is between *simple choice experiments* and *ranking experiments*. Strictly speaking, both are choice experiments as that term is used in Chapter 2. The question is whether the subject is asked to select among several outcomes or whether he is asked to rank order them, that is, to select from the set of rankings. It is generally believed that there must be some regular relation between his behavior in these two kinds of experiments when the same outcomes are involved, and the ranking theories attempt to describe these relations.

With three binary distinctions, there ought to be eight theoretical sections to the chapter, actually there are only five. The reasons are that, at present, the algebraic ranking theories are trivial extensions of the choice theories, and so do not warrant separate sections, and that no special probabilistic ranking theories have yet been developed for uncertain outcomes.

1.4 Previous Surveys of the Literature

In the last decade several excellent surveys of various aspects of the subject matter of this chapter have appeared. Edwards (1954d) and (1961b) are particularly well known to psychologists. Adams (1960) provides an excellent survey of Bernoullian utility theory (a topic discussed in Sec. 3 of this chapter) for both economists and psychologists. Arrow (1951b, 1964) and Majumdar (1958) are mainly addressed to economists, but both articles contain many useful remarks that will help psychologists who seek a general theoretical orientation in the field. Material of a similar sort is also to be found in Luce and Raiffa (1957), which is addressed to a general social science audience.

We emphasize that some of the work described and analyzed in these sources is not covered in this chapter, and the interested reader is urged to refer to them.

2 GENERAL ALGEBRAIC CHOICE THEORIES

As was noted previously, algebraic choice theories for certain outcomes have a long history in classical economics, beginning at least as early as the work of Jeremy Bentham in the eighteenth century. His definition of utility, given in the first chapter of his famous *The Principles of Morals and Legislation* (1789), still provides a good starting point for a discussion of certain theories.

By utility is meant that property in any object, whereby it tends to produce benefit, advantage, pleasure, good, or happiness (all this in the present case comes to the same thing), or (what comes again to the same thing) to prevent the happening of mischief, pain, evil, or unhappiness to the party whose interest is considered: if that party be the community in general, then the happiness of the community, if a particular individual then the happiness of that individual.

Bentham interpreted his maxim of "the greatest good for the greatest number" as meaning that utility is maximized, and he spent considerable effort in formulating a program for measuring utility.

In the classical economic literature on utility, the most important distinction is between ordinal and cardinal utility. The term *ordinal* refers to the assumption of order only, whereas *cardinal* refers to the assumption of additivity or, if not that, at least uniqueness of numerical assignment up to a linear transformation (see Chapter 1, Vol I, p. 12). During the nineteenth century it was commonly believed that little economic theory could be developed solely on the basis of ordinal utility functions. One of Pareto's (1906) great achievements was to show that much could be done with purely ordinal assumptions. It is fair to say that Pareto was the first main contributor to the theory of ordinal utility functions, to which we now turn.

2.1 Ordinal Utility Functions

Because of its structural simplicity, the theory of ordinal utility functions is a good place to begin for logical as well as historical reasons. Essentially the theory deals just with the qualitative preference for one alternative over another. To apply mathematical analysis in this setting, the primary thing that is needed is the representation of preference in the form of a numerical utility function. Once such a numerical function is available it may be used in subsequent theoretical developments, with the possibility open of applying standard mathematical tools to problems of behavior. For this reason, most of this section is devoted to a sequence of increasingly general theorems on the existence of numerical utility functions that reflect the qualitative preference structure.

Although psychologists find much of the economic literature on utility rather far removed from direct experimental or behavioral questions, the theory of ordinal utility functions was, in fact, developed to answer some rather specific questions about consumer behavior. It is appropriate to begin by sketching the setting of these questions. Most of what we say here is drawn from Wold and Jureen (1953) and Uzawa (1960).

Suppose that a consumer has income M at time t_0 (in the sequel we consider only the static case of t_0 , so we drop explicit reference to time). With his income the consumer may purchase a bundle of commodities—traditionally one of these commodities may be savings. We can describe such a bundle by a real n -dimensional vector $x = (x_1, \dots, x_n)$, where the i th component x_i specifies the amount of commodity i to be consumed. Following standard notation, we say that bundle x is greater than x' , in symbols, $x > x'$, if for every i , $x_i \geq x'_i$ and for some i , $x_i > x'_i$. As the foundation of the theory of consumer demand, a preference relation P on commodity bundles is introduced. The relation xPy is read "commodity

bundle x is preferred to commodity bundle y . The consumer's income M , the current market prices p , and the structure of his preference relation P determine his choice of commodities.

In addition to the relation P , it is also necessary to introduce a relation I of indifference, for it is not reasonable that of two distinct bundles one is necessarily strictly preferred to the other. Technical economy of formulation is achieved at many points by replacing P and I by the weak preference relation R . The relation R stands to P in the same way that the numerical relation \geq stands to $>$. The obvious equivalences are

$$\begin{aligned} xRy & \text{ if and only if } xPy \text{ or } xIy, \\ xIy & \text{ if and only if } xRy \text{ and } yRx, \\ xPy & \text{ if and only if } xRy \text{ and not } yRx, \end{aligned}$$

and we shall assume them without comment. The following postulates on the structure of the (weak) preference relation R are reasonably intuitive, and they guarantee the existence of a utility function.

Definition 1 *A relation R on the set of all commodity bundles (n -dimensional vectors) is a preference relation if the following axioms are satisfied for any bundles x , y , and z :*

- 1 Transitivity *if xRy and yRz , then xRz ,*
- 2 Connectivity *xRy or yRx ,*
- 3 Nonsatiety *if $x > y$, then xPy ,*
- 4 Continuity *if xRy and yRz , then there is a real number λ such that $0 \leq \lambda \leq 1$ and $[\lambda x + (1 - \lambda)z]Iy$*

Of these four postulates, the first two are of a very general nature, their formulation does not depend in any way on x , y and z being n -dimensional vectors. When 1 and 2 are satisfied, the relation R is called a *weak ordering*. It should be understood that the *or* of Axiom 2 is inclusive, that is, both xRy and yRx may hold.

Axioms 3 and 4 are of a more special character. Axiom 3 is called the axiom of nonsatiety because it postulates that the consumer, *ceteris paribus*, always prefers more of any commodity to less. This is obviously unrealistic when all possible commodity bundles are considered, it becomes more realistic, however, if finite bounds are imposed on each commodity.

It might be thought that Axiom 4 is not really necessary to prove the existence of a utility function u such that for all bundles x and y

$$u(x) \geq u(y) \text{ if and only if } xRy \quad (1)$$

There is, however, a classical and important counterexample that shows

at an arbitrary weak ordering cannot be represented by a numerical function

Counterexample Consider the *lexicographic ordering* of the plane $(x_1, x_2)R(y_1, y_2)$ if and only if either $x_1 > y_1$ or $x_1 = y_1$ and $x_2 \geq y_2$. Suppose that there exists a real valued function u satisfying Eq. 1. We fix x_1 and y_1 with $x_2 < y_2$ and define for each x_2

$$u(x_1) = u(x_1, x_2)$$

$$u'(x_1) = u(x_1, y_2)$$

In terms of these functions define the following function f from real numbers to intervals

$$f(x_1) = [u(x_1), u'(x_1)]$$

In the assumption that the ordering is lexicographic, f must be 1-1 since two distinct numbers are mapped into two disjoint intervals. For instance, if $x_1 > x'_1$, then $u'(x_1) = u(x_1, x_2) > u(x'_1, y_2) = u'(x'_1)$. But it is well known that it is impossible for a one-to-one correspondence to hold between the uncountable set of real numbers and a countable set of nondegenerate disjoint intervals. Thus no such function f can exist, and a fortiori there can be no function u satisfying Eq. 1 for the lexicographic ordering.

Note that Axiom 4 is not satisfied by the counterexample because, for example, the point $(4, 2)$ is between $(2, 2)$ and $(4, 4)$ in the lexicographic ordering but there is no number λ such that $\lambda(2, 2) + (1 - \lambda)(4, 4) = (4, 2)$.

We now show that the postulates of Def. 1 do guarantee the existence of a utility function.

Theorem 1 *Let R be a preference relation on the set of commodity bundles in the sense of Def. 1. Then there exists a utility function u that satisfies Eq. 1.*

PROOF We first define the function u for the "diagonal" bundles x , which are defined by the property that $x_i = x_1$ for all i by $u(x) = x_1$. Since any two indifferent vectors must have the same utility, we may extend u to any vector by constructing a diagonal vector to which it is indifferent. Let y be any vector, let y^* be the diagonal vector with all of its components equal to the smallest component in y , and let y^{**} be the diagonal vector with all its components equal to the largest component in y . Obviously, $y^{**} \geq y \geq y^*$, and therefore by the axiom of transitivity, $y^{**}RyRy^*$. By the axiom of continuity, there is a λ such that $\lambda y^{**} + (1 - \lambda)y^*$ and λ can easily be shown to be unique whenever y is not a diagonal vector. Since $\lambda y^{**} + (1 - \lambda)y^*$ is a diagonal vector from the

definition of u we have in terms of the first components of y^* and y^{**} ,

$$u(y) = \lambda y_1^{**} + (1 - \lambda)y_1^*$$

Let z be any other vector. We must show that

$$u(y) \geq u(z) \text{ if and only if } yRz$$

First, assume that yRz and $u(y) < u(z)$. Then

$$\lambda' z^{**} + (1 - \lambda')z^* > \lambda y^{**} + (1 - \lambda)y^*,$$

and therefore zRy by the nonsatiation axiom, and thus yIz . Therefore $u(y) = u(z)$, contrary to the supposition $u(y) < u(z)$, so $u(y) \geq u(z)$. Conversely, assume that $u(y) \geq u(z)$. Then by definition

$$\lambda y^{**} + (1 - \lambda)y^* \geq \lambda' z^{**} + (1 - \lambda')z^*,$$

and therefore by the axiom of nonsatiation

$$[\lambda y^{**} + (1 - \lambda)y^*]R[\lambda' z^{**} + (1 - \lambda')z^*]$$

Thus, because the ordering of R is preserved by substitution of I -indifferent vectors, it follows that yRz , which completes the proof.

Given prices p_1, \dots, p_n , a consumer with income M can afford to buy any commodity bundle $x = (x_1, \dots, x_n)$ such that

$$p_1 x_1 + \dots + p_n x_n \leq M$$

The behavioral prediction of the theory is that he will select a bundle that maximizes utility, subject to this income restraint.

The problem of the numerical representation of preferences is not completely solved by proving the existence of a numerical utility function. In order to know the extent to which numerical methods of analysis may then be applied, it is also necessary to know how unique the obtained utility function is. It is a simple matter to prove the following theorem (see Theorem 8, Chapter 1, p. 29, also see Chapter 1 for a more extensive discussion of questions of uniqueness).

Theorem 2 *Let R be a preference relation in the sense of Def. 1. Then any two utility functions satisfying Eq. 1 are related by an increasing monotone transformation.*

The restriction of the domain of utility functions to n -dimensional commodity spaces is not desirable even in economics, and it is certainly unacceptable in psychological investigations of preference and choice. Fortunately, the most general circumstances under which an ordinal utility function exists can be rather easily characterized. First, we generalize Def. 1 to say that a relation R on an arbitrary set A is a *preference relation* for A if it is a weak ordering of A , that is, if it satisfies

Axioms 1 and 2 of Def 1. Second, let B be a subset of A . Then we say that B is R -order-dense in A if and only if for every x and y in A , but not in B , such that xPy there is a z in B such that xRz and zRy . Note that the denumerable set of rational numbers is order-dense with respect to the natural numerical ordering \geq in the nondenumerable set of all real numbers. This relationship between the denumerable rational numbers and all real numbers is just the one that is necessary and sufficient for the existence of a utility function satisfying Eq 1. With respect to a preference relation that is a weak ordering a minor complication arises in applying the denumerability condition, namely, the elements of the order-dense subset must not be indifferent. This additional condition is made precise in the statement of the theorem.

Theorem 3 *Let A be an infinite set and let R be a preference relation on A . Then a necessary and sufficient condition that there exist a utility function satisfying Eq 1 is that there is a denumerable subset B of A such that (i) B is R -order-dense in A and (ii) no two elements of B stand in the relation I , that is, for any distinct x and y in B either xPy or yPx .*

PROOF [The proof is related to the classical ordinal characterization of the continuum by Cantor (1895). We do not give all details here, but sketch the main outlines, for some additional details and related theorems, see Sierpinski (1958, Chapter 11) and Birkhoff (1948, pp 31-32).]

To prove the sufficiency of the condition, let B be a denumerable subset with properties (i) and (ii). Moreover, if A has endpoints with respect to the ordering of R , we may without loss of generality include them in B . First, we know that there exists a utility function u for B , just because B is denumerable (this is Theorem 6, Chapter 1, p 26). Now by the R -order-dense condition on B , each element y of A that is not in B defines a cut in B , that is, the partition of B into two sets, $X = \{x \mid x \in B \text{ \& } xRy\}$ and $Z = \{z \mid z \in B \text{ \& } yRz\}$. Now let

$$r_1 = g \mid b_{x \in X} u(x)$$

and

$$r_2 = l \mid u_{z \in Z} u(z)$$

We then extend u to y by defining

$$u(y) = \frac{r_1 + r_2}{2}$$

It is easy to show that the utility function u thus extended from B to A satisfies Eq 1. For example, if n_1 and n_2 are in A but not in B , then if n_1Rn_2 there is a z in B such that n_1RzRn_2 and thus

$$u(n_1) \geq u(z) \geq u(n_2)$$

definition of u we have in terms of the first components of y^* and y^{**} ,

$$u(y) = \lambda y_1^{**} + (1 - \lambda)y_1^*$$

Let z be any other vector. We must show that

$$u(y) \geq u(z) \text{ if and only if } yRz$$

First, assume that yRz and $u(y) < u(z)$. Then

$$\lambda' z^{**} + (1 - \lambda') z^* > \lambda y^{**} + (1 - \lambda) y^*,$$

and therefore zRy by the nonsatiation axiom, and thus yIz . Therefore $u(y) = u(z)$, contrary to the supposition $u(y) < u(z)$, so $u(y) \geq u(z)$. Conversely, assume that $u(y) \geq u(z)$. Then by definition

$$\lambda y^{**} + (1 - \lambda) y^* \geq \lambda' z^{**} + (1 - \lambda') z^*,$$

and therefore by the axiom of nonsatiation

$$[\lambda y^{**} + (1 - \lambda) y^*] R [\lambda' z^{**} + (1 - \lambda') z^*]$$

Thus because the ordering of R is preserved by substitution of I indifferent vectors, it follows that yRz , which completes the proof.

Given prices p_1, \dots, p_n , a consumer with income M can afford to buy any commodity bundle $x = (x_1, \dots, x_n)$ such that

$$p_1 x_1 + \dots + p_n x_n \leq M$$

The behavioral prediction of the theory is that he will select a bundle that maximizes utility, subject to this income restraint.

The problem of the numerical representation of preferences is not completely solved by proving the existence of a numerical utility function. In order to know the extent to which numerical methods of analysis may then be applied, it is also necessary to know how unique the obtained utility function is. It is a simple matter to prove the following theorem (see Theorem 8, Chapter 1, p. 29, also see Chapter 1 for a more extensive discussion of questions of uniqueness).

Theorem 2 *Let R be a preference relation in the sense of Def. 1. Then any two utility functions satisfying Eq. 1 are related by an increasing monotone transformation.*

The restriction of the domain of utility functions to n dimensional commodity spaces is not desirable even in economics, and it is certainly unacceptable in psychological investigations of preference and choice. Fortunately, the most general circumstances under which an ordinal utility function exists can be rather easily characterized. First, we generalize Def. 1 to say that a relation R on an arbitrary set A is a *preference relation* for A if it is a weak ordering of A , that is, if it satisfies

Axioms 1 and 2 of Def 1. Second, let B be a subset of A . Then we say that B is R order-dense in A if and only if for every x and y in A , but not in B , such that xPy there is a z in B such that xRz and zRy . Note that the denumerable set of rational numbers is order-dense with respect to the natural numerical ordering \geq in the nondenumerable set of all real numbers. This relationship between the denumerable rational numbers and all real numbers is just the one that is necessary and sufficient for the existence of a utility function satisfying Eq 1. With respect to a preference relation that is a weak ordering a minor complication arises in applying the denumerability condition, namely, the elements of the order dense subset must not be indifferent. This additional condition is made precise in the statement of the theorem.

Theorem 3 *Let A be an infinite set and let R be a preference relation on A . Then a necessary and sufficient condition that there exist a utility function satisfying Eq 1 is that there is a denumerable subset B of A such that (i) B is R -order dense in A and (ii) no two elements of B stand in the relation I , that is, for any distinct x and y in B either xPy or yPx .*

PROOF [The proof is related to the classical ordinal characterization of the continuum by Cantor (1895). We do not give all details here, but sketch the main outlines, for some additional details and related theorems, see Sierpinski (1958, Chapter 11) and Birkhoff (1948, pp 31-32).]

To prove the sufficiency of the condition, let B be a denumerable subset with properties (i) and (ii). Moreover, if A has endpoints with respect to the ordering of R , we may without loss of generality include them in B . First, we know that there exists a utility function u for B , just because B is denumerable (this is Theorem 6, Chapter 1, p 26). Now by the R order-dense condition on B , each element y of A that is not in B defines a *cut* in B , that is, the partition of B into two sets, $X = \{x \mid x \in B \text{ \& } xRy\}$ and $Z = \{z \mid z \in B \text{ \& } yRz\}$. Now let

$$r_1 = g \mid b_{x \in X} u(x)$$

and

$$r_2 = l \mid b_{z \in Z} u(z)$$

We then extend u to y by defining

$$u(y) = \frac{r_1 + r_2}{2}$$

It is easy to show that the utility function u thus extended from B to A satisfies Eq 1. For example, if u_1 and u_2 are in A but not in B , then if u_1Ru_2 , there is a z in B such that u_1RzRu_2 and thus

$$u(u_1) \geq u(z) \geq u(u_2)$$

The remaining details of this part of the argument may be supplied by the reader

To prove the necessity of (i) and (ii), we assume that we have a function u satisfying Eq 1. The set of nonempty intervals I_i of the real numbers with rational endpoints is a denumerable set because of the denumerability of the rational numbers. We next construct corresponding intervals J_i of A by taking the inverse image under u of each interval I_i . (Note that not every point in I_i need correspond to a point in A .) From each J_i that is not empty we select an element x_i . Since the set of intervals is denumerable, the set X of elements x_i of A is denumerable. Now, let \mathcal{R} be the set of real numbers r such that for some y in A , $u(y) = r$ and

$$u(y) - \sup_{x \in Y} u(x) > 0$$

where

$$Y = \{x \mid x \in A \text{ \& } xRy\}$$

Because this set \mathcal{R} defines a set of nonoverlapping intervals of real numbers it is at most denumerable (of course, \mathcal{R} can, and in some cases would, be empty). Let X' be the inverse image under u of \mathcal{R} . Then $B = X \cup X'$ is denumerable. To show that B is order-dense in A , let t_1 and t_2 be two elements in A but not in B such that $t_1 P t_2$. Now if there are no elements of A between t_1 and t_2 , then t_1 is in X' , contrary to hypothesis. On the other hand, if there are elements between t_1 and t_2 , then at least one, say z , will be such that $u(z)$ lies in an interval with rational endpoints which is nested in the interval $[u(t_2), u(t_1)]$, and thus B is order-dense in A . This completes the proof.

It is, of course, possible to generalize Theorem 3 by no longer requiring the utility function to be real-valued. General theorems for any preference relation, that is, for any weak ordering of a set, are given in Chipman (1960a). He shows that if for the range of the utility function we replace the real numbers by transfinite sequences of real numbers under their natural lexicographic order, then such a utility function exists for any preference relation. Such generalizations are not pursued here, for they are of dubious interest for psychological theory.

2.2 Topological Assumptions³

It is easily seen that, although the utility function we constructed in the proof of Theorem 1 is continuous, the construction may be modified to obtain a discontinuous function satisfying Eq 1. For example, for a

³ This section can be omitted without loss of continuity.

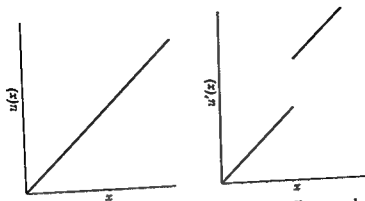


Fig. 1 Graphs of the continuous function u and the ordinally equivalent discontinuous one u'

diagonal x let

$$u'(x) = \begin{cases} x_1 & \text{if } x_1 \leq 1 \\ x_1 + 1 & \text{if } x_1 > 1 \end{cases}$$

Thus for diagonal bundles, we have the two graphs of u and u' shown in Fig. 1. In Theorem 3 it is clear that no claims of continuity can be made for the utility function considered.

For further theoretical or practical developments of the theory of preference it is almost always desirable to confine our attention to well-behaved functions, for which continuity is often a desirable if not always a sufficient restriction. Two questions naturally arise. First, once we drop the restrictive assumption of commodity spaces of Def. 1, what shall we mean by continuity and what postulates guarantee the existence of at least one continuous utility function? Second, how can we guarantee that all, not merely one, of the possible utility functions are continuous?

To answer these questions requires a brief excursion into topology. Our treatment of these matters is not extensive, even though topological considerations have been prominent in the recent economic literature on utility and preference, because it is doubtful if they are of direct significance for the research of psychologists. Our discussion is intended only to give the reader some feeling for the generality of topological methods without pretending to provide a really adequate systematic introduction to the subject. We assume no prior knowledge of topology, but we do suppose that the reader is familiar with the notion of continuity that is defined in textbooks on the calculus.

Intuitively, a function from real numbers to real numbers is continuous if its graph has no jumps or gaps in it. The rigorous definition that makes precise this intuition is easily extended to functions from n -dimensional vectors of real numbers to the real numbers. However, when we consider functions defined over arbitrary sets, there is no immediate method of

defining continuity, indeed, unless some additional structure is imposed on the set, nothing can be done. The type of structure that permits us to formulate a general definition of continuity is known as a topology.

The most important property of continuous functions is that they map points that are "near" one another into points that are again "near" one another. Put another way, the function does not create too much scrambling of points that are reasonably close to each other. In talking about functions of real numbers, this idea is adequately captured by requiring that intervals be mapped *into* intervals. Let us restrict ourselves to the open intervals, that is, to intervals that do not include their endpoints. Then the natural topology of the real line is the family of open intervals together with the sets that are formed from arbitrary unions and finite intersections of open intervals. Members of this family of sets are called open sets.

It is these last ideas that are generalized to define topologies for arbitrary sets.

Definition 2 A pair (X, \mathcal{T}) is a topological space if \mathcal{T} is a family of subsets of X , called open sets, such that

- (i) the empty set is in \mathcal{T} ,
- (ii) X is in \mathcal{T} ,
- (iii) the union of arbitrarily many sets in \mathcal{T} is also in \mathcal{T}
- (iv) the intersection of any finite number of sets in \mathcal{T} is in \mathcal{T}

We also say that \mathcal{T} is a topology for X .

Correspondingly, the ordinary ε - δ , that is, interval, definition of continuous functions is generalized to a condition in terms of open sets.

Definition 3 A function from one topological space into another is continuous if the inverse image of every open set is open. More precisely, let (X, \mathcal{T}) and (Y, \mathcal{U}) be topological spaces, and let f be a function from X into Y . Then f is \mathcal{T} - \mathcal{U} continuous if whenever $U \in \mathcal{U}$, then $f^{-1}(U) \in \mathcal{T}$.

Finally, we define the notion of separability of a topological space, for which it is useful first to introduce the notion of a base.

Definition 4 A base for the topology \mathcal{T} is a class \mathcal{B} of open sets (that is, members of \mathcal{T}) such that, for every x in X and every open set T containing x , there is a set B in \mathcal{B} such that $x \in B$ and $B \subseteq T$.

The usual topology of the real line defined above is determined by the requirement that the class of all open intervals be a base.

Definition 5 A space (X, \mathcal{T}) is called separable if \mathcal{T} has a countable base.

It is, of course, easy to construct a countable base for the real line and, in fact, for any n -dimensional Euclidean space.

In a sense the theory of ordinal utility functions would be flawed if the search for continuous functions required a topology on the set of objects

or outcomes that is extraneous to the preference relation itself. Fortunately, this is not the case. Any relation on a set generates a topology on the set in precisely the same way that the natural topology of the real line is generated from the open intervals. For a set A with relation R , we define a R -open interval to be any subset of A of the following form

$$(x, y) = \{z \mid z \in A \ \& \ x P z P y\},$$

where P is defined in terms of R as indicated earlier. The R -order topology for A is then the collection of sets generated from R -open intervals by arbitrary unions and finite intersections. It is apparent that the order topology on the set of real numbers generated by the relation \geq is the natural topology already defined in terms of open intervals.

Theorem 4 *Let R be a preference relation for the infinite set A , and let the R -order topology for A be separable. Then there exists on A a continuous, real-valued, order-preserving utility function.*

The proof of this theorem is too long to include here. A proof of essentially the same theorem is given by Debreu (1954), as far as we know, his paper contains the first discussion of the continuity of utility functions from a topological standpoint. For additional discussion see Chipman (1960a), Debreu (1963), Murakami (1961), and Newman and Read (1961).

Theorem 4 settles positively the first of the two questions raised at the beginning of this section. The second question concerned conditions that would guarantee the continuity of all the utility functions on A . It should be apparent that some condition beyond Eq. 1, that is, beyond the order-preserving property, must be imposed in order to obtain this result. The matter does not seem to have been discussed in the literature as extensively as is desirable, but additional sufficient conditions can be formulated by requiring that the utility function preserve the structure of the R -order topology for the set A .

*2.3 Additivity Assumptions

In the latter part of the nineteenth century, the economists Jevons, Walras, and, to a certain extent, also Marshall made the assumption that the utility of a commodity bundle (x_1, \dots, x_n) is just the sum of the utilities of the individual components, that is,

$$u(x_1, \dots, x_n) = u_1(x_1) + \dots + u_n(x_n) \quad (2)$$

This assumption was made in the context of the development of consumer behavior until Edgeworth, in his *Mathematical Psychics* (1881), noticed

that for the purposes of the economic theory of the time there was no need to assume Eq 2 as well as little justification for doing so. A few years later Pareto (1906) took the next step and observed that only ordinal assumptions were required. The upshot of this evolution of ideas in economics is that additivity assumptions have not been much investigated or used in the economic literature.

On the other hand, the additivity assumption represented by Eq 2 does seem to hold promise for several kinds of psychological investigations of preference and choice. The only directly pertinent studies known to us are Adams and Fagot (1959), Debreu (1960a), Edwards (1954c), Fagot (1956), and Luce and Tukey (1964). Most of the things we shall discuss in this section are to be found in the works of Adams and Fagot, Debreu, and Luce and Tukey.

To begin with, an additive utility model may be a reasonable candidate for application in any choice situation in which a selection among multi-component alternatives is required. For example, the model might apply to an attitude scaling task in which subjects are asked to choose between political candidates varying on several dimensions, say, tax policy, foreign policy, and civil liberties legislation. It should be realized that even for small sets of alternatives, such as three or four political candidates, the additivity assumption is not automatically satisfied, and thus its assumption has immediate behavioral implications. As an example, let $A_1 = \{a, b\}$, $A_2 = \{x, y\}$, and let R be a preference relation on the Cartesian product $A_1 \times A_2$. Suppose that the strict preference ordering is

$$(a, x)P(a, y)P(b, y)P(b, x)$$

For this preference ordering it is apparent at once that there are no numerical functions u_1 and u_2 such that

$$u_1(a) + u_2(x) > u_1(a) + u_2(y)$$

$$u_1(b) + u_2(y) > u_1(b) + u_2(x),$$

for the first inequality implies $u_2(x) > u_2(y)$ and the second, $u_2(x) < u_2(y)$, which is impossible.

This simple counterexample to the existence of an additive utility function raises the problem of formulating a set of conditions on the observable choice relation R that will guarantee the existence of such a function. For reasons that are presented in more detail in Sec 2.4, this problem seems to be incapable of a completely satisfactory general solution. It is perhaps desirable to be a little more explicit about what we mean by a general solution. Given the data from a particular experiment it is, of course, possible to decide whether or not there exists an

additive utility function that represents the data in the sense of Eq 2 and that is order-preserving, because the problem for any particular case is just one of solving a finite set of linear homogeneous inequalities. Adams and Fagot, in fact, give one constructive procedure for solving the inequalities.

The general problem, on the other hand, is to impose structural conditions on the relation R which guarantee a solution. Transitivity is an example of a necessary but not sufficient structural condition. Another necessary structural condition that eliminates the counterexample given previously is the condition of *independence*:

$$\begin{aligned} &\text{If } (a, x)R(a, y), \text{ then } (b, x)R(b, y), \\ \text{and} & \\ &\text{if } (a, x)R(b, x), \text{ then } (a, y)R(b, y) \end{aligned} \quad (3)$$

It is easy to see that Eq 3 follows from Eqs 1 and 2. This is called the condition of independence because interaction between the components is ruled out. In the language of economists, the two components must be neither competitive nor complementary. Adams and Fagot give a counterexample in which each set A_1 and A_2 has three elements to show that the addition of Eq 3 is not sufficient to guarantee the existence of an additive utility function.

The structural conditions we have just discussed are *open* conditions in the sense that they apply to all sets of alternatives for which additivity holds. They do not postulate the existence of any special alternatives nor do they impose on the whole pattern of alternatives a condition that is sufficient but not necessary for the existence of an additive utility function. As we have already remarked, and as we shall indicate in more detail in Sec 2.4, there are deep reasons for the difficulty, if not impossibility, of giving a fixed finite list of open structural conditions that are sufficient to guarantee the existence of an additive utility function, even for two-component alternatives that are finite in number. However, Scott (1964) has given an infinite list of open conditions, in the form of a fairly simple scheme, which are jointly necessary and sufficient. His results are discussed in a slightly different context in Sec 2.4.

Once *open* conditions are abandoned, a relatively simple set of sufficient conditions for the existence of two dimensional additive utility functions can be given by the axioms of conjoint measurement formulated by Luce and Tukey (1964). Moreover, they can be readily generalized to handle any finite number of components. For simplicity, however, we consider only the two-component case.

We begin with a preference relation R on $A_1 \times A_2$ where A_1 and A_2 are (not necessarily disjoint) sets. The first two axioms are necessary open

conditions, namely,

- 1 Weak ordering R is a weak ordering of $A_1 \times A_2$
- 2 Cancellation For all $a, b, f \in A_1$ and $p, x, y \in A_2$, if $(a, p)R(f, y)$ and $(f, x)R(b, p)$, then $(a, x)R(b, y)$

The second axiom can be cast in an apparently simpler form by defining a new relation D on $A_1 \times A_2$ which corresponds to a comparison of differences instead of sums of components (see Sec 2.4), namely,

$$(a, x)D(b, y) \text{ if and only if } (a, y)R(b, x)$$

Then the cancellation axiom is simply the assertion that D is transitive

The third axiom is not open in that it postulates that there are adequate elements in A_1 and A_2 so that certain equations can always be solved

- 3 Solution of equations For any $a, b \in A_1$ and $x, y \in A_2$, there exist $f \in A_1$ and $p \in A_2$ such that

$$(a, x)I(f, y) \text{ and } (a, x)I(b, p)$$

From these three assumptions it is not difficult to show that the condition of independence, Eq 3, holds, and so the following induced orderings R_1 on A_1 and R_2 on A_2 are well defined

$$aR_1b \text{ if and only if, for } x \in A_2, (a, x)R(b, x),$$

$$xR_2y \text{ if and only if, for } a \in A_1, (a, x)R(a, y)$$

With these definitions, we may introduce the main constructive device used by Luce and Tukey in their proof and in terms of which they introduced their final axiom. A nontrivial dual standard sequence (dss) is any set of elements of the form $\{(a_i, x_i) \mid i \text{ any positive or negative integer or } 0\}$ for which

- (i) if $i \neq j$, then not $a_i I_1 a_j$ and not $x_i I_2 x_j$,
- (ii) for all i , $(a_i, x_i)I(a_{i+1}, x_{i-1})$,
- (iii) for all i , $(a_{i+1}, x_i)I(a_i, x_{i+1})$

The final axiom is

- 4 Archimedean For any nontrivial dss $\{(a_i, x_i)\}$ and for any $(b, y) \in A_1 \times A_2$, there exist integers m and n such that

$$(a_m, x_m)R(b, y)R(a_n, x_n)$$

With these four axioms, the following result can be proved

Under these five assumptions, Debreu showed that there exist real-valued continuous functions u_i on A_i such that for $a, b \in A_1$ and $x, y \in A_2$,

$$(a, x)R(b, y) \text{ if and only if } u_1(a) + u_2(x) \geq u_1(b) + u_2(y)$$

Moreover, $u_1 + u_2$ is unique up to positive linear transformations. By suitably generalizing these assumptions (only 4 is the least bit tricky), he was able to show that the obvious generalized representation holds for any finite number of components. The proof is lengthy and is not included here.

Note added in proof J. Aczel has pointed out to us that there are results in the mathematical literature that are closely related to the question of the existence of additive utility functions. These are classed under the theory of nets or, as it was called somewhat earlier, the theory of webs (in German, the language of many of the publications, Gewebe). Roughly, the problem treated there is this: given three families of (abstract) curves in a plane for which (1) each point belongs to exactly one curve in each family (and so two different curves of the same family do not intersect) and (2) curves from two different families have exactly one point in common, when is this configuration equivalent (in either an algebraic or a topological sense) to three families of parallel straight lines? The intersection assumptions, (1) and (2), are very similar to, but stronger than, the assumption that equations can be solved, Axiom 3 above. A necessary and, with these intersection assumptions, an essentially sufficient condition for such a mapping to exist is the cancellation axiom (Axiom 2 above) with equalities replacing the inequalities. In the mathematical literature, this important property is known as the Thomsen condition. For a detailed discussion of such results, and for further references to the literature, see Aczel (1964).

2.4 Higher Ordered Metrics

Coombs (1950) introduced the term *ordered metric* to designate those scales that generate an ordering on the set of alternatives and, in addition, a partial or stronger ordering on the differences between alternatives (utility intervals). In suggesting these ordered metric scales as something stronger than ordinal scales, but not as strong as interval scales, Coombs was not thinking of the classical theory of demand, where nothing stronger than ordinal scales is needed, but rather he had in mind problems of attitude scaling and in particular the development of methods for representing the attitudes of a number of different individuals on a common scale. Further development of ordered metric scales was given

in Coombs (1952), and a few years later direct applications to choice behavior were made by Adams and Fagot (1959), Fagot (1959), Hurst and Siegel (1956), Siegel (1956), and others.

If we speak in terms of the utility difference, or the difference in preference, between pairs of alternatives, then the classical objection of economists is that choices between alternatives do not yield behavioral evidence on these differences. One can, of course, follow the classical introspective methodology of attitude tests and simply ask subjects to make judgments of the differences. Although the method of introspection has no doubt been unduly deprecated in some of the modern literature on choice behavior, it is nonetheless natural to ask if its difficulties can be overcome by designing operational procedures that will elicit relatively direct behavioral evidence on judgments of utility differences. In Sec. 3 we discuss several methods for doing this in the context of uncertain outcomes, with choices expressed between gambles. Without introducing any element of risk, however, a technique can be used that is very close to the one underlying the additivity assumptions just discussed. This is simply to present two pairs of objects and to ask for a choice between the pairs. To translate the response into an expression of utility differences, we make the following obvious transformation. If

$$u(a) + u(b) \geq u(c) + u(d),$$

then clearly

$$u(a) - u(d) \geq u(c) - u(b),$$

and this last inequality is just a way of expressing utility differences. This parallels the correspondence between the R and D relations discussed in connection with the cancellation axiom in Sec. 2.3.

Two other methods for obtaining direct behavioral evidence on utility differences which do not require uncertain outcomes are described by Suppes and Winet (1955). One interpretation, which depends on equating utility differences with different amounts of money, depends only on the assumption that amount of money is a monotonic increasing function of utility difference. A linear relationship is not required. Of course, this method is restricted to alternatives that can be thought of as objects or commodities. Suppose that the subject prefers Commodity x to Commodity y and also prefers Commodity p to Commodity q . Suppose further that he has in his possession Commodities y and q . We then present him with the opportunity of paying money to replace y by x and q by p . The utility difference between x and y is at least as great as that between p and q if and only if he will pay at least as much money to replace y by x as to replace q by p . Let it be emphasized that there is nothing special about the use of money in this method. Any sort of object or activity that may be represented on a continuum is suitable. For example,

an alternative operational definition of utility differences which has exactly the same form can be given in terms of work. In economic terms, what is needed is simply a commodity that, like money, is flexible enough to serve in different situations and such that its marginal utility is either always positive or always negative in the situations under consideration.

A second method, which eliminates the need for a commodity or activity external to the set of alternatives, is a rather slight modification of the above procedure. Following an example of Suppes and Winet, suppose that we confront a housewife with six household appliances of approximately the same monetary value and that she does not have: a mixer, a toaster, an electrical broiler, a blender, a waffle iron, and a waxer. Two of the appliances are selected at random and presented to her. Suppose that they are the toaster and the waxer. She is then confronted with the choice of trading either the toaster for the waffle iron or the waxer for the blender. Presumably, she will exchange the toaster for the waffle iron if and only if the utility difference between the waffle iron and the toaster is at least as great as the difference between the blender and the waxer (due account being taken of the algebraic sign of the difference). A sequence of such exchanges, possibly with repetitions, can be easily devised in such a way that every utility difference is compared to every other. Unfortunately, however, we know of no experiments that have attempted to apply either of these methods to the measurement of utility differences.

The formal characterization of various kinds of higher ordered metric scales is a somewhat complicated matter, and we shall not present a complete survey here. Fortunately, much of the discussion of these matters has already been covered in Chapter 1 of Vol. I, particularly in Secs. 3.3 and 3.4.

Coombs' original concept of an ordered metric scale as a simple ordering of alternatives and a partial ordering of differences between alternatives is formulated precisely by the following axioms. It is, however, convenient to replace the concept of a partial ordering by that of a quasi-ordering on the differences of alternatives (a quasi-ordering is a transitive and reflexive relation, as expressed in the first two axioms).

Definition 6 *A quaternary relation D on a set A is a weak higher ordered metric if and only if the following five axioms are satisfied for every a, b, c and d in A*

1. *if $abDcd$ and $cdDef$, then $abDef$,*
2. *$abDab$,*
3. *if $abDcd$, then $acDbd$,*
4. *if $abDcd$, then $dcDBa$,*
5. *$abDbb$ or $baDbb$*

The intended interpretation of the quaternary relation D is that $abDcd$ if and only if $u(a) - u(b) \geq u(c) - u(d)$. In terms of this intended numerical interpretation, the meanings of Axioms 3 and 4 should be clear. Axiom 3 expresses what has been called in the literature the quadruple condition, initially formulated by Marschak (first stated in print, as far as we know, in Davidson and Marschak, 1959, also see Sec. 5 4), and Axiom 4 expresses a natural condition on the sign of the differences. Axiom 5 states a restricted connectivity condition needed to obtain a weak ordering on alternatives as opposed to differences between alternatives. To show how a complete ordering may be obtained from these axioms on alternatives, we define the weak preference relation R as follows.

Definition 7 aRb if and only if $abDbb$.

The intended interpretation, of course, is that aRb just when a is weakly preferred to b . In terms of the desired numerical interpretation, aRb if and only if $u(a) \geq u(b)$. We now prove

Theorem 6 R is a weak ordering.

PROOF. (1) If aRb and bRc , then aRc .

By hypothesis aRb and bRc . In terms of the definition, this means that $abDbb$ and $bcDcc$, from Axiom 2 we have $bcDbc$ and so, by Axiom 3, $bbDcc$; from Axiom 4 and one of the hypotheses we have $ccDcb$; and thus by transitivity (Axiom 1) $abDcb$. However, by Axiom 3 this means that $acDbb$. Once again using the fact that $bbDcc$ and transitivity, we obtain $acDcc$ which, by virtue of the definition of R , is equivalent to aRc , as desired.

(2) aRb or bRa .

From Axiom 5 we have $abDbb$ or $baDbb$. If $abDbb$, then aRb from the definition of R . On the other hand, if $baDbb$, then from Axioms 2 and 3 we also have $bbDaa$ and thus by transitivity $baDaa$, and therefore by definition bRa . This concludes the proof.

Additional elementary theorems can be proved, but it is not possible to go on to prove that there exists a real-valued function u defined on the set A such that

$$abDcd \text{ if and only if } u(a) - u(b) \geq u(c) - u(d). \quad (4)$$

An immediate reason is that if such a function u were to exist, then we would necessarily have a weak ordering of differences between alternatives, but our axioms only guarantee a quasi-ordering. The implication of not having such a function u is that we cannot use numerical techniques in working with a weak ordered metric. It is necessary to restrict ourselves to elementary properties of the quaternary difference relation D and of the ordering relation R , thereby severely limiting the possibility of connecting

a weak higher ordered metric to other parts of an analysis of choice behavior

It seems a natural move, therefore, to go from a weak higher ordered metric to a strong higher ordered metric by requiring that the quaternary difference relation itself be connected. This is expressed in the following definition

Definition 8 *Let D be a weak higher ordered metric on the set A . Then D is a strong higher ordered metric if for every a, b, c , and d in A , the axiom of strong connectivity holds, that is, $abDcd$ or $cdDab$*

This last higher ordered metric is essentially the one applied by Hurst and Siegel (1956) and Siegel (1956). These experiments are discussed in Sec. 4.1

Unfortunately, even with its strong conditions, it is not possible to show that there exists a numerical representing function u preserving the structure of the difference relation D of a higher ordered metric. An example showing this was given by Scott and Suppes (1958). Moreover, by extending their initial example, they showed that no finite list of axioms that involve only open conditions (as defined in Sec. 2.3), that is, that do not involve existential assertions, are sufficient to guarantee the existence of a numerical representing function. From a psychological standpoint, this means that the structural conditions on behavior required to strengthen a strong higher ordered metric sufficiently to obtain a numerical representation and yet not sufficiently strong to guarantee an interval scale are quite complicated in character.

Because of the close connection between additivity assumptions and conditions on a difference relation, it should also be apparent that the same negative results apply a fortiori to the imposition of a fixed finite number of open structural conditions to guarantee the existence of an additive utility function.

Scott (1964) has, however, shown that an open schema that represents a bundle of axioms whose number increase with the cardinality of the set of alternatives is adequate.

Definition 9 *A quaternary relation D on a set A is a strict higher ordered metric if and only if the following axioms are satisfied*

- 1 *If $abDcd$, then $acDbd$*
- 2 *$abDcd$ or $cdDab$*
- 3 *for all sequences of elements $a_0, \dots, a_n, b_0, \dots, b_n \in A$, and all permutations π, σ of $\{0, 1, \dots, n\}$, if $a_i b_i Da_{\pi(i)} b_{\sigma(i)}$ for $0 \leq i < n$, then $a_{\pi(n)} b_{\sigma(n)} Da_n b_n$*

To make the meaning of Axiom Scheme 3 clearer, we may derive the transitivity of D . To apply Condition 3, it is convenient to formulate

transitivity as follows if $a_0b_0Da_1b_1$ and $a_1b_1Da_2b_2$, then $a_0b_0Da_2b_2$. Now let π and σ be the permutation of $\{0, 1, 2\}$ such that $\pi(0) = \sigma(0) = 1$, $\pi(1) = \sigma(1) = 2$ and $\pi(2) = \sigma(2) = 0$. Application of Condition 3 for this π and σ yields the desired result at once.

What Axiom Scheme 3 really comprises is all possible cancellation laws, some simple examples of which were discussed in Sec. 2.3. The difficulty with this axiom from a psychological standpoint is that there seems to be no simple way of summarizing what it says about choice behavior, but this we take to be an inherent complexity of the structural relations that must hold between elements of any finite set in order to guarantee the existence of a utility function that preserves the order of utility differences.

Scott established the following theorem.

Theorem 7 *A necessary and sufficient condition that a quaternary relation D on a finite set A has a representing utility function in the sense of Eq. 4 is that D be a strict higher ordered metric on A .*

The relation between strict and strong higher ordered metrics immediately follows from Theorem 7.

Theorem 8 *Every strict higher ordered metric D on a finite set A is also a strong higher ordered metric on A .*

The many conveniences that follow from having a numerical representation have led to the investigation of several methods of extending strong higher ordered metric scales to guarantee the existence of a numerical representing function. One procedure, that of Suppes and Winet (1955), is to add strong conditions that require the set of alternatives to be infinite. This amounts to adding to the axioms for strong higher ordered metrics those for infinite difference systems (Axioms 5, 6, and 7) as stated in Chapter 1 of Vol. I, p. 35. Because a detailed discussion of infinite difference structures is given there, we do not pursue them further here.

A second alternative is to build up on the concept of a finite equal difference system (Def. 16, Chapter I, Vol. I, p. 39). The intuitive idea of such an equal difference system is to keep the set of alternatives finite, but to impose the strong structural condition that alternatives be equally spaced in terms of utility differences. This means that the total structure is analogous to a single dual standard sequence in the sense of Luce and Tukey. The assumption that alternatives are equally spaced is, in general, much too special and restrictive for applications, but if we can assume the availability of a standard sequence of equally spaced alternatives, we can then use them to provide a scale for the approximate measurement of the utility of other alternatives.

The axioms for the equally spaced alternatives are extremely simple, namely, those for strong higher ordered metrics plus the assumption needed to guarantee equal spacing. Because of their simplicity, we

include them here. In order to formulate the additional axiom for the standard sequence, we need the notion that one element in the standard sequence is an immediate R -successor of another. Denoting by S the subset that is the standard sequence of alternatives, the definition of the immediate successor relation J for S is just the following

Definition 10 aJb if and only if aPb and for all c in S aPc implies bRc .

To define a finite difference system with a standard sequence, we impose only the additional assumption that any alternative is bounded from above and below by members of the standard sequence.

Definition 11 A finite difference system with a standard sequence is a triple $\mathcal{A} = \langle A, S, D \rangle$ in which the following axioms are satisfied.

- 1 the set A is finite and S is a subset of A ,
- 2 the quaternary relation D is a strong higher ordered metric on A ,
- 3 for a, b, c , and d in the standard sequence S , if aJb and cJd , then $abDcd$ and $cdDab$,
- 4 for every b in A , there are alternatives a and c in the standard sequence S such that aRb and bRc

The full proof of the following theorem, although rather long, is entirely elementary, and therefore we shall omit it (see Suppes, 1957, pp 267-274).

Theorem 9 If $\mathcal{A} = \langle A, S, D \rangle$ is a finite difference system with a standard sequence, then there is a real-valued function u defined on A such that

(i) for all a, b in A ,

aRb if and only if $u(a) \geq u(b)$,

(ii) for all a, b, c , and d in the standard sequence S ,

$abDcd$ if and only if $u(a) - u(b) \geq u(c) - u(d)$.

Moreover, any function u' satisfying (i) and (ii) is related to u by a linear transformation on S . Let u' be related to u on the set S by the transformation $u'(a) = \alpha u(a) + \beta$, for every a in S , and for any a and c in S such that aJc define the u -unit by

$$u(a) - u(c) = \Delta_u$$

Then for any b in A , u and u' are related as follows

$$\left| u(b) - \frac{u'(b) - \beta}{\alpha} \right| < \Delta_u$$

This theorem shows that the utility of alternatives in the standard sequence S are measured exactly on an interval scale. For alternatives not in the standard sequence, measurements are made to within the

accuracy of the unit interval Δ_n between adjacent members of the standard sequence. Approximations in terms of a standard sequence, but based on a somewhat different rationale, were used experimentally by Davidson, Suppes, and Siegel (1957), however, the alternatives were bets involving uncertain outcomes. The discussion of these results is given in Sec. 4.2.

We began this section on ordered metrics by mentioning Coombs' 1950 article, but as yet we have not mentioned a second important idea contained in that article. This is Coomb's concept of an unfolding technique which is designed to place both individuals and alternatives on the same scale. We may think of the position of the individual on the scale as his ideal utility point. A good example of the usefulness of this kind of model is in the problem faced by a voter when choosing between less than perfect candidates. In many situations he finds that some of the candidates are politically to the left and the others to the right of his own "ideal" position. Presumably he should vote for the one nearest to his own position, whether to the left or right of his ideal.

To sketch how these ideas may be formalized, let x be an individual and a and b two alternatives, and let $T(x, a, b)$ be the relation of x preferring a to b . We then want a utility function u to satisfy the following condition:

$$T(x, a, b) \text{ if and only if } |u(x) - u(a)| \leq |u(x) - u(b)|$$

The problem of formulating conditions on the relation T to guarantee the existence of such a utility function u is closely connected to the corresponding problem for weak, strong, and strict higher ordered metrics, and so we do not pursue it further here. Some additional remarks on the formal problem can be found in Sec. 3.6 of Chapter 1.

2.5 JND Assumptions

The algebraic choice theories discussed thus far all assume that the individual makes such a clear and definite judgment of preference that the relation of indifference is transitive, and so is an equivalence relation. The challenge to this assumption is familiar from psychophysics (see, for example, the discussion in Chapters 3 and 4, Vol. I). Economists have offered similar objections to classical utility theory. A relatively early discussion of these matters is found in Armstrong (1939, 1948, and 1951) who argues, plausibly enough, that a given alternative a may be indifferent to a second alternative b , b may be indifferent to a third alternative c , and yet a is preferred to c .

From a logical standpoint, the first and most natural question to ask

about nontransitive indifference relations concerns suitable axiomatic generalizations of simple orderings. This was the concern in Luce (1956), where the concept of a semiorder was introduced as a natural generalization of the familiar concept of a simple ordering. Luce originally stated the axioms in terms of two binary relations, one of preference and one of indifference. In Scott and Suppes (1958), the axioms were simplified and only a single binary relation of preference was used. Their axioms are stated in the following definition.

Definition 12 *A semiorder is a binary relation P on a set A that satisfies the following three axioms for all a, b, c , and d in A :*

- 1 *not aPa ,*
- 2 *if aPb and cPd , then either aPd or cPb ,*
- 3 *if aPb and bPc , then either aPd or dPc .*

In Luce (1956) a jnd function is introduced which varies with the individual elements of A , that is, the jnd function is defined on A . Intuitively it would be desirable if Luce's results were the best possible for semiorders. Unfortunately, they may be strengthened (see Scott and Suppes, 1958) to show that a numerical interpretation of P can be found which has as a consequence that the jnd function is constant for all elements of A . In particular, the following theorem may be proved (see p. 32, Chapter 1, Vol. I for the proof).

Theorem 10 *Let the binary relation P be a semiorder on the finite set A . Then there exists a real-valued function u on A such that for every a and b in A*

$$aPb \text{ if and only if } u(a) > u(b) + 1.$$

It should be noted that this theorem is restricted to finite sets of alternatives. As is evident from our earlier discussion, additional axioms are needed in order to prove such a representation theorem for infinite sets. As far as we know, no substantial investigation of semiorders for infinite sets has been made.

Matters become much more complicated if we admit subliminal differences and at the same time attempt to obtain a numerical representation stronger than an ordinal scale. Such a set of axioms was given by Gerlach (1957). To obtain more powerful numerical results she followed a proposal that originated with Wiener (1921) and introduced a four-place relation that has the following interpretation. The relation $abLcd$ holds whenever the subjective difference between a and b is algebraically sufficiently greater than the subjective difference between c and d . Gerlach's axioms on L are sufficiently strong to permit her to prove that there is a real-valued function u defined on the set of alternatives and a jnd measure Δ (that is, Δ is a real

number) such that $abLcd$ holds if and only if either

- (i) $|u(c) - u(d)| < \Delta$ and $u(a) - u(b) \geq \Delta$, or
- (ii) $|u(a) - u(b)| < \Delta$ and $u(d) - u(c) \geq \Delta$, or
- (iii) $|u(a) - u(b)| \geq \Delta$ and $|u(c) - u(d)| \geq \Delta$ and
 $[u(a) - u(b)] - [u(c) - u(d)] \geq \Delta$

Note that Condition (i) corresponds to the case when c and d are separated by less than a jnd, and a and b are separated by at least a jnd, Condition (ii) reverses these relations, and Condition (iii) covers the case when a and b as well as c and d are separated by at least a jnd. Moreover, Mrs Gerlach proved that the numerical function u is unique up to a linear transformation, with, of course, the just noticeable difference Δ being transformed not by the entire linear transformation, but only by the multiplicative part of the transformation. We do not give here the rather complicated axioms formulated by Mrs Gerlach.

A recent extensive study of jnd structures, building on the earlier work of Luce (1956), can be found in Adams (1963). Roughly speaking, Adams added to Luce's ordinal concept of a semiorder an operation of combination, with particular reference to weighing with a balance. Although most of his detailed results are restricted to this additive case, his methods of attack are of more general interest and could be applied to the higher ordered metrics discussed in Sec 2.4. Adams showed that no matter how insensitive a balance is, it is always possible, on the basis of a finite number of comparisons on the pans of the balance, to determine the weight of any object to an arbitrary degree of accuracy. Of course, for a fixed set of objects and a fixed number of observations the accuracy has a fixed limit. The primary limitation in applying his methods to psychological studies of preference is the absence of any probabilistic considerations of the sort surveyed in Secs 5 to 8 of this chapter.

3 ALGEBRAIC CHOICE THEORIES FOR UNCERTAIN OUTCOMES

3.1 The Expected Utility Hypothesis

The ordinal theory of Pareto, which dominated the economic theory of utility from the beginning of this century until the publication of von Neumann and Morgenstern's treatise on the theory of games in 1944, rested squarely on the assumption that the individual in choosing among alternatives has no uncertainty about the consequences of these alternatives. Once uncertainty in the consequences is admitted, no ordinal

theory of choice can be satisfactory. The simplest sorts of examples suffice to make this fact clear. Consider an individual who has five dollars that he is thinking about betting on the outcome of a throw of a die. Suppose that if he makes the bet and the die comes up either one or three, he will receive nine dollars, whereas, if the die comes up two, four, five, or six, he will lose his initial five dollars. His problem is to decide whether to bet the five dollars or not. Granting the ordinal assumption that the individual surely prefers more money to less—in this case, that he prefers nine to five to zero dollars—does not carry the individual very far in deciding whether or not to place the bet. Only slight reflection makes it clear that this rather artificial example of deciding whether or not to place a bet is but one of many examples, some of which are quite serious, in which decisions must be made in the face of inherently risky outcomes. One of the most common examples, one that almost all (middle and upper class) individuals in our society face, is the variety of decisions about what and how much insurance to carry. A typical middle-class member of our society now has insurance coverage for automobile collision and liability, his own death, destruction of his house by fire, loss of possessions by theft, medical and hospital insurance, and possibly accident and income insurance. Every decision to invest in such an insurance policy involves a choice in the face of uncertain outcomes, for the individual does not know what the present state of affairs will lead to in the way of future consequences for him. Insurance is a way of taking a decision against suffering a financial disaster, either to himself or to his family, in case a personal catastrophe does occur.

The expected utility hypothesis, which probably was first clearly formulated by Daniel Bernoulli (1738), is the most important approach that has yet been suggested for making decisions in the context of uncertain outcomes. The fundamental idea is exceedingly simple. The individual must make a decision from among several possible alternatives. The possible decisions may have a variety of consequences, and ordinarily the consequences are not simply determined by the decision taken but are also affected by the present state of affairs (also called the *state of nature*). We suppose that the subject has a utility function on the possible consequences and that he has a probability function on the possible states of nature. According to the expected utility hypothesis, the wise decision maker selects a decision or course of action that maximizes his expectation.

It is perhaps useful to illustrate this important complex of ideas by considering a simple example such as the decision of whether or not to go to a football game in uncertain weather. Let the set S of states of nature have as members the two possible states of raining, s_1 , or not raining, s_2 , during the game. Let the set C of possible consequences be those of being

at the game and not being rained on, c_1 , staying home, c_2 , and being at the game and being rained on, c_3 . The two decisions are going to the game, d_1 , and not going to the game, d_2 . Formally, d_1 and d_2 are functions from S to C such that $d_1(s_1) = c_3$, $d_1(s_2) = c_1$, $d_2(s_1) = d_2(s_2) = c_2$. Suppose now that the individual assigns a subjective probability of $\frac{1}{3}$ to s_1 and $\frac{2}{3}$ to s_2 , and that he prefers consequence c_1 to c_2 to c_3 . It should be evident, as we remarked earlier, that such merely ordinal preferences are insufficient to lead to a rational decision between d_1 and d_2 . Let him, however, also assign numerical values to the consequences, in particular, let his utility function u be such that $u(c_1) = 12$, $u(c_2) = 6$, $u(c_3) = -9$ (and we suppose u is unique up to a choice of unit and zero). Then the expected utility hypothesis invokes him to compute the expectation in the ordinary sense of random variables for both d_1 and d_2 , using the numerical utility function to define the values of the random variables, and then to choose the decision that has the greater expectation. He finds that

$$E(d_1) = \frac{1}{3}(-9) + \frac{2}{3}(12) = 5$$

$$E(d_2) = \frac{1}{3}(6) + \frac{2}{3}(6) = 6,$$

and so he should elect not to go to the game, d_2 .

A central problem for normative or descriptive behavior theory is to state axioms on behavior that lead to a numerical representation of utility and probability so that decisions are based on a maximization of expected utility. In the next subsection, 3.2, we consider axiom systems of this sort which assume in their statement the existence of numerical probabilities. In Sec. 3.3 we widen the context by considering axiom systems that also impose behavioral assumptions on probability, that, in essence, treat probability as a subjective degree of belief. In Sec. 3.4 we survey briefly various decision principles that have been proposed as alternatives to the expected utility hypothesis.

Before we turn to detailed analyses of behavioral assumptions that yield the expected utility hypothesis, it is natural to ask just how much beyond ordinal requirements are needed to sustain the expected utility hypothesis analysis. From the remarks made in Sec. 2.4 about representation problems for higher ordered metrics, it should be evident that the situation is not simple. Suppose, for simplicity at the moment, that we have only a finite number of states of nature. We denote by s_i the numerical probability (objective or subjective) of state i and by u_{di} the numerical utility of the consequence that results from taking decision d when state i is the true state of nature. We then express the notion that the expectation of decision d is greater than that of decision e by the following inequality

$$\sum_i s_i u_{di} > \sum_i s_i u_{ei}$$

It is simple enough to construct counterexamples to show that probability and utility must be measured on scales stronger than ordinal ones in order to preserve this inequality under admissible transformations. On the other hand, when the set of states of nature is finite and no additional assumptions are made, we cannot prove that for given decisions d and e utility must be measured on an interval scale, even if we assume that numerical probability measures are already given. For present purposes, there are really not any natural groups of transformations lying between the group of monotone increasing transformations (ordinal scales) and the group of positive linear transformations (interval scales) that permit a precise analysis of the uniqueness aspect of the expected utility hypothesis. Consequently, as we shall see, the various axiom systems developed to represent the expected utility hypothesis end up with the (unnecessarily restrictive) result that utility is measured on an interval scale. This is done, of course, by imposing rather strong structural conditions of an existential character on either the set of states of nature or on the set of consequences, for quite strong assumptions are necessary in order to show that utility *must* be measured on an interval scale.

3.2 Axiom Systems That Assume Numerical Probability

In this title we have deliberately said "numerical probability" instead of "objective probability," because the axiom systems we shall consider can be interpreted in terms of subjective probability provided that we assume a prior numerical measurement of subjective probability that satisfies the ordinary probability axioms. The scheme for measuring utility by using numerical probability originates with von Neumann and Morgenstern (1944). Because of its considerable historical interest and its essential simplicity, we state with only trivial modifications the original von Neumann-Morgenstern axiom system. Numerical probabilities enter into the statement of the axioms in the following essential way. If we have two alternatives, x and y , then we may consider the new alternative consisting of alternative x with probability α and alternative y with probability $1 - \alpha$. We denote this new alternative by $\alpha x y$, which expression is often called the α mixture of x and y . Once such mixtures are available, we may ask the subject for his preferences not only among pure alternatives but also among probability mixtures of alternatives. Thus we might ask if he prefers the pure alternative z to an α mixture of x and y , that is, to receiving x with probability α and y with probability $1 - \alpha$. It is important to realize, of course, that x and y are not numbers and that the juxtaposition " $\alpha x y$ " does not signify multiplication. The expression denotes a single

ternary operation, say h , and postulates are then required to express its properties. To make this point quite explicit, we could write in place of " xy " a more explicit functional notation such as $h(x, \alpha, y)$. In terms of this notation, the following Axioms 3 and 4 would be expressed as

$$h(x, \alpha, y) = h(y, 1 - \alpha, x),$$

$$h[h(x, \alpha, y), \beta, y] = h(x, \alpha\beta, y)$$

The von Neumann-Morgenstern axioms are incorporated in the following definition

Definition 13 *A triple $\mathcal{A} = \langle A, R, h \rangle$ is a von Neumann-Morgenstern system of utility if and only if the following axioms are satisfied for every x, y and z in A and every α and β in the open interval $(0, 1)$, where the operation h is denoted by juxtaposition*

- 1 R is a weak ordering of A ,
- 2 xy is in A
- 3 $xy = y(1 - \alpha)x$,
- 4 $(xy)\beta y = x\alpha\beta y$,
- 5 if xIy , then $xazIyaz$,
- 6 if xPy , then $xPxay$ and $xayPy$,
- 7 if xPy and yPz , then there is a γ in $(0, 1)$ such that $yPxyz$
- 8 if xPy and yPz , then there is a γ in $(0, 1)$ such that $xyzPy$

Axiom 1 just requires the familiar ordinal restriction that R be a weak ordering of A . The remaining axioms together impose much stronger requirements, which are difficult to satisfy exactly in practice. This difficulty, has in fact, been much of the source of inspiration for the subsequent developments considered in Secs. 5 to 8. The second axiom is simply a closure axiom requiring that if two alternatives are in the set A , then any probability mixture of them is also in the set of alternatives. It follows from this postulate, from the existence of $a, b \in A$ such that aPb , and from Axiom 1 that the set A is infinite. (In discussions of this sort we always implicitly require that A be nonempty and, here, we also require that there be at least two alternatives, one of which is strictly preferred to another.) The third and fourth axioms state simple assumptions about mixtures and their combinations. In more recent discussions, the properties expressed by Axioms 2, 3, and 4 are central to the characterization of what are called mixture spaces (see discussion later of the Herstein and Milnor axiomatization of utility). Notice that both Axioms 3 and 4 are trivially true if we interpret x and y as numbers. The fifth axiom asserts that if an individual is indifferent between x and y , then he is also indifferent between any probability mixture of x and z and the same

mixture of y and z . The sixth axiom asserts that if alternative x is strictly preferred to alternative y , then x is strictly preferred to any probability mixture of x and y , and any probability mixture of x and y is strictly preferred to y . The last two axioms state what amount to a continuity condition on preferences. If y is between x and z in strict preference, then Axiom 7 asserts that there is a probability mixture of x and z that is preferred to y , and Axiom 8 asserts that there is a probability mixture of x and z to which y is preferred.

The proof that these eight axioms are sufficient to guarantee the existence of a numerical utility function, unique up to a linear transformation, is rather tedious and is not given here. Readers are referred to the quite detailed proof in von Neumann and Morgenstern (1947). The precise theorem they established is the following:

Theorem 11 *Let $\mathcal{A} = \langle A, R, h \rangle$ be a von Neumann-Morgenstern system of utility. Then there exists a real-valued function u defined on A such that for every x and y in A and α in $(0, 1)$*

- (i) xRy if and only if $u(x) \geq u(y)$,
- (ii) $u(\alpha x) = \alpha u(x) + (1 - \alpha)u(y)$

Moreover, if u' is any other function satisfying (i) and (ii), then u' is related to u by a positive linear transformation.

The theorem just stated shows that the axioms are sufficient for the representation; it is also interesting to observe that they are necessary as well, once the concept of a mixture space is introduced. That is, once we require Axioms 2, 3, and 4, then the remaining axioms express necessary conditions as well as sufficient ones on any numerical utility function having the properties expressed in the theorem.

Since the publication of the von Neumann and Morgenstern axiomatization, there have been a number of modifications suggested in the literature and, more important, considerable simplifications of the proof originally given by von Neumann and Morgenstern. The intensive examination of this axiomatization is undoubtedly due to the considerable importance that has been attached to it by economists and also by mathematical statisticians. One of the first searching examinations was by Marschak (1950). Perhaps the most discussed subsequent axiomatization is that given by Herstein and Milnor (1953). The notation of Herstein and Milnor has been changed slightly to conform as much as possible to that used earlier in defining a von Neumann-Morgenstern system of utility. Their axioms are given in the following definition.

Definition 14 *A triple $\mathcal{A} = \langle A, R, h \rangle$ is a Herstein-Milnor system of utility if and only if the following axioms are satisfied for every x, y , and z*

in A and every α and β in the closed interval $[0, 1]$, where the operation h is denoted by juxtaposition

- 1 R is a weak ordering of A ,
- 2 $x\alpha y$ is in A ,
- 3 $x\alpha y = y(1 - \alpha)x$,
- 4 $(x\alpha y)\beta z = x\alpha\beta z$,
- 5 $x1y = x$,
- 6 if $x1y$, then $x\frac{1}{2}zy\frac{1}{2}z$,
- 7 the sets $\{\alpha \mid x\alpha yRz\}$ and $\{\alpha \mid zR x\alpha y\}$ are closed

The first four axioms of Herstein and Milnor, as formulated here, are precisely the same as the first four axioms of von Neumann and Morgenstern given earlier. Herstein and Milnor's fifth axiom differs because they permit probabilities to lie in the closed interval $[0, 1]$, rather than just in the open interval $(0, 1)$. A set A and a ternary operation satisfying Axioms 2 to 5 are said to constitute a *mixture-space*. The sixth axiom of Herstein and Milnor is just a weakening of von Neumann and Morgenstern's Axiom 5: from their sixth axiom one can prove Axiom 5 of the earlier definition. Finally, their seventh axiom, formulated in terms of closed sets, just imposes a condition of continuity, and it has essentially the same consequences as Axioms 7 and 8 of the earlier definition. It is pretty much a matter of mathematical taste whether one prefers the kind of elementary continuity formulation contained in the earlier definition or the very simple topological assumption made in Axiom 7 by Herstein and Milnor. Probably the main advantage of the topological version is that the resulting proof of the existence of a numerical utility function is more elegant.

Although we shall not give the complete Herstein-Milnor proof that there exists a numerical utility function for any system satisfying their axioms, we list their series of theorems and for the conceptually more important ones indicate the nature of the proof.

Theorem 12 *If x, y , and z are in A and xRy and yRz , then there exists a probability α such that $y1x\alpha z$.*

PROOF Let $T = \{\alpha \mid x\alpha zRy\}$. By virtue of Axiom 7, T is a closed subset of the unit interval $[0, 1]$. Moreover, since xRy by hypothesis and $x1z = x$ by Axiom 5, 1 is in T and thus T is not empty. Similarly, $W = \{\beta \mid yR x\beta z\}$ is closed and nonempty. By virtue of Axioms 1 and 2, every probability, that is, every number in the closed interval $[0, 1]$, is in either T or W , and therefore $T \cup W = [0, 1]$. Since the unit interval is topologically connected, that is, it cannot be decomposed into a union of closed, disjoint sets, it follows that $T \cap W$ is not empty, any element α in the intersection satisfies the requirement of the theorem.

Theorem 13 *If x and y are in A and $x1y$, then $x\alpha z1y\alpha z$.*

Note that Theorem 13 is Axiom 5 of Def. 13. This theorem shows that the weakened Axiom 6 of Def. 14 is adequate.

Theorem 14 *If xPy and if $0 < \alpha < 1$, then $x\alpha yPy$.*

This theorem is just the second part of Axiom 6 of Def. 13.

Theorem 15 *If xPy , then $x\alpha yPx\beta y$ if and only if $\alpha > \beta$*

Theorem 16 *All alternatives in A are indifferent or there are an infinite number of alternatives differing in preference.*

PROOF. If for some x and y in A , xPy , then by Theorem 15 and the density property of the real numbers there is an infinite number of preferentially distinct alternatives.

Theorem 12 is next strengthened to a uniqueness condition

Theorem 17 *If xPy and yPz , then there is a unique α such that $yIxz$.*

PROOF Use Theorem 15.

The following elementary theorem is not explicitly stated by Herstein and Milnor, but it is useful in establishing the "linearity" of the utility function, that is, property (ii) of Theorem 11

Theorem 18 $(x\beta y)\alpha(x\gamma y) = x[\alpha\beta + (1 - \alpha)\gamma]y$.

$$\begin{aligned}
 \text{PROOF} \quad x[\alpha\beta + (1 - \alpha)\gamma]y &= x\left[\frac{\alpha\beta}{\gamma} + (1 - \alpha)\right]\gamma y \\
 &= (x\gamma y)\left[\frac{\alpha\beta}{\gamma} + (1 - \alpha)\right]y \quad (\text{by Axiom 4}) \\
 &= y\left(1 - \frac{\alpha\beta}{\gamma} - 1 + \alpha\right)(x\gamma y) \quad (\text{by Axiom 3}) \\
 &= y\alpha\left(1 - \frac{\beta}{\gamma}\right)(x\gamma y) \\
 &= \left[y\left(1 - \frac{\beta}{\gamma}\right)(x\gamma y)\right]\alpha(x\gamma y) \quad (\text{by Axiom 4}) \\
 &= \left[(x\gamma y)\frac{\beta}{\gamma}y\right]\alpha(x\gamma y) \quad (\text{by Axiom 3}) \\
 &= (x\beta y)\alpha(x\gamma y) \quad (\text{by Axiom 4})
 \end{aligned}$$

Theorem 19 *Suppose xPy and define $S_{xy} = \{z \mid xRzRy\}$. Let f and g be two linear, order-preserving functions defined on S_{xy} such that for r_0 and r_1 in S_{xy} with r_1Pr_0 , we have $f(r_0) = g(r_0)$ and $f(r_1) = g(r_1)$. Then f is identical with g on S_{xy} .*

PROOF To say that f is linear and order-preserving on S_{xy} is just to say that for s and t in S_{xy}

$$f(sat) = \alpha f(s) + (1 - \alpha)f(t)$$

and

$$sRt \text{ if and only if } f(s) \geq f(t).$$

First, suppose $r_1 R z R r_0$. Then $z I r_1 \alpha r_0$ for some α by virtue of Theorem 12. Thus

$$\begin{aligned} f(z) &= \alpha f(r_1) + (1 - \alpha) f(r_0) \\ &= \alpha g(r_1) + (1 - \alpha) g(r_0) && \text{(by hypothesis)} \\ &= g(r_1 \alpha r_0) && \text{(by linearity)} \\ &= g(z). \end{aligned}$$

Suppose now that $z P r_1$. Then again by virtue of Theorem 12, for some α , $r_1 I z \alpha r_0$. Therefore

$$\begin{aligned} f(r_1) &= \alpha f(z) + (1 - \alpha) f(r_0) \\ &= g(r_1) \\ &= \alpha g(z) + (1 - \alpha) g(r_0), \end{aligned}$$

and since $z P r_1$, $\alpha > 0$ and therefore $f(z) = g(z)$. A similar proof holds for the remaining case of $r_0 P z$.

Theorem 20. *There exists a linear, order-preserving utility function on A , and this utility function is unique up to a positive linear transformation.*

PROOF. Suppose $x P y$ and define as in Theorem 19

$$S_{xy} = \{z \mid x R z R y\}.$$

By virtue of Theorem 17, for any z in S_{xy} there is a unique $\alpha_{xy}(z)$ such that

$$z I x \alpha_{xy}(z) y.$$

We choose now r_0 and r_1 with $r_1 P r_0$, and we keep them fixed for the rest of the proof. We consider only pairs x, y such that

$$r_0, r_1 \in S_{xy},$$

and for any $z \in S_{xy}$, we define

$$u(z) = \frac{\alpha_{xy}(z) - \alpha_{xy}(r_0)}{\alpha_{xy}(r_1) - \alpha_{xy}(r_0)}.$$

Now by virtue of Theorem 15, $\alpha_{xy}(z) > \alpha_{xy}(u)$ if and only if $z P u$, and thus

$$u(z) > u(u) \text{ if and only if } z P u.$$

The linearity of u is established by the following argument. Consider $z \beta u$. Then by Theorem 17

$$\begin{aligned} z I x z(z) y \\ u I x z(u) y, \end{aligned}$$

and therefore

$$z \beta u I [x z(z) y] \beta [x z(u) y],$$

and thus by Theorem 18

$$z\beta uIx[\beta\alpha(z) + (1 - \beta)x(u)]y$$

Now by definition,

$$\begin{aligned}\beta u(z) + (1 - \beta)u(w) &= \beta \left[\frac{\alpha(z) - \alpha(r_0)}{\alpha(r_1) - \alpha(r_0)} \right] + (1 - \beta) \left[\frac{\alpha(w) - \alpha(r_0)}{\alpha(r_1) - \alpha(r_0)} \right] \\ &= \frac{[\beta\alpha(z) + (1 - \beta)\alpha(w)] - \alpha(r_0)}{\alpha(r_1) - \alpha(r_0)} \\ &= u\{x[\beta\alpha(z) + (1 - \beta)\alpha(w)]y\} \quad (\text{by definition}) \\ &= u[z\beta w] \quad (\text{by above equivalence})\end{aligned}$$

Obviously $u(r_1) = 1$ and $u(r_0) = 0$, and in view of Theorem 19, the numerical value of u for any $z \in S_{xy}$ is independent of x and y . Thus u may be extended without ambiguity or inconsistency over the whole of A .

The proof that u is unique up to a positive linear transformation we leave as an exercise.

Generalizations of the von Neumann-Morgenstern approach to utility have been made in several directions. Hausner (1954) dropped the Archimedean properties, expressed by Axioms 7 and 8 of Def 13, and he obtained a representation in terms of vectors rather than real numbers (His axioms are not precisely the remaining Axioms, 1 to 6, of Def 13, but they are essentially equivalent.)

A different generalization has been pursued by Aumann (1962), who investigated the consequences of dropping the connectivity axiom which demands that any two alternatives be comparable in preference, that is, that one must be weakly preferred to the other. Aumann weakened Eq 1 of Sec 2.1 to the implication

$$\text{if } xRy, \text{ then } u(x) \geq u(y) \quad (5)$$

In addition to the axioms on a mixture space (Axioms 2 to 5 of Def 14), he postulated that

- (i) R is transitive and reflexive on the mixture space,
- (ii) if $0 < \alpha < 1$, then xRy if and only if, for all z , $\alpha xzRy\alpha z$,
- (iii) if αyPz for all $\alpha > 0$, then not xPy

Without assuming connectivity he proved that there exists a linear utility function satisfying Implication 5.

An earlier result of the same sort, but in the context of a finite set of alternatives for which subjective probability is also derived from behavioral assumptions, was given by Davidson, Suppes, and Siegel (1957, Chapter 4).

Still a third sort of generalization has been pursued by Pfanzagl (1959a, b). He introduced the general concept of a "metric" operation in an

ordered set and developed the consequences of having two such operations. Roughly speaking, a binary operation \circ is a metric operation in Pfanzagl's sense if it is monotonic, continuous, and bisymmetric in the ordering relation [it is bisymmetric if $(x \circ y) \circ (z \circ w) I (x \circ z) \circ (y \circ w)$]. Moreover, he showed that two metric operations \circ and $*$ on the same ordered set lead to scales identical up to linear transformations if the isometry-relation

$$(x \circ y) * (z \circ w) I (x * z) \circ (y * w)$$

holds for any elements x, y, z and w in the set. The mixture space operation $x \alpha y$ is a special case of his metric operation, and two distinct probabilities α and β satisfy the isometry-relation, that is,

$$(x \alpha y) \beta (z \alpha w) I (x \beta z) \alpha (y \beta w)$$

Among other things, to construct a utility scale unique up to a linear transformation it is sufficient to consider only mixtures with a fixed probability α . The reader is referred to Pfanzagl's monograph (1959a) for full details on these matters.

3.3 Axiom Systems for Subjective Probability

In this subsection we consider the most important direction in which the von Neumann-Morgenstern approach has been generalized, namely, the derivation of a numerical subjective probability function, as well as a utility function, from qualitative postulates on preferences among choices. In many situations in which an individual must decide or choose among alternatives, there is no quantitative measure of probability at hand, ready to be used to evaluate the expected utility of the various alternatives. Thus it is natural to try to extend the axiom systems considered in the previous section to include axioms on decisions that yield measures of both utility and probability, thereby extending the domain of applicability of the expected utility hypothesis. Because it is intended that such enlarged axiom systems should apply when a relative frequency characterization of probability is either not available or meaningful, it is customary in the literature to refer to the probability that is derived as *subjective probability* in contrast to the objective probability defined in terms of relative frequency. We shall say more about the subjective character of probability as we consider various proposals that have been made.

A little reflection on the problem of jointly axiomatizing structural conditions that are sufficient to yield measurement of both utility and subjective probability suggests two different ways to proceed. One is to attempt to state axioms in such a way that we obtain first a measure of

utility, which is then used to obtain a measure of subjective probability. The other approach proceeds in the reverse order—we state axioms that permit us first to obtain a measure of subjective probability, which is then used to measure utility along the line of argument described in the preceding subsection. The earliest approach (Ramsey, 1931)⁴ followed the first tack, that is, utility is measured first. Ramsey's essential idea was to find a chance event with subjective probability of one-half, then to use this event to determine the utilities of outcomes, and, finally, to apply the constructed utility function to measure the subjective probabilities of the states of nature.

Because this approach has been used extensively in choice experiments by a number of people (its experimental application originates with Davidson, Suppes, & Siegel, 1957), we describe it in greater detail. The first thing to make clear is that one can determine if a chance event has subjective probability $\frac{1}{2}$ without first having a quantitative measure of probability. Suppose a subject is choosing between the two options shown in the following matrix:

	Option 1	Option 2
E	a	c
\bar{E}	b	d

If the subject chooses Option 1 and if event E happens, then he receives outcome a . If, however, the complementary event \bar{E} happens, he receives outcome b . On the other hand, if he chooses Option 2 and event E occurs, then he receives outcome c , whereas, if \bar{E} occurs, he receives d . If the subject chooses Option 1 over Option 2, and if we had a subjective probability function s and a utility function u with the (subjective) expected utility property, then we would express his choice by the following inequality:

$$s(E)u(a) + s(\bar{E})u(b) \geq s(E)u(c) + s(\bar{E})u(d) \quad (6)$$

If the subject were indifferent between the options, the inequality would become the following equality:

$$s(E)u(a) + s(\bar{E})u(b) = s(E)u(c) + s(\bar{E})u(d) \quad (7)$$

The notion of indifference is critical to Ramsey's theory.

We now attempt to find an E^* such that for every pair of alternatives a and b we have

$$s(E^*)u(a) + s(\bar{E}^*)u(b) = s(E^*)u(b) + s(\bar{E}^*)u(a) \quad (8)$$

⁴ In actual fact Ramsey's two essays on this matter were written in 1926 and 1928, but they were not published until after his death in 1931.

We shall not go more deeply into the formal aspects of the Ramsey approach, partly because the axioms are very similar in spirit to those discussed in Sec 2.4 dealing with utility differences. Readers are referred to Davidson and Suppes (1956) and Suppes (1956).

The approach that begins with a consideration of probability rather than utility originated with de Finetti, a comprehensive statement of this viewpoint is found in de Finetti (1937), which includes extensive references to his earlier works on the foundations of probability. The most important recent work on these matters is Savage's (1954) book, which extended de Finetti's ideas, in particular by paying greater attention to the behavioral aspects of decisions, although from the standpoint of experimental psychology Savage's approach is still very far from a thorough-going behavioral one. Six relevant articles, including an English translation of de Finetti (1937), have been reprinted in Kyburg and Smokler (1964).

We do not discuss here the many important ideas of de Finetti and Savage concerning the foundations of probability and statistics, but we do want to emphasize those ideas that seem especially important for the psychological theory of preference and choice.

Perhaps the best place to begin is with de Finetti's axioms for qualitative probability. In spirit, these axioms are similar to those we considered earlier for higher ordered metrics. Suppose we ask our subject to tell us for a variety of pairs of events which of each pair he believes to be the more probable. The question then arises: how complicated must the conditions be on the qualitative relation *more probable than* in order to obtain a numerical probability measure over events?

The question to be asked of our subject has been formulated in a non-behavioral fashion, however, it is easy to give a behavioral method that leads to the kinds of responses we wish to obtain. Suppose we want behavioral evidence as to whether the subject thinks event E is more or less probable than event F . Consider two options with outcomes a , b , or c and assume that a is preferred to b . For example, outcome a might be winning five cents, b losing five cents, and c winning or losing nothing. The matrix of the game we present him has the following form

	Option 1	Option 2
E	a	b
F	b	a
$\widetilde{E \cup F}$	c	c

where $\widetilde{E \cup F}$ is the event that neither E nor F occurs. Thus our three events are exhaustive, although not necessarily mutually exclusive.

Suppose that the subject chooses Option 1. Under the hypothesis that he is maximizing subjective utility we have the following inequality

$$s(E)u(a) + s(F)u(b) + s(\widetilde{E \cup F})u(c) \\ \geq s(E)u(b) + s(F)u(a) + s(\widetilde{E \cup F})u(c) \quad (11)$$

On the assumption that a is preferred to b , that is that $u(a) > u(b)$, it follows immediately from Eq. 11 that

$$s(E) \geq s(F)$$

Thus from behavioral data on choices between options and the hypothesis that subjects are choosing so as to maximize expected utility, we obtain inequalities on subjective probability that are not based on introspective data.

A discussion of experiments whose objective has been the measurement of subjective probability is reserved until Sec. 4.3.

Corresponding to the analysis of various higher ordered metrics in Sec. 2.4 it is natural to ask what formal requirements must be placed on the qualitative relation *more probable than* in order to guarantee the existence of a probability measure that reflects the order structure of the relation.

Let \geq be the relation of (weakly) *more probable than*. For the formal statement of the axioms, it is convenient to assume that the relation \geq holds between events that are subsets of a given sample space X . In other words we use the usual set theoretical notions for representing events.

Definition 15 A pair (X, \geq) is a qualitative probability structure if the following axioms are satisfied for all subsets A, B and C of X :

- 1 if $A \geq B$ and $B \geq C$, then $A \geq C$
- 2 $A \geq B$ or $B \geq A$
- 3 if $A \cap C = \phi$ and $B \cap C = \phi$ then $A \geq B$ if and only if $A \cup C \geq B \cup C$
- 4 $A \geq \phi$
- 5 not $\phi \geq X$

The first two axioms just assert that \geq is a weak ordering of the subsets of X . The third axiom formulates in qualitative terms the important and essential principle of additivity of mutually exclusive events. The fourth axiom says that any event is (weakly) more probable than the impossible event and the fifth that the certain event is strictly more probable than the impossible event. Defining the strict relation $>$ in the customary fashion,

$$A > B \text{ if and only if not } B \geq A.$$

We shall not go more deeply into the formal aspects of the Ramsey approach, partly because the axioms are very similar in spirit to those discussed in Sec 2.4 dealing with utility differences. Readers are referred to Davidson and Suppes (1956) and Suppes (1956).

The approach that begins with a consideration of probability rather than utility originated with de Finetti, a comprehensive statement of this viewpoint is found in de Finetti (1937), which includes extensive references to his earlier works on the foundations of probability. The most important recent work on these matters is Savage's (1954) book, which extended de Finetti's ideas, in particular by paying greater attention to the behavioral aspects of decisions, although from the standpoint of experimental psychology Savage's approach is still very far from a thorough-going behavioral one. Six relevant articles, including an English translation of de Finetti (1937), have been reprinted in Kyburg and Smokler (1964).

We do not discuss here the many important ideas of de Finetti and Savage concerning the foundations of probability and statistics, but we do want to emphasize those ideas that seem especially important for the psychological theory of preference and choice.

Perhaps the best place to begin is with de Finetti's axioms for qualitative probability. In spirit, these axioms are similar to those we considered earlier for higher ordered metrics. Suppose we ask our subject to tell us for a variety of pairs of events which of each pair he believes to be the more probable. The question then arises: how complicated must the conditions be on the qualitative relation *more probable than* in order to obtain a numerical probability measure over events?

The question to be asked of our subject has been formulated in a non-behavioral fashion, however, it is easy to give a behavioral method that leads to the kinds of responses we wish to obtain. Suppose we want behavioral evidence as to whether the subject thinks event E is more or less probable than event F . Consider two options with outcomes a , b , or c and assume that a is preferred to b . For example, outcome a might be winning five cents, b losing five cents, and c winning or losing nothing. The matrix of the game we present him has the following form

	Option 1	Option 2
E	a	b
F	b	a
$\widetilde{E \cup F}$	c	c

where $\widetilde{E \cup F}$ is the event that neither E nor F occurs. Thus our three events are exhaustive, although not necessarily mutually exclusive.

Suppose that the subject chooses Option 1. Under the hypothesis that he is maximizing subjective utility we have the following inequality:

$$s(E)u(a) + s(F)u(b) + s(\widetilde{E \cup F})u(c) \geq s(E)u(b) + s(F)u(a) + s(\widetilde{E \cup F})u(c) \quad (11)$$

On the assumption that a is preferred to b , that is, that $u(a) > u(b)$, it follows immediately from Eq. 11 that

$$s(E) \geq s(F)$$

Thus from behavioral data on choices between options and the hypothesis that subjects are choosing so as to maximize expected utility, we obtain inequalities on subjective probability that are not based on introspective data.

A discussion of experiments whose objective has been the measurement of subjective probability is reserved until Sec. 4.3.

Corresponding to the analysis of various higher ordered metrics in Sec. 2.4, it is natural to ask what formal requirements must be placed on the qualitative relation *more probable than* in order to guarantee the existence of a probability measure that reflects the order structure of the relation.

Let \geq be the relation of (weakly) *more probable than*. For the formal statement of the axioms, it is convenient to assume that the relation \geq holds between events that are subsets of a given sample space X . In other words, we use the usual set-theoretical notions for representing events.

Definition 15 A pair (X, \geq) is a qualitative probability structure if the following axioms are satisfied for all subsets A, B , and C of X :

- 1 if $A \geq B$ and $B \geq C$, then $A \geq C$,
- 2 $A \geq B$ or $B \geq A$,
- 3 if $A \cap C = \phi$ and $B \cap C = \phi$, then $A \geq B$ if and only if $A \cup C \geq B \cup C$,
- 4 $A \geq \phi$,
- 5 not $\phi \geq X$.

The first two axioms just assert that \geq is a weak ordering of the subsets of X . The third axiom formulates in qualitative terms the important and essential principle of additivity of mutually exclusive events. The fourth axiom says that any event is (weakly) more probable than the impossible event, and the fifth that the certain event is strictly more probable than the impossible event. Defining the strict relation $>$ in the customary fashion,

$$A > B \text{ if and only if not } B \geq A,$$

we may state the fifth axiom as $\lambda > \phi$. It is a simple matter to interpret behaviorally each of the axioms in terms of the option scheme already described.

To give a somewhat deeper sense of the structure imposed by the axioms, we state some of the intuitively desirable and expected consequences of the axioms. It is convenient in the statement of theorems to use the (weakly) less probable relation, defined in the expected manner

$$A \leq B \text{ if and only if } B \geq A$$

The first theorem says that \leq is an extension of the subset relation

Theorem 21 *If $A \subseteq B$, then $A \leq B$*

PROOF Suppose, on the contrary, that not $A \leq B$, that is, that $A > B$. By hypothesis $A \subseteq B$, so there is a set C disjoint from A such that $A \cup C = B$. Then, because $A \cup \phi = A$, we have at once

$$A \cup \phi = A > B = A \cup C,$$

and therefore by contraposition of Axiom 3, $\phi > C$, which contradicts Axiom 4.

Theorem 22 *If $\phi < A$ and $A \cap B = \phi$, then $B < A \cup B$*

Theorem 23 *If $A \geq B$, then $\bar{B} \geq \bar{A}$*

Theorem 24 *If $A \geq B$, $C \geq D$, and $A \cap C = \phi$, then $A \cup C \geq B \cup D$*

Theorem 25 *If $A \cup B \geq C \cup D$ and $C \cap D = \phi$, then $A \geq C$ or $B \geq D$*

Theorem 26 *If $B \geq \bar{B}$ and $\bar{C} \geq C$, then $B \geq C$*

Because it is relatively easy to prove that a qualitative probability structure has many of the expected properties, as reflected in the preceding theorems, it is natural to go on and ask the deeper question whether or not it has all of the properties necessary to guarantee the existence of a numerical probability measure P such that for any subsets A and B of X

$$P(A) \geq P(B) \text{ if and only if } A \geq B \quad (12)$$

If X is an infinite set, it is moderately easy to show that the axioms of Def. 15 are not strong enough to guarantee the existence of such a probability measure. General arguments from the logical theory of models in terms of infinite models of arbitrary cardinality suffice; a particular counterexample is given in Savage (1954, p. 41). de Finetti (1951) stressed the desirability of obtaining an answer in the finite case. Kraft, Pratt, and Seidenberg (1959) showed that the answer is also negative when X is finite; in fact, they found a counterexample for a set X having five elements (and thus, 32 subsets), but it is too complicated to present here.

It is, of course, apparent that by adding special structural assumptions to the axioms of Def. 15 it is possible to guarantee the existence of a

probability measure satisfying Eq 12. In the finite case, for example, we can demand that all the atomic events be equiprobable, although this is admittedly a very strong requirement to impose.

Fortunately, a simple, general solution of the finite case has recently been found by Scott (1964). (Necessary and sufficient conditions for the existence of a probability measure in the finite case were formulated by Kraft et al., but their multiplicative conditions are difficult to understand. Scott's formulation represents a real gain in clarity and simplicity.) The central idea of Scott's formulation is to impose an algebraic condition on the characteristic functions of the events. Recall that the characteristic function of a set is just the function that assigns the value 1 to elements of the set and the value 0 to all elements outside the set. For simplicity of notation, if A is a set we denote its characteristic function by A^c . Scott's conditions are then embodied in the following theorem, whose proof we do not give.

Theorem 27 *Let X be a finite set and \geq a binary relation on the subsets of X . Necessary and sufficient conditions that there exist a probability measure P on X satisfying Eq 12 are the following for all subsets A and B of X ,*

- 1 $A \geq B$ or $B \geq A$,
 - 2 $A \geq \phi$,
 - 3 $X \geq \phi$,
 - 4 for all subsets $A_0, \dots, A_n, B_0, \dots, B_n$ of X , if $A_i \geq B_i$ for $0 \leq i < n$, and
- $$A_0^c + \dots + A_n^c = B_0^c + \dots + B_n^c,$$
- then $A_n \leq B_n$.

To illustrate the force of Scott's Condition 4, we may see how it implies transitivity. First, for any three characteristic functions it is necessary that

$$A^c + B^c + C^c = B^c + C^c + A^c,$$

that is, for all elements x

$$A^c(x) + B^c(x) + C^c(x) = B^c(x) + C^c(x) + A^c(x)$$

By hypothesis, $A \geq B$ and $B \geq C$, and therefore by virtue of Condition 4, $C \leq A$, and thus, by definition, $A \geq C$, as desired. The algebraic equation of Condition 4 just requires that any element of X , that is, any atomic event, belongs to exactly the same number of A_i and B_i , for $0 \leq i \leq n$. Obviously, this algebraic condition cannot be formulated in the simple set language of Def 15, and thus it represents quite a strong condition. It is, however, an open condition in the sense of Sec 2.3.

Where X is infinite, a number of strong structural conditions have been shown to be sufficient but not necessary. For example, de Finetti (1937) and independently Koopman (1940a, 1940b, 1941) used an axiom to the effect that there exist partitions of X into arbitrarily many events equivalent in probability. This axiom, together with those of Def. 15, is sufficient to prove the existence of a numerical probability measure. Other related conditions of a similar existential sort are discussed in Savage (1954). Extending the methods used to prove Theorem 27, Scott has improved on these earlier results for the infinite case and has found properties that are both necessary and sufficient, but we do not state his somewhat complicated conditions here.

Thus far we have not explicitly considered sets of axioms that characterize utility and subjective probability together, although we began this subsection with a discussion of the alternative approaches that originate with Ramsey and de Finetti. We conclude with a sketch of the best known set of axioms, namely, those of Savage (1954). Savage has a single primitive relation \geq of weak preference on decisions or acts, and decisions are themselves functions from the set S of states of nature to the set A of consequences. In terms of \geq it is fairly straightforward to define a corresponding relation of preference on consequences and one of comparative probability on the states of nature. Assuming these two additional relations, Savage's seven postulates may be formulated verbally as follows:

- 1 The relation \geq is a weak ordering of the set D of decisions.
- 2 Given two decisions restricted to a subset of the states, then one is weakly preferred to the other.
- 3 Given two decisions, each of which has a constant outcome or consequence on a subset X of states of nature, then one decision is weakly preferred to the other given the set X if and only if the constant outcome of the first decision is weakly preferred to that of the second decision.
- 4 For any two sets of states of nature, X and Y , one is (weakly) more probable than the other, that is, $X \geq Y$ or $Y \geq X$.
- 5 All consequences are not equally preferred.
- 6 If decision f is strictly preferred to g , then there is a partition of S so fine that if f' agrees with f and g agrees with g , except on one element of the partition, then f is strictly preferred to g' and f' is strictly preferred to g .
- 7 If every possible consequence of the decision f is at least as attractive as the decision g considered as a whole, then f is weakly preferred to g .

Postulate 7 formulates the sure-thing principle about which we shall have more to say in the next subsection.

Because the formal statement of these seven axioms requires several

rather complex definitions and to avoid overlong and conceptually opaque qualifying conditions, we do not give a technical formulation here, but refer the reader to Savage's book or to the summary in Luce and Raiffa (1957). On the basis of these axioms, Savage proves the following

Theorem 28 *If $\mathcal{S} = \langle S, A, D, \geq \rangle$ is a system satisfying Axioms 1 to 7, then there exists a subjective probability function s on subsets of S and there exists a utility function u on A such that for any finite partition X_1, \dots, X_n of S and any functions f and g of D constant on elements of the partition*

$$f \geq g$$

if and only if

$$\sum_i s(X_i)u[f(X_i)] \geq \sum_i s(X_i)u[g(X_i)]$$

From a behavioral standpoint the most serious weakness of Savage's system, and all those similar to it, is the essential use of the constant decisions, that is, those decisions that have a constant outcome independent of the state of nature. In actual practice it is rare indeed for such decisions to be realizable, each outcome cannot be realized for some possible state of nature. Unfortunately, there seems to be no simple method of getting around their use. In Savage's own technical discussion, the constant decisions are used both to define the ordering on consequences and the one on states. (For additional discussion of these matters, see Suppes, 1956.)

3.4 Other Decision Principles

A great deal of the modern theoretical literature about decisions made in uncertain situations has concentrated on decision principles other than the maximization of expected utility. The basic motivation for this work is the recognition that, in general, a decision maker does not have information adequate to assign probabilities to the uncertain events that, in part, determine the outcome that ultimately results. This is especially a problem when the uncertainty arises not only from random factors in the environment but also from the more or less rational decisions that are made by other people or organizations. If a decision maker is unable, or unwilling, to act as if he knows the probability of each event occurring, then he must invoke some weaker decision principle than the maximization of expected utility—some principle that depends on incomplete information about the relative likelihoods of the several events. The formulation and mathematical exploration of these different principles are the main foci of the theory of games and of much of statistical decision theory. For more extensive treatments of these ideas than will be presented here, see Blackwell

and Girshick (1954), Luce and Raiffa (1957), Raiffa and Schlaifer (1961), and von Neumann and Morgenstern (1944, 1947, 1953)

Beyond any doubt, the simplest and least controversial principle of rational behavior—one that does not presume any knowledge at all about the relative likelihoods of the relevant events—is the *sure-thing principle*. It asserts that if two strategies (that is, possible decisions or acts available to the decision maker) a and a' are such that for each possible event (that is, state of nature or strategies of other decision makers), the outcome that results from the choice of a is at least as desirable as the one that results from a' and that for at least one event the outcome from a is strictly preferred to that from a' , then strategy a is *better than* strategy a' . A rational person is assumed to choose a over a' .

The main weakness of the sure-thing principle as a guide to, or description of, behavior is that it can so rarely be applied, in general, of two strategies neither is better than the other, in the preceding sense. To show, however, that it is not totally without force, consider the famous game known as the prisoner's dilemma. There are two players (as decision makers are called in game theory), A and B , each with two strategies, and the payoffs (in money or utility units, as the case may be) are given by

		Player B	
		b_1	b_2
Player A	a_1	(5, 5)	(-10, 10)
	a_2	(10, -10)	(-1, -1)

where player A receives the first payoff in the cell selected and player B the second one. Now, looking at the situation from A 's point of view, if B chooses b_1 , then clearly a_2 is better than a_1 since, by definition, 10 is preferred to 5, and equally well, if b_2 is chosen, a_2 is better than a_1 since -1 is preferred to -10. Hence, by the sure-thing principle, A should choose a_2 . An exactly parallel argument leads B to choose b_2 , and so the resulting payoff is (-1, -1).

This is an odd, somewhat disturbing result from a social standpoint since it is evident that the strategy choice $\langle a_1, b_1 \rangle$, which leads to the outcome (5, 5), is preferable to both players. Indeed, we can formulate a social version of the sure-thing principle that dictates this choice. A strategy pair $\langle a, b \rangle$ is *better than* the pair $\langle a', b' \rangle$ provided that the outcome from $\langle a, b \rangle$ is at least as good for both players and is strictly preferred by one of them to the outcome from $\langle a', b' \rangle$. Any strategy pair such that no other pair is better in this sense is said to be *Pareto optimal*. It is easy to see in the prisoner's dilemma that $\langle a_1, b_1 \rangle$ is Pareto optimal. A rational social decision principle should, it is generally agreed, result in Pareto

optimal strategies (This notion can be generalized in the obvious way to games involving three or more players, we do not spell out the details)

The prisoner's dilemma makes it quite clear that the sure-thing principle for individuals is not generally consistent with the sure-thing principle (Pareto optimality) for social conflicts. Since it is not usually possible to enforce social principles without a good deal of machinery beyond the decision situation itself and since the existing empirical data (Deutsch, 1958, 1960, Lutzker, 1960, Rapoport, 1963, pp 558-561, Scodel, Minas, Ratoosh, & Lipetz, 1959) strongly suggests that under normal circumstances the individual principle overrides the social one, it seems appropriate to attempt to generalize the individual sure-thing principle to cover decision situations to which it cannot itself be applied.

One idea is to search for strategies, one for each player, such that no person acting alone can benefit himself by a change of strategy. We make this precise for two players, the generalization to any number is obvious. If $\langle a, b \rangle$ is a pair of strategies, let $u_i(a, b)$ denote the corresponding outcome to player i . The strategy pair $\langle a, b \rangle$ is said to be in *equilibrium* if for any other strategy a' of player A ,

$$u_A(a, b) \geq u_A(a', b),$$

and if for any other strategy b' of player B ,

$$u_B(a, b) \geq u_B(a, b')$$

In the prisoner's dilemma, $\langle a_2, b_2 \rangle$ is in equilibrium. In the game

$$\begin{array}{cc} & b_1 & b_2 \\ \begin{array}{c} a_1 \\ a_2 \end{array} & \left[\begin{array}{cc} (2, 1) & (-1, -1) \\ (-1, -1) & (1, 2) \end{array} \right] \end{array}$$

the sure-thing principle is powerless, but it is easy to see that both $\langle a_1, b_1 \rangle$ and $\langle a_2, b_2 \rangle$ are in equilibrium. Thus the notion of an equilibrium pair extends the class of games to which considerations of individual rationality can be applied. Nevertheless, it has two failings. First, as the preceding example shows, it does not tell an individual player what to do to get one of the desired outcomes: if player A were to choose a_1 in an attempt to get the $(2, 1)$ outcome and player B were to choose b_2 in an attempt to get $(1, 2)$, then the nonequilibrium pair $\langle a_1, b_2 \rangle$ would result, and the players would receive the undesired outcome $(-1, -1)$. Second, the equilibrium notion is incapable of resolving all games since, for example,

$$\left[\begin{array}{cc} (1, -1) & (-1, 1) \\ (-1, 1) & (1, -1) \end{array} \right]$$

has no pair of strategies that are in equilibrium, as is easily verified. We take up the second problem first

The major resolution that has been suggested does not involve a further weakening of the decision principle, but rather the given discrete game is extended to a continuous one by enriching greatly the strategies available to each player. von Neumann (1928, see also von Neumann and Morgenstern, 1944, 1947, 1953) pointed out that whenever a set of strategies is available, then so are all the probability distributions over it. A player can generate any distribution he chooses by using suitable auxiliary devices. Thus, if his preferences satisfy the axioms for expected utility when the probabilities are known, as they are when he generates them himself, then the expected payoff can be calculated for each probability distribution over the given strategies, that is, for each *mixed strategy*, as such a distribution over strategies is called. This was, in fact, the main purpose for which von Neumann and Morgenstern first developed their theory of expected utility. Within the context of mixed strategies, it can be shown that every game has at least one pair of (mixed) strategies that are in equilibrium (Nash, 1950). This by no means trivial result is a partial generalization of the famous minimax theorem of von Neumann (1928) which not only established the existence of mixed strategy equilibrium pairs in those two-person games for which the payoff to one player is always the negative of that to the other (the so called zero-sum games), but also showed that if $\langle a, b \rangle$ and $\langle a', b' \rangle$ are in equilibrium, then so are $\langle a, b' \rangle$ and $\langle a', b \rangle$, and that the payoffs to a player are the same for all pairs of strategies that are in equilibrium. Neither of these last two statements is generally true for nonzero-sum games, as can be seen in the second game, nor for games with more than two players. Thus, although the first problem mentioned about equilibrium strategies, namely, that they do not prescribe rational behavior for the individual, is not a problem when we restrict our attention to two-person zero-sum games, it is for other classes of games.

An alternative approach to solving the two weaknesses of the equilibrium notion is suggested by an important property of equilibrium strategies in the two-person zero-sum case. For each of a player's strategies, suppose that we determine what is the worst (minimum) outcome that can occur, and then find those strategies for which this worst outcome is as good as possible (that is, find the strategy having the maximum value of the minimum outcome). Such a strategy is called, for obvious reasons, a *maximin strategy*. For two person zero-sum games, it can be shown that a pair $\langle a, b \rangle$ of mixed strategies is in equilibrium if and only if a and b are both maximin mixed strategies. Observe that the notion of a maximin strategy is defined for any game and that such a strategy always exists whether we restrict ourselves to pure strategies or whether mixed strategies are

admitted, however, except for two-person zero sum games, the notion of maximin strategies does not have any simple relation to that of equilibrium strategies. The principle underlying maximin strategies is conservative in the extreme: concern is focused exclusively on the worst possible outcome from any course of action, no matter how improbable that outcome may be.

Savage (1951) suggested that this last notion should be applied not to the given matrix of payoffs, or to its mixed strategy extension, but rather to what he called the *regret matrix*, which is constructed as follows. For a given player, replace the outcome in a given cell by the following number: find the best outcome (or expected outcome, as the case may be) that he could have achieved by using any of his (mixed) strategies, on the assumption that the choices of chance and other players are fixed, and from it subtract the actual value in that cell. For the prisoner's dilemma, this yields the transformation

$$\begin{bmatrix} (5, 5) & (-10, 10) \\ (10, -10) & (-1, -1) \end{bmatrix} \rightarrow \begin{bmatrix} (5, 5) & (9, 0) \\ (0, 9) & (0, 0) \end{bmatrix}$$

For example, the -10 in the (a_1, b_2) cell becomes 9 because, with b_2 fixed, the best player A could have done is -1 by choosing a_2 , and that minus the -10 of the entry yields 9 , which is his "regret" in having chosen a_1 rather than a_2 when B chooses b_2 . The 10 in the same cell for player 2 becomes 0 because the best he could do with a_1 fixed is 10 , which minus the 10 of the cell yields 0 .

The proposed decision rule is to choose any strategy that minimizes the maximum regret. In the prisoner's dilemma this rule again dictates (a_2, b_2) , the solution previously obtained by the sure thing principle. In the following game

$$\begin{bmatrix} (2, 1) & (-1, -1) \\ (-1, -1) & (1, 2) \end{bmatrix} \rightarrow \begin{bmatrix} (0, 0) & (2, 2) \\ (3, 3) & (0, 0) \end{bmatrix},$$

the rule yields (a_1, b_2) , which is not one of the two equilibrium pairs of the game, but rather what one would expect if each player attempted to get his preferred equilibrium pair. Thus, at least in this case, the minimum regret criterion seems more descriptive than the equilibrium notion.

Savage's proposal, although intuitively reasonable, suffers from the defect that no theory of utility now in existence is adequate to justify calculating the differences required for the regret matrix and, at the same time, to justify the expected utility property which is needed if mixed strategies are to be used.

Before turning to other ideas, it should be mentioned that several authors, most notably Milnor (1954), have attempted to gain a better

understanding of these various decision criteria by means of an axiomatic approach. They have listed various "elementary" axioms that a decision principle might satisfy, and they have shown which sets of these axioms are necessary and sufficient to characterize a given principle. These results give some insight into the various concepts of rationality that are embodied and excluded from the different decision principles. For a summary of these results, see Luce and Raiffa (1957, pp. 286-309).

The decision principles that we have just discussed all differ sharply from the principle of expected utility maximization in that they presume absolutely no information about the relative likelihoods of the events that affect the outcome, whereas the latter principle can be applied only when the decision maker is willing to commit himself in advance to relative likelihoods. Neither extreme seems to capture the reality of most decision problems, which characteristically involve some qualitative information about the relative likelihoods of events, but not enough to justify the assignment of a unique probability measure to them. Little, if any, satisfactory axiomatic theory yet exists to deal with decisions in the presence of some but incomplete information, however, some relevant ideas have been discussed, and so we now turn our attention to them.

One framework of discussion centers about the concept of *level of aspiration*, initially introduced by Dembo (1931) and now familiar to most psychologists. The first direct comparison of it with a utility analysis of choice behavior was made by Simon (1955, 1956). Simon illustrated his ideas by the familiar example of an individual selling his house. Each day he sets an acceptance price based on various factors: bids he has received in the past, rumors about prevailing market conditions, his urgency in consummating a sale, etc. If during the day he receives one or more offers above this price, he accepts the highest. If not, he sets a new acceptance price the next day and awaits further offers. The acceptance price he sets each day is his *level of aspiration* (for selling the house) at that moment in time. Simon contends that this kind of model, including the mechanisms for changing the level of aspiration with experience, is much closer than the expected utility model to the decision process actually used when the alternatives are complex and information about them is far from complete. Simon has given a certain amount of theoretical development in the two articles cited, but at present the formal aspects, especially the mechanism for processing information, are far from fully stated.

Some possible theoretical connections between utility theory and level of aspiration were postulated by Siegel (1957). For him, the level of aspiration is a point on the individual's utility curve. All outcomes below this point have negative utility and all above it, positive utility. When the number of outcomes is finite, Siegel defines the level of aspiration to be

"that goal which has the largest difference in utility between it and the next lower goal" However, the rationale for this definition is far from clear to us. In any case, some experimental applications of these ideas are given in Siegel and Fouraker's (1960) book on experiments in bilateral monopoly.

Although the concept of level of aspiration has great face validity and probably must ultimately be incorporated in some form into any adequate theory of decision making, it is fair to say that so far it has not been developed in any very thorough theoretical fashion or tested extensively—certainly not to the extent that the concepts previously mentioned in this subsection have been.

A quite different concept for dealing with uncertainty is Shackle's (1949, 1955) *potential surprise*. But again, these ideas neither have been stated with any great formal clarity nor have they been applied to any experiments, they were developed to handle various difficulties that arise in connection with expectations in economic theory. In spite of this separation from most psychological work on choice behavior, Shackle's writings include a number of psychologically insightful and original remarks about choice behavior.

To each possible future event he assumes that there can be assigned a degree of potential surprise, and rules are given for combining potential surprises. These differ from the usual rules for dealing with probabilities mainly in being nonadditive. Specifically, if $\eta(E)$ denotes the potential surprise of event E , then the combining rules are essentially the following:

(i) if $E_1 \cap E_2 = \phi$, then $\eta(E_1 \cup E_2) = \min [\eta(E_1), \eta(E_2)]$,

(ii) $\eta(E \cap F) = \max [\eta(E|F), \eta(F)]$ where $\eta(E|F)$ is the potential surprise of E given that F has occurred.

Thus, when E and F independent, that is, $\eta(E|F) = \eta(E)$, rule (ii) reduces to

$$\eta(E \cap F) = \max [\eta(E), \eta(F)]$$

Arrow (1951a, b, 1964) pointed out that it is a consequence of these rules that "there is no law of large numbers or elimination of risks by consolidation of independent events, for the potential surprise attached to a sequence of unfavorable outcomes is as large as to any one." Just how strange such a calculus can be is illustrated by a simple example. Ten successive heads in as many flips of a fair coin seems, intuitively, to have positive potential surprise, yet in Shackle's calculus this event is assigned a potential surprise of zero. We can see this as follows. Consider first one flip, and let H be the event of a head and T the event of a tail. Since $H \cup T$ is certain, $\eta(H \cup T) = 0$, because $H \cap T = \phi$, rule (i) yields

$$\eta(H \cup T) = \min [\eta(H), \eta(T)]$$

Since surely $\eta(H) = \eta(T)$, we have

$$\eta(H) = 0$$

Now, making the usual assumption of independence,

$$\eta(H_1 \cap \dots \cap H_{10}) = \max [\eta(H_1), \dots, \eta(H_{10})] = 0$$

Clearly, 10 plays no role in this argument — any finite number of successive heads has zero potential surprise!

Shackle has a good deal to say in favor of his rules in spite of these consequences. Essentially, his defense rests on the claim that in real life situations we are never in a position to make the probability computations of textbook examples — such computations are possible only for divisible and repeatable experiments. Indeed, he has argued against the general applicability of either a subjective or frequency theory of probability, but his remarks were mainly directed toward the frequency theory and he has not dealt in an explicit and technical way with the many persuasive arguments put forward by subjectivists, such as de Finetti.

4 EXPERIMENTAL TESTS OF ALGEBRAIC MODELS

Over the past fifteen years a number of experimental studies have been performed that are more or less relevant to the algebraic theories of preference discussed in Secs. 2 and 3, but unfortunately few bear directly on the empirical validity of the various theories discussed. There are several reasons for this. The most important is that the algebraic theories of preference are special cases of algebraic theories of measurement, and as such do not readily lend themselves to an exact statistical analysis of their relation to experimental data. For example, the axioms for the various ordered metrics are formulated in such a fashion that it is natural simply to ask whether or not a set of data satisfies the axioms, not to attempt a statistical goodness-of-fit evaluation. In fact, it is not entirely clear how the theory of goodness-of-fit should be stated for the various kinds of higher ordered metrics. In models similar to those of von Neumann and Morgenstern the basic closure postulate requires an infinity of objects, and again it is not clear when we should be willing to regard a set of data as satisfying or definitely not satisfying the axioms. We would not expect, of course, any finite sample of data to satisfy the axioms exactly, and the difficult question is how to formulate an appropriate criterion

In view of such difficulties, it is not surprising that the bulk of the experimental literature is concerned with studies designed to test certain limited aspects of choice behavior. Many of the main points that have been established are rather far removed from the theoretical issues we have discussed in the preceding sections. In the interest of completeness we describe the studies of greatest theoretical import, but because the accumulated literature is relatively large we do not attempt a really complete survey. The reader interested in additional references should consult the survey articles mentioned in Sec 1.4.

Our summary of these experimental studies is organized in terms of the major headings of Secs 2 and 3, not in strict chronology. Under a given heading, however, the order is primarily chronological. The first section deals with higher ordered metrics, the second with the measurement of utility, the third with the measurement of subjective probability, and the fourth with other models that have been proposed as a result of experimental conjectures or results but which we have not discussed with any thoroughness in the preceding sections.

4.1 Higher Ordered Metrics

Coombs' (1950) concept of a higher ordered metric has been applied to experimental studies of preference by Coombs and his collaborators in a number of studies and also by Siegel (1956) and Hurst and Siegel (1956).

Coombs (1954) applied his ideas about ordered metrics and the unfolding technique to judgments of the esthetic quality of a series of isosceles triangles all of which had a base of 1 in. and an altitude that varied from 0.25 to 2.5 in. in steps of 0.25 in. Sets of three triangles were presented to subjects who judged both the most and least preferred in each triad. All 120 possible sets of three triangles formed from a set of 10 were presented. (It should be emphasized that this use of the method of triads is not essential to the application of Coombs' ordered metric concepts.)

In analyzing the data, it was assumed that the most preferred triangle was located nearest the subject's own "ideal" triangle and that the least preferred triangle was located furthest away. The data may be described as follows. The unfolded data from 17 of the 31 subjects satisfied a common interval scale for the 10 stimulus triangles. The responses of 8 additional subjects satisfied the ordinal properties of this common scale, but they required different metric relations. The 6 remaining subjects failed to satisfy the ordinal properties of the common scale. It is to be emphasized that the imposition of the possibility of unfolding the data to

get a common ordering of stimuli and subjects is, of course, a much stronger assumption than simply assuming that each subject has his own weak higher ordered metric. Considering the subjectivity usually associated with such esthetic preferences, it is somewhat surprising that the data from 17 of the subjects could be interpreted in terms of a common strict higher ordered metric.

A useful discussion of some of the problems and mathematical consequences of determining an ordered metric from experimental data was given by Coombs and Beardslee (1954), but the only data they reported are from a pilot experiment with one subject, so we do not review the results here.

In moving from a strong higher ordered metric to a strict higher ordered metric, in the sense of the definitions given in Sec. 2.4, it is necessary but not sufficient for adjacent utility intervals to be additive. Coombs and Komorita (1958) investigated the extent to which additivity is satisfied in an experimental ordered metric for the utility of money. Only three graduate student subjects participated in the experiment, but a fairly large number of preferences were required from each. A method of triads was used that was similar to the one just described. The experimental data quite strongly supported the additivity hypothesis. In particular, of 30 predictions tested, 29 were confirmed. It should be mentioned that one can expect the results to be best when the outcomes are money, because the natural order of preference is very strongly entrenched in all subjects. One never expects a subject to prefer less to more money, but inconsistencies of simple preference are common with nonmonetary outcomes.

Two other papers by Coombs (1958, 1959) are also concerned with the problems of applying the unfolding technique to obtain an ordered metric, but because the preferences were mostly probabilistic, the summary of this experiment is reserved until Sec. 8.2.

Siegel (1956) described a method for obtaining a strong ordered metric scale in the sense of Def. 8 of Sec. 2.4. He used simple gambles constructed from events with subjective probabilities of one half, following the procedure suggested in Davidson, Suppes, and Siegel (1957), to obtain not only an ordering of alternatives but also a complete ordering of the differences between alternatives. The ordinal comparison of utility intervals in Siegel's method is based on the kinds of options and the resulting equations discussed in Sec. 3.3 (see, in particular, Inequality 6). Because the original article presented only a pilot study based on one subject, we do not consider it further. However, Hurst and Siegel (1956), using the same methodology, ran 30 prison inmates with cigarettes as outcomes rather than money. Of the 30 subjects, 29 completed the experiment. The

following is a typical offer

	Option 1	Option 2
<i>QUJ</i>	+30	+5
<i>QUG</i>	-10	+5

where *QUJ* is a nonsense syllable printed on three sides of a die and *QUG* is a nonsense syllable printed on the other three sides. Sufficient options were used in order to construct an ordered metric (in the sense of Def. 8, Sec. 2.4) for each subject, and from this metric a number of choices was predicted between additional options.

The main hypothesis tested was the accuracy of these predictions as compared with those based on the model in which it is assumed the subject maximizes the expected number of cigarettes. For 6 subjects the two models gave identical predictions, which means that their subjective utility scales were essentially linear in the number of cigarettes. For 15 subjects the ordered metric model yielded better predictions than the simple expectation model, but data for 8 subjects fell in the opposite direction. For an over-all statistical test, the total number of erroneous predictions for each of the two models was determined for each subject, and the differences between these two totals were ranked. A Wilcoxon matched-pairs, signed-ranks test indicated that the observed rank sums differed significantly from what would be expected under the null hypothesis and favored the ordered metric model ($P < 0.025$).

Because of the close relation between the additivity assumptions discussed in Sec. 2.3 and the ordered metric models, we mention at this point an experiment by Fagot (1956), reported in Adams and Fagot (1959), which was designed to test additivity assumptions. Each of 24 subjects was instructed to take the role of a personnel manager for a corporation and choose among hypothetical applicants, two at a time, for an executive position of a specific nature. Applicants were described in terms of just two characteristics, intelligence and ability to handle people, with four levels of ability admitted for each characteristic. Each choice of a subject was represented as an inequality in utility values as described in Sec. 2.3. The additive model is satisfied if and only if the entire set of 77 inequalities has a solution (43 of the 120 applicant pairs were omitted because one weakly dominated the other on both characteristics). Analysis of the data showed that 6 subjects satisfied perfectly the additive model. The remaining 18 subjects who did not satisfy the additive model surprisingly enough did not satisfy the simple ordinal model, that is, each of the 18 violated the transitivity axiom at least once. More detailed analysis

suggested that the deviations from either the ordinal or additive model were due mainly to occasional inattention, for 80% of the subjects had at most three such "errors" or deviations, that is, at least 74 of the 77 observed choices satisfied the additive model. Only two of the subjects had more than four such errors

4.2 Measurement of Utility

Although discussions of the measurement of utility have had a relatively long history both in economics and psychology, and the proposal to extend classical psychophysical methods to the measurement of utility goes back at least to Thurstone (1945), the first real experimental attempt to use any of these ideas in an actual measurement of utility was by Mosteller and Nogee (1951). Their experiment was designed to test the empirical validity of the von Neumann and Morgenstern axiomatization as applied to alternatives consisting of the gain and loss of small amounts of money, together with probability combinations of such outcomes. The subject was presented with bets which he could accept or refuse (the bets were presented to four or five subjects at a time, but each subject made his own decisions). If he refused the bet, no money changed hands, if he accepted it, he either lost 5 cents or won some amount x of money, which one depending on whether or not a specified chance event E occurred. When the subject was indifferent between the two options, that is, when he accepted the bet approximately half the time, then the following equation was used to calculate the relative utilities of the outcomes

$$u(0\text{¢}) = pu(-5\text{¢}) + (1 - p)u(x),$$

where p is the probability of E occurring. Since u is an interval scale, $u(0\text{¢})$ and $u(-5\text{¢})$ can be fixed. By setting them to be 0 and -1 , respectively, $u(x) = p/(1 - p)$. Note that p is an objective probability, not a measured subjective one. Thus, by selecting an event with probability p and varying the amount x of money until the subject was indifferent between the options, it was possible to find the amounts of money corresponding to any fixed utility $u = p/(1 - p)$. Nine utility points were determined for each subject ranging from $u(-5\text{¢}) = -1$ to 101. This constructed utility function was then used to predict the choices of each subject when he was faced with somewhat more complex options.

Fourteen subjects completed the experiment, of whom nine were Harvard undergraduates and five were from the Massachusetts National Guard. Each subject was given \$1.00 at the beginning of each hour of

play with which to gamble Testing took place over a period of about four months

The bets were constructed from possible hands of poker dice After three initial sessions, each group of subjects was instructed in how to calculate the true odds for any hand, and in addition, each was given a sheet with the true odds for all the poker dice hands used and was, moreover, "required to keep this sheet in front of him in all future sessions" Additional information was provided to the subjects about the number of times a particular hand had been previously played and the number of times players had won when they selected it during the initial "learning" sessions

In order to obtain utility curves for individual subjects, the first step in the analysis requires that the point of indifference be determined for each subject for each hand of poker dice used This means that the amount of money must be found for which the subject is indifferent as to whether he accepts or rejects the bet Mosteller and Nogee (1951, p 383) described in clear fashion their procedure for finding this point

For each hand a range of offers had been made The proportion of times the subject elected to play each offer was calculated, and these points were plotted on ordinary arithmetic graph paper with vertical axis as per cent participation and horizontal axis as amount of offer in cents A freehand curve or a broken-line curve was then fitted to the points The abscissa value of the point where this curve crossed the 50 per cent participation line gave in cents the subject's indifference offer for that hand In other words for that hand this calculation yielded an interpolated offer which the subject would be equally likely to accept or reject if given the opportunity

Since the probability p of winning is fixed for a given hand, the utility of the indifference offer is just $(1 - p)/p$

Figure 2 shows how this indifference point was constructed for subject B-1 and the poker dice hand 55221 (Amounts of money in cents are plotted on the abscissa) Note that Fig 2 approximates rather closely the perfect step function that is required theoretically by the von Neumann-Morgenstern axiomatization and, in fact, by all of the standard algebraic theories for choice among uncertain outcomes

Utility curves constructed in this manner for two of the student subjects are shown in Fig 3 Both these curves support the classical assumption of decreasing marginal utility, but, as Mosteller and Nogee pointed out, such apparent support for this economic idea must be interpreted cautiously For one thing, the range of bets for the students ran only from 5 cents to about \$5.50, and only up to about \$1.00 for the Guardsmen (For obvious financial reasons, this restriction of the range is not peculiar to the Mosteller and Nogee experiment, but is true of all the experiments

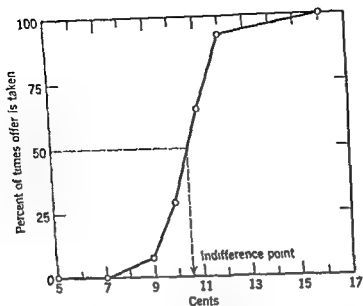


Fig 2 The data for one subject and hand 55221 are plotted to show how the indifference point is actually obtained. Adapted by permission from Mosteller & Nogee (1951, p 385)

that have been performed) For another, the utility curves of the Guardsmen generally supported the assumption of increasing marginal utility in the range of bets considered

Mosteller and Nogee gave the subjects additional offers of a more complicated sort which they called doublet hands. In a doublet hand the subject is presented with two hands of poker dice: if he beats the low hand, he gets a certain reward, if he beats the high hand, he receives an additional reward as well, and if he fails to beat either hand, he loses 5 cents. As the authors remarked, the introduction of these doublet hands caused considerable confusion in the betting patterns of the subjects. The most definite evidence of this confusion was the increase in what the authors called the zone of inconsistency, that is, in the zone in which the subject did not consistently choose to accept or refuse a bet. The presence of a fairly wide zone of inconsistency, both for the original bets and for the doublet hands, of course, is not compatible in a literal sense with the von Neumann-Morgenstern axiomatization. This incompatibility is particularly reflected by Theorem 17 of Sec 3.2, which asserts that there is a unique probability combination of alternatives that is indifferent to a given alternative. As was already remarked, the existence of zones of inconsistency is precisely the reason for the development of the probabilistic theories discussed in Secs 5 to 8 of this chapter. Apart from the existence of a larger zone of inconsistency, the constructed utility curves were fairly successful in predicting behavior with respect to the doublet hands. We do

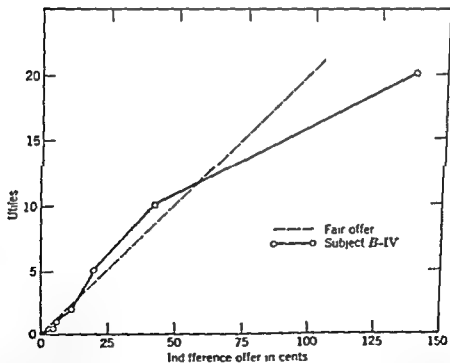
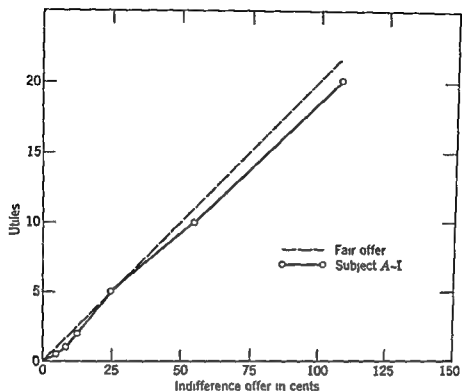


Fig 3 Utility curves for two student subjects and straight line showing mathematically fair offer. Offer for 101 utiles is not shown. Adapted by permission from Mosteller & Nogee (1951, p. 387)

not attempt a quantitative summary of the data here. The reader is urged to refer to the original Mosteller and Nogee article, which contains a large number of interesting side remarks and insights into the behavior of the subjects, as well as a good deal of additional systematic material we have not attempted to summarize. Mosteller and Nogee summarize (p. 403) their experiment in the following fashion:

- 1 that it is feasible to measure utility experimentally,
- 2 that the notion that people behave in such a way as to maximize their expected utility is not unreasonable,
- 3 that on the basis of empirical curves it is possible to estimate future behavior in comparable but more complicated risk taking situations

We next turn to the experimental studies of Davidson, Suppes, and Siegel (1957), which in part were undertaken to meet the following three important criticisms of the Mosteller and Nogee experiment. First, Mosteller and Nogee failed to check systematically the adequacy of their utility curves in terms of simple bets of the same type as those used to construct them. Such checks using simple hands would have provided a better test of the von Neumann Morgenstern axioms than those based on the more complicated doublet hands which may introduce complications not embodied in the axiom system. Because of this lacuna, it is somewhat difficult to evaluate precisely whether or not the Mosteller and Nogee experiment substantiates a claim that utility can be measured on an interval scale, for some recent remarks on this point, see Becker, DeGroot, and Marschak (1964).

The second criticism is that the choices used by Mosteller and Nogee to estimate the utility functions involved either accepting or rejecting a gamble. One option always involved playing and therefore taking a risk, whereas the other resulted in no play. Thus, if participation itself involves either a negative or a positive utility, the experiment was so designed that this special feature could produce maximum distortion.

The third criticism is that their analysis, which to some degree was necessitated by the experimental design, assumed that subjective probability is equal to objective probability. In spite of the fact that they went to some pains to brief subjects on the probability considerations involved in the offers with which they were faced, no clear evidence exists to show that they were wholly successful in equating the two probabilities from the subject's standpoint.

In designing a study to meet these three criticisms, Davidson, Suppes, and Siegel (1957) essentially followed the Ramsey (1931) line of development for measuring utility and subjective probability. For the moment we restrict ourselves to their efforts to measure utility.

To begin with, it was necessary to find an event whose subjective

for these remeasured utility curves were, in general, quite similar to those determined in the initial session. This evidence is encouraging, particularly in the light of the concern expressed by several investigators that experimentally determined utility curves are temporally unstable.

It is natural to ask whether the upper and lower bounds found for the utility points f , c , d , and g in Table 1 enclose the actuarial value, that is, to the accuracy of the measurement scheme, is the utility function simply

Table 1 Summary of Data Determining Bounds in Cents for Fixed Points on the Utility Scale with $u(-4¢) = -1$ and $u(6¢) = 1$. (Davidson, Suppes, & Siegel, 1957, p. 62)

Subject	Bounds for f Where $u(f) = -5$	Bounds for c Where $u(c) = -3$	Bounds for d Where $u(d) = 3$	Bounds for g Where $u(g) = 5$
1	-18 to -15¢	-11 to -10¢	11 to 12¢	14 to 18¢
2	-34 to -30	-12 to -11	12 to 18	31 to 36
3	-18 to -11	-8 to -7	10 to 13	14 to 22
4	-29 to -24	-15 to -14	14 to 17	25 to 31
5	-21 to -14	-10 to -9	10 to 12	16 to 24
6	-25 to -21	-14 to -13	13 to 15	19 to 23
7	-18 to -7	-7 to -6	7 to 14	10 to 23
8	-25 to -21	-14 to -13	14 to 17	23 to 28
9	-35 to -29	-12 to -11	16 to 18	43 to 50
10	-26 to -20	-15 to -14	14 to 15	20 to 27
11	-22 to -19	-14 to -13	11 to 13	18 to 22
12	-21 to -13	-12 to -11	8 to 12	11 to 15
13	-34 to -23	-14 to -13	13 to 17	23 to 32
14	-16 to -13	-10 to -9	12 to 15	20 to 24
15	-12 to -8	-8 to -7	8 to 10	11 to 15

linear in money? Of the 60 pairs of bounds shown in Table 1, 19 of the intervals are entirely above the linear money value, 21 are entirely below it, and 20 include it.

Finally, to avoid the second criticism of the Mosteller and Nogee experiment, choices in the experiment of Davidson et al. were primarily between pairs of options, both of which involved uncertainty. It is interesting, therefore, to note the strong similarity between the shape of the utility curves obtained and the quality of predictions made in the two experiments. This similarity is, in fact, rather promising for experimental results in this area.

Davidson et al (pp 75-79) stated two important criticisms of their experiment. One is that the method is restricted to objects that are equally spaced in utility, which severely limits its applicability. It is suitable for essentially continuous commodities, such as money, but it is quite unsatisfactory for many interesting alternatives that are essentially not divisible. The other criticism is that the method of approximation used is cumbersome, and it becomes especially unsatisfactory when a large number of utility points are sought because the width of the bounds tends to spread as the number of points is increased.

In the next chapter of their book, Davidson et al attempted to meet these two criticisms by using a linear programming model to measure utility. As was already remarked in Sec 3.3, when a chance event E^* of subjective probability $\frac{1}{2}$ is used, a choice of Option 1 over Option 2 in

$$\begin{array}{cc} \text{Option 1} & \text{Option 2} \\ E^* \left[\begin{array}{cc} a & c \\ b & d \end{array} \right] \end{array}$$

may be represented by the inequality that is based on the maximization of expected utility

$$u(a) + u(b) \geq u(c) + u(d) \quad (13)$$

The essential idea is to apply linear programming methods to solve a set of inequalities of this form. In the experiment, options were built up from all possible pairs of six different outcomes, but excluding those for which a prediction follows from ordinal preferences alone. The subject's choices in the remaining 35 pairs of options therefore determined 35 inequalities in six variables. Only rarely does this set of inequalities have a solution. Therefore, Inequality 13 was replaced by

$$u(a) + u(b) + \theta \geq u(c) + u(d) \quad (14)$$

where θ is a constant to be determined.

Using linear programming methods, a solution is subject to a set of inequalities of this form such that θ is a minimum and such that the normalizing restriction is met that the smallest interval be of length one. Intuitively, θ may be thought of as the threshold of preference: if the utility difference between a and c differs by more than θ from the difference between d and b , then the larger interval must be chosen, that is, Option (a, b) is chosen over (c, d) . Some additional probabilistic postulates for the behavior when the difference between two utility intervals is less than θ were also formulated by these authors, but we do not consider them here.

It must be emphasized that the utility function obtained by this linear programming approach is not unique: the minimum θ is compatible with

a convex polyhedron of solutions. Because of limited computational facilities, Davidson et al. reported only the first solution reached, although, as they remark, the best single choice is probably the centroid of the polyhedron.

To test this linear programming model, the following three-session experiment was performed with ten music students as subjects and long-playing records as outcomes. In the first session a utility curve for six records was determined by the methods just described. In the second session a utility curve was found for another six records, two of which were also in the first set, thus permitting the construction of a joint utility curve. This joint curve could then be used to predict choices

Table 2. Summary of Predictions of Linear Programming and Ordinal Models (Davidson, Suppes, & Siegel, 1957, p. 92)

Subject	Linear Programming Model				Ordinal Model		
	Clear Predictions Correct	Clear Predictions Wrong	Predictions Within θ	Total	Predictions Correct	Predictions Wrong	Total
1	30	1	24	55	6	0	6
2	24	7	24	55	18	2	20
3	21	0	34	55	11	2	13
4	23	15	17	55	21	11	32
5	21	19	15	55	27	19	46
6	4	0	51	55	0	0	0
7	10	13	32	55	7	12	19
Total	133	55	197	385	90	46	136

between untested combinations of the full set of ten records used in the first two sessions, these predictions were tested in the third session.

Of the 10 subjects, only 7 were able to complete the entire experiment. The predictions for these 7 subjects are summarized in Table 2. The first column gives the number of predictions that were clearly correct—"clearly" because the utility differences were greater than the estimated θ . The second column gives the number of predictions that were clearly wrong, and the third, those that fell within θ and therefore need a more detailed probabilistic analysis. The right-hand side of the table gives the accuracy of the predictions that follow from the simple ordinal model that is obtained in the obvious way from the rankings given by the subjects. Unfortunately, a rather large proportion of predictions fell within θ , thus making it difficult to evaluate the adequacy of the model. If θ is ignored and one simply predicts the option having the larger of two utility intervals, then 254 are correct and 126 are wrong. In only five cases is no prediction made because of exact equality of the intervals.

Perhaps the major difficulty encountered in applying the linear programming model to data from this experiment is the fact that only one subject satisfied the purely ordinal axiom of transitivity, that is, the ordinal predictions of how the long-playing records would be ordered in preference in Session 3 was completely verified for only one subject. Lack of transitivity accounts for the large θ 's and the resulting insensitivity of the model which is reflected in Table 2.

Davidson et al. pointed out two undesirable features of the model itself. In the first place, when the inequalities have integer coefficients, as these do, the application of linear programming methods almost always yields very simple values for the utility intervals, in fact, there is a strong tendency for the utility intervals to become identical, a tendency which seems psychologically unrealistic. For example, under the normalizing assumption stated previously that the minimum interval be of length one, all utility intervals of adjacent alternatives in Session 1 were either 1, $\frac{5}{2}$, 2, or 3. For five atomic intervals and seven subjects, that is, for a total of 35 intervals to be measured, this restricted set of numerical results certainly seems far too simple.

The second criticism is that the model leads to simple violations of the sure-thing principle when it is applied to monetary outcomes, specifically, it predicts that a larger interval will not be chosen with probability one when both utility differences are less than θ . Thus, to use their example, when the subject is presented with the following pair of options

	Option 1	Option 2
E^*	50¢	45¢
E^*	-48¢	-50¢

the sure thing principle and common sense choose Option 1. However, for a θ of any size that choice is not predicted with probability one.

This failing was eliminated in a closely related nonlinear programming model proposed by Suppes and Walsh (1959). In their setup inequalities of the form of Eq. 14 were replaced by nonlinear inequalities of the following sort

$$\eta[u(a) + u(b)] \geq u(c) + u(d)$$

and the problem was to minimize the threshold parameter η subject to the restriction that $\eta \geq 1$. Their experimental procedure was very similar to that of Davidson et al., just described, except that the outcomes were monetary. Their subjects were eight sailors from an air base. Without going into their experimental results in detail, suffice it to say that they did find some predictive superiority of the nonlinear utility model over the actuarial model. A rather thorough critique of their interpretation

of their results was given by DeGroot (1963), who makes a good case for claiming that their model shows little real superiority over the actuarial model

A probabilistic version of the linear programming model has recently been developed and applied by Dolbear (1963). His results are discussed in Sec 8 4

In an early experiment—the next following the classic Mosteller and Nogee study—Edwards (1955) was concerned with the confounding of utility and subjective probability, and he proposed to test a subjective expected utility model. Rather than apply one of the rather elaborate methods of measurement discussed in Sec 3, he invoked a convenient but highly simplifying assumption, which, in the notation of Sec 3 2, may be expressed as follows. There is an integer N such that

$$u(y\frac{1}{N}x_0) \approx Nu(\epsilon\frac{1}{N}x_0), \quad (15)$$

where y is a reasonably large amount of money like \$5 50, ϵ is a small amount of money like 10 cents, and x_0 is the outcome of neither winning nor losing any money. Naturally we do not expect exact equality to hold but a good approximation is feasible. Applying now the expected utility property to Eq 15 and choosing the scale so that $u(x_0) = 0$, we obtain

$$u(y) \approx Nu(\epsilon) \quad (16)$$

If ϵ is fixed throughout the experiment, it is natural to set $u(\epsilon) = 1$. Then the utility of the alternative y is uniquely determined under the standard assumption of an interval scale. Unfortunately, as Edwards (1961b) has subsequently pointed out himself, if the N bets assumption is taken in full generality, it may be shown to imply that utility is linear in money.

The subjects were five male undergraduates each of whom was run for a total of 50 sessions. The first 12 sessions were devoted to the N -bets method of utility measurement, the technique used for determining indifference was similar to that of Mosteller and Nogee. The most outstanding thing about the resulting utility curves is their relatively high degree of linearity, which is much greater than that found by Mosteller and Nogee and by Davidson, Suppes, and Siegel.

We reserve consideration of the rest of this experiment, in particular, the number of correct and incorrect predictions based on the subjective expected utility model, until we have discussed various methods for measuring subjective probability (Sec 4 3).

In concluding this subsection on the measurement of utility, we wish once again to emphasize to those readers mainly oriented in psychology and not overly familiar with economics that an extensive and in many

points a very penetrating literature exists on the measurement of utility within economics [See the bibliography of Majumdar (1958) for a guide to these papers and books] However, much of this literature, in spite of its acuity, differs considerably from the usual psychological article On the one hand, no systematic experimental or empirical results are presented, and, on the other hand, no systematic formal methods or results are set forth A high proportion of these articles consists of acute observations on what should be the case in the behavior of any reasonable person, and, as might be expected, much of the discussion is devoted to normative rather than descriptive concerns It is nonetheless a trifle startling to find that Majumdar's entire book, which is devoted to the measurement of utility, contains no reference to any experiments aside from a very casual one to Mosteller and Nogee

4.3 Measurement of Subjective Probability

The bifurcation of these measurement problems into utility and subjective probability is artificial for several of the studies that we are summarizing because both have been measured within a single study, sometimes even within the same experimental session, and predictions have been made from both measures On the other hand, several of the studies about subjective probability evidenced no concern with utility whatsoever, and so we have been led to make a simple distinction between the measurement of utility and subjective probability Moreover, as far as we know, no experimental studies exist in which extensive measurements of subjective probability were made first, following the development of Savage (1954) which was discussed in Sec 3.3, and then followed by the measurement of utility In all the studies with which we are familiar the order has been reversed utility measurement has preceded subjective probability measurement (In making this statement we do not count as a quantitative measurement of probability the determination of an event with subjective probability $\frac{1}{2}$)

The first attempt to measure subjective probability experimentally was apparently made by Preston and Baratta (1948) Subjects were run in groups of two or more, and they used play money to bid for gambles in a simple auction game The successful bidder was permitted to roll a set of dice after the bidding was completed The probability of winning with the dice corresponded exactly to the probability stated on the card presented for auction For example, on a given play subjects might bid for a prize of 250 points with probability 0.25 of winning If for this gamble the average successful bid was 50 then the authors computed the

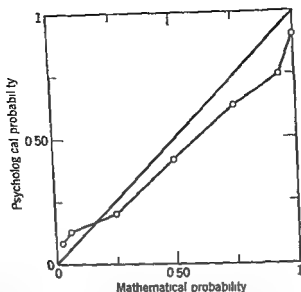


Fig 4 Functional relationship between psychological and mathematical probability
Adapted with permission from Preston & Baratta (1948, p 188)

psychological probability to be $50/250 = 0.20$. From our earlier discussion, it is clear that this method of computing subjective probability assumes that the utility of play money is linear with the value assigned to the money. There is little evidence to support this assumption, but, in spite of this, the experimental results are of considerable interest.

Using this method of computation, perhaps their most interesting conclusion was that objective probabilities less than 0.20 are systematically overestimated and objective probabilities greater than 0.20 are systematically underestimated. The mathematical or objective probabilities of the gambles they offered and of the psychological probabilities computed from the bid data are shown in Fig 4. The point of intersection of the subjective and objective probability curves is about 0.20. Other related results are referred to shortly.

Two other conclusions are noteworthy. First, by comparing the data of sophisticated subjects with those of subjects who were relatively naive about the facts of probability, they concluded that the underestimating and overestimating effects just described exist in both kinds of subjects. Second, an increase in the number of players, and therefore in the competition among them, tended to increase the underestimation of high probabilities and the overestimation of low ones, which is a somewhat surprising result.

A readymade, real life comparison of subjective and objective probabilities is provided by the betting on horses under the pari-mutuel system as compared with the objective probabilities of winning as determined a posteriori after the races. The total amount of money bet on a horse

divided by the net total bet on all horses in a race determines the odds and thus the collective subjective probability that the horse will win. Griffith (1949) examined data from 1386 races run in 1947, his results are in remarkable qualitative agreement with those obtained by Preston and Baratta. Low objective probabilities are systematically overestimated by the subjective probabilities, and high objective probabilities are systematically underestimated. For Griffith's data, objective and subjective probability are equal at 0.16, which is close to the 0.20 value obtained by Preston and Baratta. Griffith remarks in a final footnote that the same computations were carried through for all races run at the same tracks in August 1934, and essentially the same results were obtained. In this case the indifference point fell at 0.18 rather than at 0.16. The invariance of the results from the depression economy of 1934 to the relatively affluent one of 1947 increases their significance.

A still more extensive study, which is very similar to Griffith's, was made by McGlothlin (1956) of 9605 races run from 1947 to 1953, mostly on California tracks. As one would expect, the general character of his results agree closely with those of the earlier study. The objective and subjective probability curves intersect between 0.15 and 0.22. McGlothlin also found some interesting tendencies during the course of the usual eight races on a given day. First, there was an increasing tendency to overestimate low probabilities, and this phenomenon was particularly striking for the final race of the day. There seems to be evidence that on the last race of the day many bettors are uninterested in odds that are not sufficient to recoup their losses from the earlier races. This phenomenon is so striking that bets placed on low odds horses had a net positive expectation even after subtracting the tracks' take, which is 13% in California. This kind of dynamic phenomena is not easily incorporated into an expected utility model. There was also some evidence that losing bettors increased the size of their wagers on the next race more than did winning bettors.

As has already been remarked, Mosteller and Nogee could and did interpret their experimental results in terms of subjective probability rather than utility. Their comparison of objective and subjective probabilities for their two groups of students, as well as the data from the Preston and Baratta study, are shown in Table 3. These results are not in qualitative agreement with those of Preston and Baratta and of Griffith. The Harvard students systematically underestimated objective probabilities over the entire range of values. (Naturally, this could not happen if subjective probabilities were required to add up to 1, but this constraint was not built into the studies we have been considering.) The Guardsmen, on the other hand, equated subjective and objective probability at about

$p = 0.50$, which is considerably higher than the corresponding points obtained by Preston and Baratta, Griffith, and McGlothlin. We join Mosteller and Noguee in being at a loss to explain the differences in these results.

Table 3 Comparison of True and Psychological Probabilities from Preston and Baratta (1948) and for the Student and Guardsmen Groups of the Mosteller and Noguee Experiment (1951, p. 397)

Approximate True Probability	P & B	Students $N = 10$	Guardsmen $N = 5$
0.667	0.55	0.54	0.56
0.498	0.42	0.47	0.50
0.332	0.26	0.30	0.36
0.167	0.15	0.16	0.28
0.090	0.14	0.081	0.18
0.047	0.12	0.038	0.083
0.010	0.07	0.0085	0.052

We noted in Sec. 4.2 that Edwards (1955) measured subjective probability by using the utility curves constructed by his N bets method and the familiar equation for indifference between a certain gain of a fixed amount and a probability p of winning another amount or obtaining nothing. Under the utility normalization chosen (see p. 320), the subjective probability of the event having objective probability p is just the obvious ratio of the two utilities. He devoted sessions 13 to 18 to subjective probability measurements, using both negative and positive outcomes. Edwards' results were different from any previously mentioned. When the outcomes were positive, the five subjects almost uniformly overestimated objective probabilities in the probability range tested, namely, from $\frac{1}{4}$ to $\frac{3}{4}$. When the outcomes of the bets were negative (losses), the subjective values were extremely close to the objective ones.

Edwards used the constructed utility and subjective probability functions to predict choices in Sessions 19 to 22. The subjective expected utility model he applied, which was described on p. 320, predicted significantly better than chance the choice behavior of subjects in these four sessions. One hundred and twelve offers were made of each of four bets at each of five levels of expected value as determined by their monetary outcomes and objective probabilities. Of the resulting 2240 offers, the subjective

expected utility (SEU) model correctly predicted the choice made in 1629 cases. Edwards compared these predictions with those obtained from the expected utility (EU) model that utilizes objective probability and measured utility and with the subjective expected monetary (SEM) model that utilizes monetary value and subjective probability. The results for the EU model were slightly worse than chance, and therefore very unsatisfactory. The SEM model, on the other hand, did slightly better than the SEU model, but not at a level that is statistically significant. These results argue that the difference between subjective and objective probability is of considerably greater importance than that between utility and monetary outcome. Some extensions of Edwards' findings are reported by Becker (1962).

Davidson, Suppes, and Siegel (1957, Chapter 2) used their constructed utility curves (Sec 4.2) and Eq 10 of Sec 3.3 to measure subjective probability. For seven of their subjects they made three different determinations of the subjective probability of an event whose objective probability was 0.25. This event was generated by a four-sided die—two opposite ends were rounded so that it always landed on one of the other four sides. Using the method of approximation described in Sec 4.2, they found upper and lower bounds for each subject and for each combination of utilities used. For five of the seven subjects the three sets of lower and upper bounds had a nonempty intersection, and for two they did not. For four of the five subjects with a nonempty intersection, the upper bound was below 0.25 and for the remaining subject the lower bound was below and the upper bound above 0.25. These results agree with those of Preston and Baratta, Griffith, and McGlothlin concerning the underestimation of probabilities above 0.20.

Among the experimental studies devoted to the measurement of subjective probability, this is the only one in which the procedure used could admit the conclusion that subjective probability cannot be consistently measured, although a number of experimental studies are cited in Sec 4.4 which cast doubt on the possibility of such measures. The point is that a claim of fundamental measurement can only really be made when it is clear that enough data have been collected so that it is possible to reject the model. The systematic pursuit of the cross-checks implied by any model of fundamental measurement seems to have been unduly neglected in most experimental studies of subjective probability.

Toda (1951, 1958) has proposed a two-person game method for measuring subjective probability which is very similar to the auction procedure of Preston and Baratta. One of its more interesting applications was made by Shuford (1959) to obtain extensive measurements of subjective probability of both elementary and compound events. Subjects rolled a

20 face die (with the integers 1 to 20 on the faces) twice to select the row and column of a 20×20 matrix of vertical and horizontal bars. The subjects, 64 airmen in training at an airbase, were run in pairs. The sequence of events on a given trial was as follows. The matrix of horizontal and vertical bars was projected on a screen. Subject A wrote down his bid x that a horizontal bar, say, would be selected ($0 \leq x \leq 10$), subject B decided to "buy" or "sell" A's bid, the die was rolled by the experimenter to decide the bet, that is, which element of the projected matrix was selected, and finally the subjects scored themselves for the play. It can be shown that A's minimax strategy in this game is to set x equal to 10 times his subjective probability of the favorable outcome.

Two games were played with each pair of subjects. In the first game the payoff was determined by the occurrence of a horizontal or vertical bar, as the case might be, in the position of the matrix selected by the two rolls of the 20 face die. In the second game, the bet paid off if two successive selections of elements of the matrix made with the die resulted in two bars of the same type.

Shuford presented the results for individual subjects, but as these are too complicated to give here, we content ourselves with some summary observations. Confirming the earlier findings of Preston and Baratta, Griffith, and McGlothlin, a fairly large fraction of the subjects overestimated low probabilities and underestimated high ones. This was true for both the elementary and the compound events. On the other hand, Shuford found that the subjective probability estimates of a number of the subjects were fit quite well by a linear function of objective probability, although the slope and intercept of this function varied from one subject to another. His findings about the estimation of compound events are particularly interesting, for, to our knowledge, this is the only experiment explicitly concerned with this issue. A majority of the subjects approximated the correct rule, that is, they estimated the probability of the compound event as approximately the square of the probability of the elementary event. That the application of the correct rule was so common is surprising because when the subjects were asked at the end of the series of trials what rule they had used, only two stated the correct one. Those who did not approximate the correct rule came very close to approximating the probability of a single elementary event. For at least two subjects, however, no simple rule seemed to account for their estimates of the probability of compound events. As Shuford remarked, the investigation of the compounding rules used in other experimental situations is of considerable importance.

Finally, we should like merely to mention a series of studies that, although not explicitly concerned with the measurement of subjective

probability, bear a rather close relation to it. The common thread in this work is a scale of proportion that is based on the responses of subjects to randomly composed arrays of two or more types of elements, usually in a relatively difficult perceptual situation. The specific references are Philip (1947), Shuford (1961a, b), Shuford and Wiesen (1959), Stevens and Galanter (1957), Wiesen (1962), and Wiesen and Shuford (1961). The three papers involving Wiesen are of particular interest because of the emphasis on Bayesian estimation schemes.

4.4 Experimental Studies of Variance-Preference, Utility-of-Gambling, and Other Models

A number of investigators have attempted either to show that models of choice behavior based only on the concepts of subjective probability and utility cannot possibly work, because, for example, of an interaction between utility and probability, or to establish that alternative models, in fact, do a better job of prediction. We review briefly the studies most relevant to the central concerns of this chapter.

Probably the most extensive set of empirical studies that have attempted to isolate factors in gambling that are not easily accounted for by an expected utility model were performed by Edwards. In the first paper of the series (1953), probability preferences in gambling were studied using 12 undergraduates as subjects who selected between bets of equal expected monetary value. The chance events were generated by a pinball machine, the objective probabilities varied by $\frac{1}{8}$ steps from $\frac{1}{8}$ to $\frac{7}{8}$. Edwards found that two main factors determined choices between bets of equal expected value. First, there was a general tendency either to prefer or to avoid long shots, that is, bets with a low probability of winning or losing, and second, there were personal preferences for specific probabilities, the two most important being a definite preference for the $\frac{3}{8}$ probability and an aversion to the $\frac{5}{8}$ probability of winning. The existence of specific probability preferences certainly raises difficulties for any axiomatization of utility and subjective probability along the lines discussed in Sec. 3. In Edwards (1954a) the methods of the earlier study were extended to bets having different expected monetary values. The results tended to show that subjects' choices were influenced by preferences for bets with higher expected values (or lower negative expected values) as well as by preferences among the probabilities. Edwards (1954b) replicated and extended slightly the findings of the two earlier studies. In particular, one experiment showed a similar pattern of preferences among probabilities in a nongambling game that involved uncertainty. The exact details of the

game are too complicated to describe here, suffice it to say that it was conceptualized to the subjects as a strategic game of military warfare in which they were to act as the head of an amphibious task force whose mission was to destroy ammunition stored at one of eight ports

Edwards (1954c) asked whether or not variance preferences exist in gambling. Using the pinball game mentioned, some preferences for variances were exhibited when the probabilities were held constant, but on the whole these seemed to be secondary in importance to probability preferences. The same pattern of probability preferences found in Edwards' earlier work was again evidenced.

These studies of Edwards are essentially negative in character in the sense that no well formulated alternative models are set forth to account for the experimental findings. Moreover, most of the experiments involved the assumption either of objective probabilities or of utilities linear with money, and, therefore, a more detailed critique is necessary to refute the subjective expected utility model. On the other hand, some of Edwards' data on probability preferences are so striking that it seems clear that no simple SEU model can adequately explain his findings.

Let us turn in more detail to the question of variance preferences. The suggestion is an old one (Fisher, 1906) do individuals make choices on the basis not only of expectation but also on the dispersion of the possible outcomes? That variance preferences are a significant variable has been argued especially by Allais (1953) in his criticisms of the "American school" that advocates the analysis of choice behavior in the presence of uncertainty in terms of expected utility. The issues here are thoroughly tangled. Although a number of possible classifications can be devised for analyzing the problem, such as the variance of the objective distribution, the variance of the subjective distribution, a linear function of the mean and variance of the objective distribution, a linear function of the mean and variance of the subjective distribution, etc., perhaps the most constructive approach is to postulate preferences in terms of the objective probabilities and to compare that with the expected utility model. This tack was followed by Coombs and Pruitt (1960) who suggested, as an alternative to the expected utility model, that a gamble be characterized in terms of the expectation, variance, and skewness of its objective probability distribution over money. They make the valid point that although a variance preference usually can be interpreted alternatively in terms of a utility function for money, the number of admissible preference orderings is much reduced when a gamble is characterized as a function of expectation, variance, and skewness than when arbitrary monotonic utility functions are permitted. However, this criticism of utility theory is faulted when parametric cardinal utility functions are used.

For example, a polynomial utility function with three degrees of freedom would make a natural comparison to their function of expectation, variance, and skewness

They conducted a study with 99 undergraduates who chose among bets having a constant expectation but different skewnesses and variances. A typical choice was between

Bet *A* $\frac{1}{2}$ to win \$1 40, $\frac{2}{3}$ to lose 70¢,
 Bet *B* $\frac{1}{2}$ to win \$1 00, $\frac{1}{2}$ to lose \$1 00

All bets had an expected monetary value of 0. From the analysis of their data, Coombs and Pruitt drew two principal conclusions of relevance here. First, variance preference orderings were limited almost exclusively to those that could be generated by unfolding the natural ordering of the variance scale, using the unfolding technique described at the end of Sec. 2.4. Approximately one-third of the subjects preferred low variance, one third high variance, and one-third intermediate degrees of variance. Second, the great majority of subjects had their ideal positions at one end or the other of the skewness scale.

Pruitt (1962) extended the Coombs and Pruitt approach to a pattern and level of risk (PLR) model. According to Pruitt's definition, two bets have the same pattern if one may be obtained from the other by multiplying the outcomes by a positive constant. Thus, the following two bets have the same pattern:

A $\frac{1}{2}$ chance to win \$1 00
 $\frac{1}{3}$ chance to lose 75¢
 $\frac{1}{6}$ chance to lose 60¢
 $\frac{1}{2}$ chance to win \$3 00
B $\frac{1}{3}$ chance to lose \$2 25
 $\frac{1}{6}$ chance to lose \$1 80

His definition of the level of risk of a bet is the sum of its negative outcomes weighted by their respective probabilities of occurrence. For instance, the level of risk of Bet *A* is 35 cents, and of Bet *B*, \$1 05. Pruitt argued that these two aspects of bets are the ones that people usually perceive, and he was lead to define the utility, $u(A)$, of an Alternative or Bet *A* by the following equation:

$$u(A) = r(A) \cdot p(A) + g(A),$$

where $r(A)$ is the level of risk of *A*, $p(A)$ is the utility of the pattern of *A*, and $g(A)$ is the utility of risk of *A*. Pruitt did not attempt to derive this somewhat arbitrary equation from simple behavioral assumptions analogous to those considered in Sec. 3. Rather, he derived several

consequences of this equation combined with some Coombsian assumptions about the individual possessing an "ideal" level of risk. Typical examples are the following two propositions which he tested on the Coombs and Pruitt (1960) data mentioned previously and on the data from a similar experiment with 39 undergraduates as subjects. The first proposition asserts that the order of preference among patterns is independent of the level of risk. The second proposition asserts that the more preferred a pattern of risk, the higher will be the ideal level of risk for that pattern. Both propositions are firmly supported by the data from the two experiments. Pruitt's article is also recommended to the reader because of its useful and incisive review of the strengths and weaknesses of various expected utility models, all of which have already been mentioned in this chapter.

A specific utility-of-gambling model has been proposed by Royden, Suppes, and Walsh (1959). The behavioral postulate of the model is that individuals choose among options so as to maximize the sum of the expected monetary value and the utility of gambling. For bets or options with two outcomes and with subjective probability $\frac{1}{2}$ for each outcome, each bet is uniquely determined by its mean and the absolute difference between the two outcomes, the absolute difference is itself twice the standard deviation. The utility of gambling of a bet was defined then as a function of the mean and the difference, and the two-dimensional surface corresponding to the function was experimentally determined by an experimental procedure similar to that of Davidson, Suppes, and Siegel (1957). The utility of gambling surfaces obtained for individual subjects turned out to be sufficiently complex to cast doubt on the possible adequacy of any simple theory of probability and variance preferences as complete determiners of choices.

The subjects were eight sailors from a naval airbase and eight university undergraduates. The utility of gambling surface constructed for each subject was used to predict choices in an additional experimental session. In comparison with the actuarial model—maximization of expected monetary value—the utility-of-gambling model was significantly better in predicting the choices of the sailors and significantly worse with the students as measured by a chi-square test that combined individual significance levels.

Finally, we mention the empirical studies of risk and gambling by John Cohen and his associates at the University of Manchester [for an overview see either the book by Cohen and Hansel (1956) or the book by Cohen (1960)]. Although some interesting findings are reported, the experiments are not quantitatively oriented and so fall outside the main concern of this chapter.

GENERAL PROBABILISTIC CHOICE THEORIES⁵

Paralleling our presentation of the algebraic theories of preference, we first examine those probabilistic theories that impose no mathematical structure (other than the set structure) over the outcome set. Following this, in Sec. 6, we take up the general probabilistic theories that attempt to describe how subjects rank order sets of outcomes. In Sec. 7, we discuss theories that deal explicitly with outcomes that are probability distributions over a set of "elementary" outcomes. And finally, in Sec. 8, we examine the experiments performed to test these models.

5.1 Response Probabilities

As we emphasized in Sec. 1, theorists are currently trying to cope with the class of preference experiments for which the subject has no discrimination problems. In such experiments his responses simply inform the experimenter which of the several offered outcomes the subject prefers at that instant. That being so, it has generally been taken for granted that one can suppress all the notational apparatus for the stimuli and responses and act as if what is chosen is an element from the unordered set X of presented outcomes, where X is a subset of the set A of all outcomes used or potentially available in the experiment. Actually, this oversimplifies matters a great deal by ignoring the possibility of response biases, but because all of the existing theories are of this character we follow the convention that we need only consider the set X of outcomes offered and the one, $x \in X$, chosen by the subject.

We shall suppose in these sections that the subject's response to the same choice situation is governed by a probability mechanism, and so in general he exhibits inconsistencies. Indeed, it was the empirical observation of inconsistencies (Sec. 4.2) that led to the study of probability models in the first place. The assumption of a simple probability mechanism to explain behavioral inconsistency is certainly not the only possibility. For example, a number of authors believe that inconsistencies arise from

⁵ Many of the ideas and most of the theorems that we discuss in this section and the next are due to Marschak and his collaborators. The reader interested in several notions that we shall not cover here should consult Becker, DeGroot, and Marschak (1963a), Block and Marschak (1960) and Marschak (1960). Some of the proofs that we give are taken directly from these papers; the remainder are somewhat different because we have chosen to organize the results somewhat differently. We make no attempt to indicate which is which.

the subject attending to different aspects of the choice situation. Were it possible to specify what it is that causes him to attend to one aspect rather than another aspect, a deterministic model might very well be appropriate. As long as these causes are unknown, we may very well have to be satisfied with probability models.

In contrast to algebraic utility theory, which largely has not been the product of psychologists, probabilistic theories have had a long development in psychology and only recently have begun to be welded together with the algebraic utility ideas. The Fechner-Thurstone development of probability models for psychophysics and psychoacoustics corresponds to what in the present context are called strong and random utility models. The recent history of developments in probability models is rather tangled, but much of it can be recovered from our references to specific ideas and results.

The probability models all assume that when X is presented on trial n , there exists a probability $p_{Xn}(x)$ that $x \in X$ is chosen. The models in this section include the added assumption that these probabilities are independent of the trial on which x is presented, so the subscript n is suppressed. The assumption that these are probabilities and that a choice is made at each opportunity is summarized by

$$\begin{aligned} p_X(x) &\geq 0, & \text{for all } x \in X, \\ \sum_{x \in X} p_X(x) &= 1 \end{aligned} \tag{17}$$

In practice, these probabilities are estimated by presenting X repeatedly to the subject, often interleaved among presentations of a number of other subsets of A , and the proportion of times x is chosen when X is presented is taken as the estimate of $p_X(x)$.

The theoretical problem is to discover what mathematical constraints the subject imposes on the probabilities beyond those automatically implied by probability theory. Each theory makes such a proposal, and the empirical problem, discussed in Sec. 8, is to decide how accurate the proposals actually are. A number of the theories do not attempt to describe the probability structure for all subsets of A but just for the two element sets. In these binary preference theories it is convenient to write $p(x, y)$ for $p_{\{x, y\}}(x)$.

5.2 Constant Utility Models

Although much data suggest that algebraic utility models are inadequate, the intuitive idea of representing the strength of preference in terms of a numerical utility function—in terms of a subjective scale—is much too

appealing to be abruptly dropped. Indeed, no one has yet dropped it at all, every theory that we examine includes such a notion, although in at least one case it can be dispensed with easily if one chooses.

Basically, at present, there are only two approaches to how response probabilities and utility scales relate. In the approach characterized by the constant utility models discussed here, the utility function is a fixed numerical function over the outcomes, and the response probabilities are some function of the scale values of the relevant outcomes. The choices are assumed to be governed directly by these response probabilities. In the other approach, defined by the random utility models of Sec 5.3, the utility function is assumed to be randomly determined on each presentation, but once selected the subject's decision is unequivocally determined by the relevant utilities just as in the algebraic models. Probabilistic behavior arises from the randomness of the utility function, not from the decision rule. We first present the constant utility models.

Perhaps the weakest probabilistic generalization of the fundamental decision rule of the algebraic models is the one that simply says that a tendency to choose x over y exists if and only if the utility of x is greater than the utility of y . This we formalize as

Definition 16 *A weak (binary) utility model is a set of binary preference probabilities for which there exists a real-valued function w over A such that*

$$p(x, y) \geq \frac{1}{2} \text{ if and only if } w(x) \geq w(y), \quad x, y \in A \quad (18)$$

If no other requirements are imposed, then it is not difficult to see that w must be an ordinal scale, that is, w is unique up to strictly monotonic increasing transformations.

Every algebraic theory of utility provides us with a weak binary utility model, and conversely, by means of the following definition. If $x, y \in A$, then

$$x \geq y \text{ if and only if } p(x, y) \geq \frac{1}{2}$$

A set of axioms on \geq that are sufficient to establish the existence of a real-valued, order preserving utility function w can be immediately restated into an equivalent set of axioms about the binary probabilities using Def 16. If the axioms on \geq yield a scale stronger than ordinal, then clearly the corresponding weak utility model has a scale of exactly the same strength. Thus, for example, if $A = A_1 \times A_2$ and if \geq satisfies the axioms of Luce and Tukey (1964) (see Sec 2.3), then the equivalent axioms for binary probabilities yield a representation theorem of the form: there exist real-valued functions u_1 on A_1 and u_2 on A_2 such that

$$p[(a, x), (b, y)] \geq \frac{1}{2} \text{ if and only if } u_1(a) + u_2(x) \geq u_1(b) + u_2(y),$$

and u_1 and u_2 are interval scales with a common unit.

Next in level of strength is the Fechnerian model, most familiar from classical psychophysics (see Chapter 4, Sec 2)

Definition 17 A strong (or Fechnerian) (binary) utility model is a set of binary preference probabilities for which there exist a real-valued function u over A and a cumulative distribution function ϕ such that

$$(i) \phi(0) = \frac{1}{2}, \text{ and}$$

$$(ii) \text{ for all } x, y \in A \text{ for which } p(x, y) \neq 0 \text{ or } 1,$$

$$p(x, y) = \phi[u(x) - u(y)] \quad (19)$$

It is clear that by an appropriate change in ϕ , u can be transformed by positive linear transformations without affecting the representation, and it can be shown (Block & Marschak, 1960, p 104) that these are the only possible transformations—that u is an interval scale—when A is a continuum and ϕ is strictly increasing

Theorem 29 Any strong (binary) utility model in which the preference probabilities are different from 0 or 1 is also a weak utility model, but not conversely

PROOF Suppose that $p(x, y) \geq \frac{1}{2}$, then the monotonicity of ϕ and $\phi(0) = \frac{1}{2}$ imply, by (ii) of Def 17, that $u(x) - u(y) \geq 0$. So u is the weak utility function

The failure of the converse is easily shown by examples

Although the idea that strength of preference, as measured by the binary choice probabilities, should be a monotonic function of a scale of strength of preference, of utility, is an appealing one, examples have been suggested that make it dubious as a general hypothesis. The following was offered by L. J. Savage⁶ as a criticism of the next model to come, but it applies equally well to the strong binary model

Suppose that a boy must select between having a pony x and a bicycle y and that he wavers indecisively between them. According to the strong utility model, $u(x)$ must be approximately equal to $u(y)$. The bicycle dealer, trying to tip the scale in his favor, suddenly brings forth another bicycle z which, although basically the same as y , is better in minor ways, such as having a speedometer. One can well believe that the boy will still be indifferent or nearly so, between x and z . The minor differences between the two bicycles do not help much to resolve his indecision between a pony and a bicycle—any bicycle within reason. According to the model, we are forced to conclude that the utility difference between the two bicycles, $u(y) - u(z)$, is small when evaluated in terms of preferences relative to x .

⁶ In correspondence with Luce

Nevertheless, were the boy forced by his impatient father to confine his choice to the two bicycles, he might very well exhibit a strong preference for one over the other, forcing us to conclude that the utility difference $u(y) - u(z)$ is not small, contrary to the earlier conclusion. If this can happen—and the introspections of several people suggest that it can—then the strong binary utility model is too strong to describe these preferences.

We return to this problem after first stating an even stronger model.

Definition 18 *A strict binary utility model is a set of binary preference probabilities for which there exists a positive real valued function v over A such that for all $x, y \in A$ for which $p(x, y) \neq 0$ or 1 ,*

$$p(x, y) = \frac{v(x)}{v(x) + v(y)} \quad (20)$$

The strict utility scale is determined up to multiplication by a positive constant, that is, it is a ratio scale.

The strict binary utility model has been investigated by a number of authors: Abelson & Bradley (1954), Becker, DeGroot, & Marschak (1963a), Block & Marschak (1960), Bradley (1954a,b, 1955), Bradley & Terry (1952), Ford (1957), Gulliksen (1953), Luce (1959), Marschak (1960), Thurstone (1930), and Tornqvist (1954).

Theorem 30 *Any strict binary utility model is also a strong utility model, but not conversely.*

PROOF Let v be the strict utility function. Define $u = \log v + b$, and let ϕ be the logistic distribution function, $\phi(\zeta) = 1/(1 + e^{-\zeta})$, then

$$\begin{aligned} p(x, y) &= \frac{1}{1 + v(y)/v(x)} \\ &= \frac{1}{1 + \exp\{-[u(x) - u(y)]\}} \\ &= \phi[u(x) - u(y)] \end{aligned}$$

The failure of the converse is easily shown by examples.

These last two models have two rather peculiar features. The first, which is also shared by the weak utility model, is their restriction to binary choices. Although many experiments involve only binary presentations, larger presentation sets could easily be used, and one would expect an adequate theory to treat all possible presentation sets. Second, these theories are stated only for probabilities different from 0 and 1, nothing is asserted about how the 0 and 1 probabilities relate to the others or to each other, which is the reason we were able to omit the restriction to probabilities different from 0 or 1 in the hypothesis of Theorem 30, but not in that of Theorem 29. Some authors have suggested that the non-0

or 1 models are suitable idealizations to reality, on a par with the continuous idealizations to discrete phenomena in physics, and so there really is no problem. Others, for example, Block and Marschak (1960, p. 122), have cited simple economic examples where small changes in one component of a commodity bundle change a choice probability from about $\frac{1}{2}$ to 1 in a highly discontinuous fashion. Many feel that these apparent discontinuities are a deep problem in the understanding of preferences.

A generalization of the strict binary model has been suggested that attempts to overcome both these problems (Luce, 1959). In the original presentation, the 0-1 difficulty was dealt with in what, to some critics, seemed a distinctly artificial manner, however, as was shown in Chapter 4, Sec. 3.2, this apparently artificial formulation is implied by the following simpler assumption.

Definition 19 *A set of preference probabilities defined for all the subsets of a finite set A satisfy the choice axiom provided that for all x , Y , and X such that $x \in Y \subseteq X \subseteq A$,*

$$p_X(x) = p_X(x | Y),$$

whenever the conditional probability exists

What the choice axiom says is that a choice from Y is just that, independent of what else may have been available. That is to say, even though X may have been presented, if we look only at those occasions when the choices were made from Y , then the probability of choosing x from Y , $p_X(x | Y)$, is exactly the same as the probability of choosing x from Y , $p_Y(x)$, when only Y was presented in the first place.

Theorem 31 *If the choice axiom holds and A is finite, then there exists a ratio scale v on A such that for any $p_Y(x) \neq 0, 1$,*

$$p_Y(x) = \frac{v(x)}{\sum_{y \in Y} v(y)} \quad (21)$$

PROOF Equation 24 of Sec. 3.2 of Chapter 4 (p. 221)

We see that for non 0 or 1 probabilities, the choice axiom generalizes the strict binary utility model, indeed, the property asserted in Theorem 31 has been called the *strict utility model* by Block and Marschak (1960).

Of course, the pony bicycle example discussed is applicable to the strict utility models because they are special cases of the strong ones. Nevertheless, it is instructive to see how it violates the choice axiom. [Substantially, this same example was discussed by Debreu (1960b) in his review of Luce (1959).] Let x again denote the pony and let y and z be two nearly identical bicycles with compensating features so that the boy is indifferent

between each of the pairs, hence $v(x) = v(y) = v(z)$. According to Theorem 31, the probability of $\frac{1}{2}$ of choosing the pony x when only one bicycle is offered is dropped to $\frac{1}{3}$ when they are both offered. Savage and Debreu feel that this conclusion is incorrect—that the two bicycles would be treated as a single alternative, later to be analyzed into a final choice if the pony is rejected.

The argument seems convincing, and it is closely related to the observation (Luce, 1959, pp. 132–133) that we cannot expect the choice axiom to hold for over-all decisions that are divided in some manner into two or more intermediate decisions. It appears that such criticisms although usually directed toward specific models, are really much more sweeping objections to all our current preference theories. They suggest that we cannot hope to be completely successful in dealing with preferences until we include some mathematical structure over the set of outcomes that, for example, permits us to characterize those outcomes that are simply substitutable for one another and those that are special cases of others. Such functional and logical relations among the outcomes seem to have a sharp control over the preference probabilities and they cannot long be ignored.

5.3 Random Utility Models

The random utility models are more similar in spirit to the algebraic models than are the constant utility ones. They are the same to this extent: the subject is assumed to choose the outcome that has the largest utility value at the time of choice. They differ in that the utilities are no longer assumed to stay put. Each time a choice is to be made, a utility function is selected according to some postulated probability mechanism, and the outcome is determined algebraically by it. The most familiar psychological mechanism of this type is Thurstone's model in which the utilities are determined by multivariate normal distribution functions, but more complex mechanisms are possible.

In the next definition we use the concept of a (finite dimensional) random vector. Let A be the finite set of alternatives and let U be a function defined on A such that, for each x in A , $U(x)$ is a random variable. Then we call U a *random vector on A* . In general, no assumption is made about dependency relations between the random variables. Thus without further assumptions the following equality cannot be simplified

$$Pr[U(x) \geq U(y), y \in Y] = \int_{-\infty}^{\infty} Pr[U(x) = t, U(y) \leq t, y \in Y] dt$$

If the random variables are independent, then the right-hand side simplifies to

$$\int_{-\infty}^{\infty} \Pr[U(x) = t] \prod_{y \in Y - \{x\}} \Pr[U(y) \leq t] dt$$

Definition 20 A random utility model is a set of preference probabilities defined for all subsets of a finite A for which there is a random vector U on A such that for $x \in Y \subseteq A$,

$$p_1(x) = \Pr[U(x) \geq U(y), y \in Y] \quad (22)$$

If the definition is only asserted for the binary preference probabilities, that is,

$$p(x, y) = \Pr[U(x) \geq U(y)],$$

then the model is called a binary random utility model. If the random vector U consists of components that are independent random variables, then we say the model is an independent random utility model.

The primary theoretical results so far obtained about random utility models establish relations with the constant utility models, which we now present, and with certain observable properties, which we discuss in Sec 5.6.

Theorem 32 For choice probabilities different from 0 and 1, any strict utility model is an independent random utility model, but not conversely.
PROOF ⁷ Suppose that v is the scale of the strict utility model. Define U to be the random vector whose components are independent random variables with probability densities

$$\Pr[U(x) = t] = \begin{cases} v(x)e^{v(x)t}, & \text{if } t \leq 0 \\ 0, & \text{if } t > 0 \end{cases}$$

Consider

$$\begin{aligned} \Pr[U(x) \geq U(y), y \in Y] &= \int_{-\infty}^{\infty} \Pr[U(x) = t] \prod_{y \in Y - \{x\}} \Pr[U(y) \leq t] dt \\ &= \int_{-\infty}^0 v(x)e^{v(x)t} \prod_{y \in Y - \{x\}} \left[\int_{-\infty}^t v(y)e^{v(y)\tau} d\tau \right] dt \\ &= \int_{-\infty}^0 v(x)e^{v(x)t} \prod_{y \in Y - \{x\}} e^{v(y)t} dt \\ &= \int_{-\infty}^0 v(x) \exp \left(\sum_{y \in Y} v(y)t \right) dt \\ &= \frac{v(x)}{\sum_{y \in Y} v(y)} \end{aligned}$$

⁷ This proof is due to Eric Holman and A. A. J. Marley. It has the advantages over Block's and Marschak's (1960, p. 110) proof that it does not use any ranking notions that it is constructive and that it is a good deal shorter.

Note that any monotonic transformation f of U in the preceding example again provides a random utility interpretation of a given strict utility model provided that we replace t by $f^{-1}(t)$ in the right side of the defining equation. It is conjectured that these are the only reasonably well-behaved examples, but no proof has yet been devised. In any event, just about any random utility model one cares to write down is not a strict utility model, so the failure of the converse of the theorem is easy to establish.

Although certain other relations exist among the random, weak, and strong utility models that are easy to work out directly, we shall not do so because they can be deduced from later results using only the transitivity of implication. It is sufficient to note the results here.

There are (binary) random utility models that are not weak, and therefore not strong, utility models

There are strong, and therefore weak, utility models that are not random utility models

The question also arises how the strong and the random models are related when both are binary. No complete answer seems to be known, however, the following is of interest.

Theorem 33 *Any strong utility model for which the distribution function ϕ is the difference distribution of two independent and identically distributed random variables is a binary random utility model*

PROOF Let u be the strong utility function and define the random variable $U(x) = u(x) + \epsilon(x)$ where for $x \neq y$, $\epsilon(x)$ and $\epsilon(y)$ are independent and identically distributed and ϕ is the distribution function of $\epsilon(x) - \epsilon(y)$. Then

$$\begin{aligned} Pr[U(x) \geq U(y)] &= Pr[u(x) + \epsilon(x) \geq u(y) + \epsilon(y)] \\ &= Pr[\epsilon(y) - \epsilon(x) \leq u(x) - u(y)] \\ &= \phi[u(x) - u(y)] \\ &= p(x, y) \end{aligned}$$

5.4 Observable Properties

None of the models just described has ever been tested experimentally in any complete fashion. Indeed, save for the choice axiom, they are all stated in terms of nonobservable utility functions, and so it is impossible to test them completely until we know conditions that are necessary and sufficient to characterize them in terms of the preference probabilities themselves, for only these can be estimated from data. No results of this

generality are now known, and so experimenters have been content to examine various observable properties that are necessary consequences of at least some of the models

In this subsection we list several of the more important properties, although not all that Marschak and his colleagues have investigated theoretically, in Sec 5.5 we cite some of the interrelations among them, and in Sec 5.6, the relations between them and the several models are described

Consider three outcomes x , y , and z and the corresponding binary probabilities $p(x, y)$, $p(y, z)$, and $p(x, z)$. If we think of these as represented in the plane by arrows of lengths $p(x, y)$, $p(y, z)$, and $p(x, z)$, then it is not unreasonable that we should be able to construct a triangle from them. That is to say, the length corresponding to the x - z pair should not exceed the sum of the lengths corresponding to the x - y and y - z pairs. If so, then the following property must be met by the data

Definition 21 *A set of binary preference probabilities satisfies the triangle condition if for every $x, y, z \in A$,*

$$p(x, y) + p(y, z) \geq p(x, z) \quad (23)$$

Marschak (1960, p. 317) attributes the triangle condition to Guilbaud (1953). It is in many ways the weakest property that has been suggested.

The next group of ideas are all attempts to extend to the probabilistic situation the notion that preferences are transitive. The term "stochastic" used in the definition is inappropriate (see the discussion in Sec. 1.3), but it is so ingrained in the literature that we make no attempt to change it here.

Definition 22 *Whenever $\min [p(x, y), p(y, z)] \geq \frac{1}{2}$, the binary preference probabilities are said to satisfy*

(i) weak (stochastic) transitivity provided that

$$p(x, z) \geq \frac{1}{2}.$$

(ii) moderate (stochastic) transitivity provided that

$$p(x, z) \geq \min [p(x, y), p(y, z)], \quad (24)$$

(iii) strong (stochastic) transitivity provided that

$$p(x, z) \geq \max [p(x, y), p(y, z)] \quad (25)$$

Marschak (1960, p. 319) attributes the definitions of weak and strong transitivity to Valavanis-Vail (1957) and that of moderate (sometimes called *mild*) transitivity to Georgescu-Roegen (1958) and Chipman (1958).

Menger (1951) proposed a multiplicative condition on triples of binary probabilities which Morrison (1963) modified slightly by requiring the

same hypothesis as in the stochastic transitivity conditions. We adopt Morrison's form of the definition.

Definition 23. *A set of binary preference probabilities satisfy the multiplicative condition if for every $x, y, z \in A$ such that $\min[p(x, y), p(y, z)] \geq \frac{1}{2}$, then $p(x, y)p(y, z) \leq p(x, z)$.*

We turn next to a binary property, due to Davidson and Marschak (1959), which involves four elements and which has already been discussed for algebraic models in Sec. 2.4. It is suggested by the strong utility model. Suppose that that model holds, then by the strict monotonicity of ϕ , $p(w, x) \geq p(y, z)$ is equivalent to $u(w) - u(x) \geq u(y) - u(z)$, which in turn is equivalent to $u(w) - u(y) \geq u(x) - u(z)$ and that is equivalent to $p(w, y) \geq p(x, z)$. Thus we are led to consider the following property.

Definition 24. *A set of binary preference probabilities satisfy the quadruple condition provided that $p(w, x) \geq p(y, z)$ implies $p(w, y) \geq p(x, z)$.*

Our final binary property was pointed out by Luce (1959). It says, in essence, that the probability of an intransitive cycle of independent choices $x > y$, $y > z$, $z > x$ is exactly the same as the probability of $x > z$, $z > y$, $y > x$.

Definition 25. *A set of binary preference probabilities not equal to 0 or 1 satisfy the product rule^a if for every set of three distinct elements^b $x_1, x_2, x_3 \in A$,*

$$p(x_1, x_2)p(x_2, x_3)p(x_3, x_1) = p(x_1, x_3)p(x_3, x_2)p(x_2, x_1). \quad (26)$$

^a This term appears to be due to Estes (1960, p. 272). Suppes and Zinnes (1963, p. 49) introduced the term "multiplication condition," which, however, is too similar to the "multiplicative condition" of Def. 23.

^b One might expect Def. 25 to have been stated for n elements rather than three, however, the following result, which was pointed out to us by R. M. Rose, shows that this is unnecessary.

If a set of binary preference probabilities satisfies the product rule, then for any set of $n \geq 3$ distinct elements $x_1, x_2, \dots, x_n \in A$

$$p(x_1, x_2)p(x_2, x_3) \dots p(x_{n-1}, x_n) = p(x_1, x_n)p(x_n, x_{n-1}) \dots p(x_2, x_1) \quad (27)$$

PROOF. By hypothesis it is true for $n = 3$. We proceed inductively, assuming Eq. 27 for n and showing it for $n + 1$. Let $x_1, x_2, \dots, x_n, x_{n+1}$ be distinct elements of A . By the induction hypothesis, Eq. 27 can be written

$$\frac{p(x_1, x_2)p(x_2, x_3) \dots p(x_{n-1}, x_n)}{p(x_n, x_{n-1}) \dots p(x_2, x_1)} = \frac{p(x_1, x_n)}{p(x_n, x_1)}$$

Substituting,

$$\frac{p(x_1, x_2)p(x_2, x_3) \dots p(x_{n-1}, x_n)p(x_n, x_{n+1})p(x_{n+1}, x_1)}{p(x_1, x_{n+1})p(x_{n+1}, x_n)p(x_n, x_{n-1}) \dots p(x_2, x_1)p(x_1, x_2)} = \frac{p(x_1, x_n)p(x_n, x_{n+1})p(x_{n+1}, x_1)}{p(x_1, x_{n+1})p(x_{n+1}, x_n)p(x_n, x_1)} = 1,$$

by the product rule

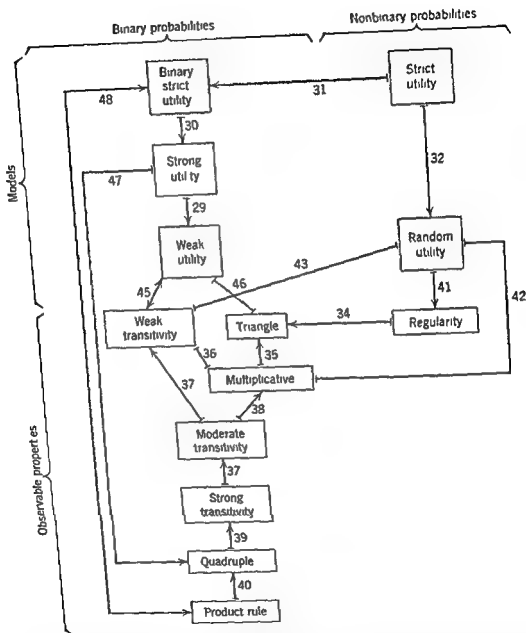


Fig 5 A summary of the implications (\rightarrow) and failures of implication (\nrightarrow) among models and observable properties on the assumption that the probabilities are different from 0 and 1 and that the set of all alternatives is finite. The number beside a line indicates the relevant theorem. The relation between any two concepts can be deduced using only the transitivity of implication from these relations plus the fact that a binary model or property never implies a nonbinary one (see text)

hence $p(z, y) > \frac{1}{2}$. There are, therefore, two possible (nonvacuous) cases of the multiplicative condition to consider.

(i) $p(x, z) \geq \frac{1}{2}$, in which case our hypothesis implies

$$\begin{aligned} p(x, z)p(z, y) &> [p(x, y) + p(y, z)]p(z, y) \\ &= p(x, y)[1 - p(y, z)] + p(y, z)p(z, y) \\ &= p(x, y) + p(y, z)[p(z, y) - p(x, y)] \\ &\geq p(x, y), \end{aligned}$$

because, by hypothesis, $p(x, y) < 1 - p(y, z) = p(z, y)$. Since this contradicts the multiplicative condition, we turn to the other possibility.

(ii) $p(y, x) \geq \frac{1}{2}$, in which case we have

$$\begin{aligned} p(z, y)p(y, x) &= [1 - p(y, z)]p(y, x) \\ &> [1 + p(x, y) - p(x, z)]p(y, x) \\ &= [p(z, x) + p(x, y)][1 - p(x, y)] \\ &= p(z, x) + p(x, y)[1 - p(z, x) - p(x, y)] \\ &= p(z, x) + p(x, y)[p(x, z) - p(x, y)] \\ &\geq p(z, x), \end{aligned}$$

because by hypothesis $p(x, z) - p(x, y) > p(y, z) > 0$. Since this contradicts the multiplicative condition, we have shown that the triangle condition holds.

The converse does not hold, for example, when

$$p(x, y) = p(y, z) = p(z, x) = \frac{2}{3}$$

Theorem 36 *Neither the multiplicative condition nor weak stochastic transitivity implies the other*

PROOF. The binary probabilities $p(x, y) = p(y, z) = 0.60$ and $p(x, z) = 0.36$ satisfy the multiplicative condition but not weak transitivity.

The binary probabilities $p(x, y) = p(y, z) = 0.9$ and $p(x, z) = 0.6$ satisfy weak transitivity but not the multiplicative condition.

Theorem 37 *Strong transitivity is strictly stronger than moderate transitivity, which in turn is strictly stronger than weak transitivity*

PROOF. The implications follow immediately from the definitions. The failure of the opposite implications is easily shown by simple examples.

Theorem 38 *Moderate stochastic transitivity implies the multiplicative condition, but not conversely.*

PROOF. If $\min [p(x, y), p(y, z)] \geq \frac{1}{2}$, then moderate transitivity implies

$$\begin{aligned} p(x, z) &\geq \min [p(x, y), p(y, z)] \\ &\geq p(x, y)p(y, z) \end{aligned}$$

The converse is false, for if it were true, then by Theorem 37 the multiplicative condition would imply weak transitivity, contrary to Theorem 36.

Theorem 39 *The quadruple condition implies strong stochastic transitivity, but not conversely*

PROOF Suppose that $p(x, y) \geq \frac{1}{2}$, then since $p(z, z) = \frac{1}{2}$ for any $z \in A$, the quadruple condition implies $p(x, z) \geq p(y, z)$. A similar argument using $p(y, z) \geq \frac{1}{2} = p(x, x)$ establishes strong transitivity.

The converse does not hold because $p(x, y) = p(x, z) = \frac{1}{3}$ and $p(y, z) = \frac{2}{3}$ satisfy strong transitivity but not the quadruple condition.

$$p(x, y) = \frac{1}{3} = p(x, z), \text{ but } p(x, x) = \frac{1}{2} < p(y, z) = \frac{2}{3}$$

Theorem 40 *For binary probabilities different from 0 and 1, the product rule implies the quadruple condition, but not conversely*

PROOF Suppose that $p(w, x) \geq p(y, z)$, then rewriting the product rule on $\{w, x, y, z\}$,

$$\begin{aligned} \frac{p(z, x)p(w, y)}{p(x, z)p(y, w)} &= \frac{p(w, x)p(z, y)}{p(x, w)p(y, z)} \\ &\geq 1, \end{aligned}$$

from which it follows that $p(w, y) \geq p(x, z)$.

Simple counterexamples show that the converse is false.

5.6 Relations between the Models and the Observable Properties

Following the plan shown in Fig. 5, we next establish the basic connections between the models and the observable properties. We begin with the weaker models and work up to the stronger ones.

Theorem 41 *Any random utility model is regular, but not conversely*

PROOF Let $x \in X \subseteq Y$, then by Def. 20,

$$\begin{aligned} p_X(x) &= \Pr[U(x) \geq U(y) \mid y \in X \subseteq Y] \\ &\geq \Pr[U(x) \geq U(y) \mid y \in Y] \\ &= p_Y(x) \end{aligned}$$

The proof that the converse is false is given as a corollary to Theorem 49 of Sec. 6.1.

Theorem 42 *Neither being a random utility model nor satisfying the multiplicative condition implies the other*

PROOF. Consider $A = \{x, y, z\}$, let ϵ be a number such that $0 < \epsilon < 1$, and let $U = \langle U(x), U(y), U(z) \rangle$ be the random vector with the following distribution

$$Pr(U = u) = \begin{cases} \epsilon & \text{if } u = \langle 3, 2, 1 \rangle \\ \frac{1-\epsilon}{2} & \text{if } u = \langle 1, 3, 2 \rangle \text{ or } \langle 2, 1, 3 \rangle \\ 0 & \text{otherwise} \end{cases}$$

Observe that

$$\begin{aligned} p(x, y) &= Pr[U(x) \geq U(y)] \\ &= Pr[U = \langle 3, 2, 1 \rangle] + Pr[U = \langle 2, 1, 3 \rangle] \\ &= \epsilon + \frac{1-\epsilon}{2} \\ &= \frac{1+\epsilon}{2} \\ &> \frac{1}{2} \end{aligned}$$

In like manner,

$$\begin{aligned} p(y, z) &= \frac{1+\epsilon}{2} \\ p(x, z) &= \epsilon \end{aligned}$$

Thus

$$\begin{aligned} p(x, y)p(y, z) - p(x, z) &= \left(\frac{1+\epsilon}{2}\right)^2 - \epsilon \\ &= \left(\frac{1-\epsilon}{2}\right)^2 \\ &> 0, \end{aligned}$$

and the multiplicative condition is violated

The implication the other way fails because the multiplicative condition restricts only the binary probabilities

Theorem 43 *Neither being a random utility model nor satisfying weak stochastic transitivity implies the other*

PROOF. Consider $A = \{x, y, z\}$, let ϵ be a number such that $0 < \epsilon < \frac{1}{2}$, and let $U = \langle U(x), U(y), U(z) \rangle$ be a random vector with the following distribution

$$Pr(U = u) = \begin{cases} \epsilon + \frac{1}{2} & \text{if } u = \langle 3, 2, 1 \rangle, \langle 1, 3, 2 \rangle, \text{ or } \langle 2, 1, 3 \rangle \\ -\epsilon + \frac{1}{2} & \text{if } u = \langle 1, 2, 3 \rangle, \langle 2, 3, 1 \rangle, \text{ or } \langle 3, 1, 2 \rangle \\ 0 & \text{otherwise} \end{cases}$$

Then,

$$\begin{aligned}
 p(x, y) &= \Pr[U(x) \geq U(y)] \\
 &= \Pr(u = \langle 3, 2, 1 \rangle) + \Pr(u = \langle 2, 1, 3 \rangle) + \Pr(u = \langle 3, 1, 2 \rangle) \\
 &= \epsilon + \frac{1}{6} + \epsilon + \frac{1}{6} - \epsilon + \frac{1}{6} \\
 &= \epsilon + \frac{1}{2}
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 p(y, z) &= \epsilon + \frac{1}{2} \\
 p(x, z) &= -\epsilon + \frac{1}{2}
 \end{aligned}$$

Clearly, weak transitivity is not met

Although it is not indicated on Fig 5, perhaps it is worth noting that by imposing some fairly strong regularity conditions on the random utility model it is possible to show that strong stochastic transitivity must hold and that under slightly stronger conditions we can obtain an expression for the weak utility function, which by Theorems 37 and 45 must exist. It will be convenient to use the notation

$$\begin{aligned}
 F(t, x, y) &= \Pr[U(x) \leq t] - \Pr[U(y) \leq t] \\
 f(t, x, y) &= \frac{dF(t, x, y)}{dt} = \Pr[U(x) = t] - \Pr[U(y) = t]
 \end{aligned}$$

Theorem 44 Suppose that a set of binary preference probabilities is a binary independent random utility model

- (i) If for $x, y \in A$, $F(t, x, y)$ is either nonnegative for all t or nonpositive for all t , then strong stochastic transitivity is satisfied
- (ii) if, in addition, $\lim_{t \rightarrow \infty} tF(t, x, y) = \lim_{t \rightarrow -\infty} tF(t, x, y) = 0$, then

$$w(x) = E[U(x)] = \int_{-\infty}^{\infty} t \Pr[U(x) = t] dt$$

is a weak utility function

PROOF We first note that

$$\begin{aligned}
 p(x, z) - p(x, y) &= p(y, x) - p(z, x) \\
 &= \int_{-\infty}^{\infty} \Pr[U(x) = t] F(t, z, y) dt
 \end{aligned}$$

Because $F(t, z, y)$ has the same sign for all t and $\Pr[U(x) = t] \geq 0$, it follows that $p(x, z) - p(x, y)$ has the same sign as $F(t, x, y)$

(i) Suppose $p(x, y) \geq \frac{1}{2}$ and $p(y, z) \geq \frac{1}{2}$. Thus

$$0 \leq p(x, y) - \frac{1}{2} = p(x, y) - p(x, x)$$

$$0 \leq p(y, z) - \frac{1}{2} = p(y, z) - p(y, y),$$

and so $F(t, y, x) \geq 0$ and $F(t, z, y) \geq 0$. Therefore $p(x, z) - p(x, y) \geq 0$ and $p(x, z) - p(y, z) = p(z, y) - p(z, x) \geq 0$, which proves strong stochastic transitivity.

(ii) Consider

$$w(x) - w(y) = E[U(x) - U(y)]$$

$$= \int_{-\infty}^{\infty} t f(t, x, y) dt$$

$$= t F(t, x, y) \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} F(t, x, y) dt$$

$$\geq 0 \quad \text{if and only if} \quad -F(t, x, y) = F(t, y, x) \geq 0,$$

where we have integrated by parts and used the hypothesis of part (i). But

$$p(x, y) - \frac{1}{2} = p(x, y) - p(x, x) \geq 0 \quad \text{if and only if} \quad F(t, y, x) \geq 0,$$

which proves that

$$p(x, y) \geq \frac{1}{2} \quad \text{if and only if} \quad w(x) \geq w(y)$$

It should be noted that Thurstone's Case V model (see Vol I, p 55), which assumes independent normal distributions with equal variances, satisfies all the conditions of this theorem.

Theorem 45 *A weak utility model satisfies weak stochastic transitivity.*

The converse is not true in general, but it is when A is finite.

PROOF If $p(x, y) \geq \frac{1}{2}$ and $p(y, z) \geq \frac{1}{2}$, then in a weak utility model there is a function w such that $w(x) \geq w(y) \geq w(z)$, and therefore $p(x, y) \geq \frac{1}{2}$.

To show that the converse is false, consider points in the plane, $x = (x_1, y_1)$ and for $\alpha, \beta > \frac{1}{2}$, $\alpha \neq \beta$, define

$$p(x, y) = \begin{cases} \alpha & \text{if } x_1 > y_1 \\ \beta & \text{if } x_1 = y_1 \quad \text{and } x_2 > y_2 \\ \frac{1}{2} & \text{if } x_1 = y_1 \quad \text{and } x_2 = y_2 \end{cases}$$

We first show that these probabilities satisfy moderate stochastic transitivity and, therefore by Theorem 37, weak transitivity. Assuming $p(x, y) \geq \frac{1}{2}$ and $p(y, z) \geq \frac{1}{2}$, there are nine cases to consider, we look at only three because the rest are similar.

- 1 $p(x, y) = \alpha = p(y, z)$, then $x_1 > y_1 > z_1$, so $p(x, z) = \alpha$
- 2 $p(x, y) = \alpha, p(y, z) = \beta$, then $x_1 > y_1 = z_1$, so $p(x, z) = \alpha$
- 3 $p(x, y) = \beta, p(y, z) = \frac{1}{2}$, then $x_1 = y_1, x_2 > y_2, y_1 = z_1$, and $y_2 = z_2$, so $x_1 = z_1$ and $x_2 > z_2$, hence $p(x, z) = \beta$

Now, the relation \geq defined by $x \geq y$ if and only if $p(x, y) \geq \frac{1}{2}$ holds if and only if $x_1 > y_1$ or $x_1 = y_1$ and $x_2 \geq y_2$. Thus, if these probabilities were to satisfy the weak utility model, there would have to be an order-preserving function of this relation, contrary to what we showed in Sec 2.1, but when A is finite, it is immediately evident that an order-preserving function exists.

Theorem 46 *Neither being a weak utility model nor satisfying the triangle condition implies the other*

PROOF For $A = \{x, y, z\}$ and $p(x, y) = 0.1, p(y, z) = 0.6$, and $p(x, z) = 0.8$, the weak utility model holds with $u(x) = 1, u(y) = 2$, and $u(z) = 0$, but the triangle condition does not because $p(x, y) + p(y, z) = 0.7 < 0.8 = p(x, z)$.

If the converse held, then we would have the chain

$$\text{multiplicative} \xrightarrow{35} \text{triangle} \longrightarrow \text{weak utility} \xrightarrow{45} \text{weak transitivity},$$

contrary to Theorem 36.

Theorem 47 *A strong utility model satisfies the quadruple condition, but not conversely*

PROOF A proof of the first part was given in the discussion leading to the quadruple condition.

Debreu (1958) showed that the quadruple condition does not imply the strong utility model in general, but that it does when the following condition of *stochastic continuity* is met: for every $x, y, z \in A$ and every real α such that $p(x, y) < \alpha < p(x, z)$, there exists a $w \in A$ such that $p(x, w) = \alpha$.

Theorem 48 *A strict utility model satisfies the product rule, but not conversely except in the binary case when they are equivalent*

PROOF To show that being a (binary) strict utility model implies satisfaction of the product rule, substitute Eq. 20 (p. 335) into both sides of Eq. 26 (p. 341) and cancel the denominators. The converse, which is not true in general, is true in the binary case. For $a \in A$, define $v(x) = p(x, a)/p(a, x)$, then from $p(x, y)p(y, a)p(a, x) = p(x, a)p(a, y)p(y, x)$, we see that

$$\begin{aligned} p(x, y) &= \frac{p(x, a)/p(a, x)}{p(x, a)/p(a, x) + p(y, a)/p(a, y)} \\ &= \frac{v(x)}{v(x) + v(y)} \end{aligned}$$

6 GENERAL PROBABILISTIC RANKING THEORIES

In the algebraic studies of preference, no distinction is currently drawn between theories of choice and of ranking. In the simplest models, knowing which outcome is best is equivalent to knowing the utilities of all the outcomes of the presented set, and because utilities are numerical this is equivalent to a ranking of the presentation set. However, once we entertain the possibility that decisions may be governed by a probability process, the connections between choices and ranking are no longer at all obvious, it is a theoretical and experimental problem of some magnitude to try to find out what they are.

Very little theoretical work has yet been done, and to our knowledge there are no relevant experiments. The theoretical work is due to Block and Marschak (1960), Luce (1959, p. 68-74), and Marschak (1960).

Two general modes of attack on the theoretical problem have been tried. One may assume either that the choice probabilities can be expressed as some relatively simple and "natural" function of the ranking probabilities or that the ranking probabilities can be expressed simply in terms of the choice probabilities. Our next two subsections present the only serious ideas that have been put forward.

6.1 A Function Relating Choice to Ranking Probabilities

We begin with a simple example. Suppose that $X = \{x, y, z\}$ is presented many times and that the subject is required to rank order the three outcomes. This gives us estimates of the ranking probabilities $p(xyz)$, $p(yxz)$, etc., where xyz denotes the ranking in which x is first, y second, and z third, etc. If we only knew these probabilities and we were asked what is the probability that the subject will choose x over y , the most obvious thing to do is to calculate the total probability of all the rankings in which x is rated better than y , that is, to assume

$$p(x, y) = p(xyz) + p(xzy) + p(zxy) \quad (28)$$

Indeed, Coombs (1958) invoked just this assumption (applied to sets of four elements) to estimate binary choice probabilities (see Sec. 8.2).

As has been pointed out, this is not a necessary connection, and models are easily devised in which it does not hold. For example, suppose that the rankings of three alternatives are generated in the following way from the choice probabilities. A pair of outcomes is chosen at random, that is,

with probability $\frac{1}{2}$, and they are ordered according to the relevant binary choice probability. One of these two outcomes is then chosen at random, that is, with probability $\frac{1}{2}$, and it is compared with the third outcome according to the relevant binary choice probability. This may or may not produce a ranking of the three outcomes, if not, the two not previously compared are now ordered, which produces a ranking of all three. It is not difficult to show that

$$p(x, y) = \frac{p(xyz) - p(zxy)}{p(xyz) - p(zxy) + p(yxz) - p(zyx)},$$

which is obviously different from Eq. 28.

Nevertheless, the idea embodied in Eq. 28 seems worth exploration. To do so, some notation is needed.

Let A denote the finite set of all n outcomes, and let R denote the set of $n!$ rankings of A . Thus $\rho \in R$ is a particular ranking of A . Denote by ρ_i the element of A that is ranked in the i th position by ρ , and denote by $\rho(x)$ the rank order position of $x \in A$ under the ranking ρ . The set of rankings for which $x \in A$ is placed above all the elements in $Y \subseteq A$ is denoted by $R(x, Y)$, that is,

$$R(x, Y) = \{\rho \mid \rho \in R \text{ and } \rho(x) > \rho(y), \text{ all } y \in Y - \{x\}\}$$

Block and Marschak (1960, p. 107) proved the following interesting result.

Theorem 49 *A set of preference probabilities p_Y , $Y \subseteq A$ is a random utility model (Def. 20, p. 338) if and only if there exists a probability distribution p over the set R of rankings of A such that*

$$p_Y(x) = \sum_{\rho \in R(x, Y)} p(\rho) \quad (29)$$

PROOF Suppose, first, that the preference probabilities satisfy a random utility model. Define

$$p(\rho) = Pr[U(\rho_1) > U(\rho_2) > \dots > U(\rho_n)],$$

then

$$\begin{aligned} p_Y(x) &= Pr[U(x) > U(y), y \in Y - \{x\}] \\ &= \sum_{\rho \in R(x, Y)} Pr[U(\rho_1) > U(\rho_2) > \dots > U(\rho_n)] \\ &= \sum_{\rho \in R(x, Y)} p(\rho) \end{aligned}$$

Conversely, because we can place A into one-to-one correspondence with the set I_n of the first n integers, we may just as well assume $A = I_n$ and that R is the set of permutations of I_n ordered by magnitude. Now,

for any vector $u = (u_1, u_2, \dots, u_n)$ with real components u_i , we define the random vector U by

$$Pr[U = u] = \begin{cases} p(u) & \text{if } u \in R \\ 0 & \text{if } u \notin R \end{cases}$$

It is easy to see that

$$\begin{aligned} p_Y(x) &= \sum_{u \in R(x, y)} p(u) \\ &= \sum_{u \in R(x, Y)} Pr[U = u] \\ &= Pr[U(x) > U(y), y \in Y - \{x\}] \end{aligned}$$

Corollary *A model that satisfies the regularity condition need not be a random utility model*

PROOF For $A = \{1, 2, 3, 4\}$, it is easy to see from Eq. 29 that a necessary condition for the random utility model to hold is that

$$\begin{aligned} p_{(2,3)}(2) + p_A(2) - p_{(1,2,3)}(2) - p_{(2,3,4)}(2) \\ = p(1423) + p(4123) \\ \geq 0 \end{aligned}$$

Let $0 < \epsilon < \frac{1}{6}$ and let

$$\begin{aligned} p_A(2) &= \epsilon \\ p_A(i) &= (1 - \epsilon)/3, \quad i = 1, 3, 4 \\ p_S(i) &= \begin{cases} \frac{1}{3} & \text{if } S \text{ has 3 elements} \\ \frac{1}{2} & \text{if } S \text{ has 2 elements} \end{cases} \end{aligned}$$

It is clear that these choice probabilities satisfy regularity, but that

$$\begin{aligned} p_{(2,3)}(2) + p_A(2) - p_{(1,2,3)}(2) - p_{(2,3,4)}(2) &= \frac{1}{2} + \epsilon - \frac{1}{3} - \frac{1}{3} \\ &< -\frac{1}{6} + \frac{1}{6} \\ &= 0, \end{aligned}$$

and so the random utility model is not satisfied

6.2 A Function Relating Ranking to Choice Probabilities

Now we look at the same problem of how choice and ranking probabilities are related from the other point of view, namely, how the ranking probabilities can be expressed as a function of the choice probabilities. And again there is a natural model. It seems most reasonable that a person

might rank order a set of outcomes by first selecting what strikes him as the best outcome and assigning it rank one, this outcome is then discarded, and he selects the best outcome from the remaining set and assigns it rank two, and so on, until the set is exhausted. Once stated, this seems like the only sensible mechanism, but there are alternatives. For example, he might simply reverse the procedure, first select the least satisfactory outcome and rank it last, then the least satisfactory from the remaining set and rank it next to last, etc. The relation between these two procedures is considered shortly. Or he might proceed as in the example of Sec 6.1 by choosing a pair of elements at random, order them, and then compare one of these with another outcome chosen at random, etc. Note that this last scheme is prone to give a partial order or an intransitive relation when there are more than three elements, but it is not obvious that this is descriptively wrong. To generate a consistent ranking of numerous alternatives, for example, of student applications, is not considered particularly easy by most people who have ever had to do it.

Recall that if ρ is a rank ordering of A , then ρ_i denotes the element of A that has the i th rank. We may formulate the first model proposed previously as asserting that for all $\rho \in R$,

$$p(\rho) = p_A(\rho_1)p_{A-(\rho_1)}(\rho_2) \cdots p_{A-(\rho_1, \dots, \rho_{n-1})}(\rho_n) \quad (30)$$

Nothing much seems to be known about Eq. 30 by itself, but when coupled with Eq. 29 of Sec 6.1, Block and Marschak (1960, p. 109) have proved the following strong result, the first part of which generalizes a result in Luce (1959, p. 72).

Theorem 50 *If a set of preference probabilities p_Y , $Y \subseteq A$, is a strict utility model (Theorem 31, p. 336), then there exist ranking probabilities p such that p and p_Y satisfy both Eqs. 29 and 30, conversely, if there exist preference and ranking probabilities that satisfy both Eqs. 29 and 30, then the set of preference probabilities is a strict utility model.*

PROOF Suppose that the preference probabilities form a strict utility model. If we define the ranking probabilities by Eq. 30, then it is sufficient to show that Eq. 29 holds. Observe, first, that the set $R(x, Y)$ is the union of the following disjoint sets: $R(x, A)$, the sets of rankings where $x_1 \in A - Y$ is ranked first and x second, the sets where $x_1 \in A - Y$ is ranked first, $x_2 \in A - Y - \{x_1\}$ is ranked second, and x third, etc.

$$\begin{aligned} \sum_{\rho \in R(x, Y)} p(\rho) &= p_A(x) + \sum_{z_1 \in A - Y} p_A(z_1)p_{A-(z_1)}(x) \\ &\quad + \sum_{z_1 \in A - Y} \sum_{z_2 \in A - Y - \{z_1\}} p_A(z_1)p_{A-(z_1)}(z_2)p_{A-(z_1, z_2)}(x) \\ &\quad + \sum_{z_1 \in A - Y} p_A(z_1) p_{Y \cup \{z_1\}}(z_2)p_Y(x) \end{aligned}$$

For any Z such that $Y \subseteq A - Z$, the strict utility model implies

$$\begin{aligned} p_{A-Z}(x) &= p_Y(x)p_{A-Z}(Y) \\ &= p_Y(x)[1 - p_{A-Z}(A - Z - Y)] \end{aligned}$$

Substituting these expressions in the previous equation,

$$\begin{aligned} \sum_{\rho \in R(x, Y)} p(\rho) &= p_Y(x) \left\{ 1 - p_A(A - Y) + \sum_{z_1 \in A - Y} p_A(z_1) \right. \\ &\quad \times [1 - p_{A-\{z_1\}}(A - Y - \{z_1\})] \\ &\quad + \sum_{z_1 \in A - Y} \sum_{z_2 \in A - \{z_1\}} [p_A(z_1)p_{A-\{z_1\}}(z_2)] \\ &\quad \times [1 - p_{A-\{z_1, z_2\}}(A - Y - \{z_1, z_2\})] \\ &\quad + \\ &\quad \left. + \sum_{z_1 \in A - Y} p_A(z_1) p_{\{z_1\} \cup \{z_k\}}(z_k) \right\} \\ &= p_Y(x) \left[1 - p_A(A - Y) + p_A(A - Y) \right. \\ &\quad - \sum_{z_1 \in A - Y} p_A(z_1)p_{A-\{z_1\}}(A - Y - \{z_1\}) \\ &\quad \left. + \sum_{z_1 \in A - Y} p_A(z_1)p_{A-\{z_1\}}(A - Y - \{z_1\}) - \right] \\ &= p_Y(x) \end{aligned}$$

Conversely, if Eqs 29 and 30 hold, then we have

$$\begin{aligned} p_Y(x) &= p_A(x) + \sum_{z_1 \in A - Y} p_A(z_1)p_{A-\{z_1\}}(x) \\ &\quad + \sum_{z_1 \in A - Y} \sum_{z_2 \in A - \{z_1\}} p_A(z_1)p_{A-\{z_1\}}(z_2)p_{A-\{z_1, z_2\}}(x) \\ &\quad + \sum_{z_1 \in A - Y} p_A(z_1) p_{\{z_1\} \cup \{z_k\}}(z_k)p_Y(x) \end{aligned}$$

where $p_Z(u) = 0$ when $u \notin Z$. Clearly, for $Y = A$, the strict utility model holds. To prove it generally, we perform a decreasing induction on the size of Y . By the induction hypothesis,

$$p_{A-\{z_1\}}(x) = \frac{p_A(x)}{p_A(A - \{z_1\})}, \quad p_{A-\{z_1, z_2\}}(x) = \frac{p_A(x)}{p_A(A - \{z_1, z_2\})}, \quad \text{etc}$$

Substituting these into the preceding equation yields

$$\begin{aligned}
 p_Y(x) &= p_A(x) \left[1 + \sum_{z_1 \in A-Y} \frac{p_A(z_1)}{p_A(A - \{z_1\})} \right. \\
 &\quad \left. + \sum_{z_1 \in A-Y} \sum_{z_2 \in A-Y - \{z_1\}} \frac{p_A(z_1)p_{A-\{z_1\}}(z_2)}{p_A(A - \{z_1, z_2\})} + \right. \\
 &\quad \left. + \left[\sum_{z_1 \in A-Y} p_A(z_1) \quad p_{Y \cup \{z_1\}}(z_k) \right] p_1(x) \right] \\
 &= p_A(x)F(Y) + p_1(x)G(Y),
 \end{aligned}$$

where $F(Y)$ and $G(Y)$ are independent of x . Rewriting,

$$p_Y(x) = p_A(x) \left[\frac{F(Y)}{1 - G(Y)} \right],$$

and so

$$\frac{p_Y(x)}{p_Y(x)} = \frac{p_A(x)}{p_A(x)},$$

which yields the strict utility model if we set $v(x) = p_A(x)$

6.3 An "Impossibility" Theorem

Our final result about rankings concerns the relation of the ranking model embodied in Eq. 30 and the corresponding model based upon choices of least preferred outcomes. We let p_Y denote the usual preference probabilities and p_Y^* the choice probabilities for the least preferred element of Y . Exactly what empirical interpretation should be given to "choosing the least preferred element of Y " is not clear. If in estimating p_Y we simply ask the subject to select from Y the element he most prefers, then it seems reasonable to estimate p_Y^* by asking which element he least prefers. However, if we estimate p_Y by giving him the element that he selects, then what is a comparable payoff procedure to estimate p_Y^* ? One possibility is to reward him with an element chosen at random from $Y - \{x\}$ when he says he least prefers x . In any event, let us suppose that p_1^* exists.

Now suppose that a single random utility model underlies both of these judgments. The probability that he gives the ranking ρ when he is asked to order the outcomes from the most preferred to the least preferred is, as before,

$$p(\rho) = \Pr[U(\rho_1) > U(\rho_2) > \dots > U(\rho_n)]$$

The probability of the ranking ρ when he is asked to order the elements from the least preferred to the most preferred is

$$p^*(\rho) = Pr[U(\rho_1) < U(\rho_2) < \dots < U(\rho_n)]$$

Observe that if we let ρ^* denote the reverse of ranking ρ , that is,

$$\rho_1^* = \rho_n, \dots, \rho_i^* = \rho_{n+1-i}, \dots, \rho_n^* = \rho_1, \text{ then} \quad (31)$$

$$p^*(\rho^*) = p(\rho)$$

A similar argument based upon Eq 29 yields

$$p(x, y) = p^*(y, x), \quad \text{for } x, y \in A. \quad (32)$$

Finally, let us suppose that Eq 30 holds both for the starred and the unstarred probabilities, that is,

$$p(\rho) = p_A(\rho_1)p_{A-\{\rho_1\}}(\rho_2) \cdot p_{\{\rho_{n-1} \rho_n\}}(\rho_{n-1}) \quad (33)$$

$$p^*(\rho) = p_A^*(\rho_1)p_{A-\{\rho_1\}}^*(\rho_2) \cdot p_{\{\rho_{n-1} \rho_n\}}^*(\rho_{n-1}) \quad (34)$$

Theorem 51 Suppose that the preference and ranking probabilities for the most preferred outcome, p_Y and p , and for the least preferred outcome, p_Y^* and p^* , satisfy the same random utility model (and hence satisfy Eqs. 31 and 32). If the two ranking probabilities are given by Eqs 33 and 34, that is, if both p_Y and p_Y^* satisfy strict utility models, then for all $x \in Y \subseteq A$,

$$p_Y(x) = p_Y^*(x) = \frac{1}{|Y|},$$

where $|Y|$ denotes the number of elements in Y .

PROOF. Let the two strict utility scales be v and v^* , respectively, where with no loss of generality we may assume

$$\sum_{x \in A} v(x) = \sum_{x \in A} v^*(x) = 1$$

For any $x, y \in A$, let ρ be that ranking of A for which $\rho_1 = x$, $\rho_2 = y$, and let σ be that ranking of A for which $\sigma_1 = y$, $\sigma_2 = x$, and $\sigma_i = \rho_i$, $i = 3, \dots, n$. By Eq 33,

$$\begin{aligned} \frac{p(\rho)}{p(\sigma)} &= \frac{p_A(x)p_{A-\{x\}}(y)p_{A-\{x,y\}}(\rho_3)}{p_A(y)p_{A-\{y\}}(x)p_{A-\{x,y\}}(\rho_3)} \\ &= \frac{\frac{v(x)}{\sum_{z \in A} v(z)} \frac{v(y)}{\sum_{z \in A-\{x\}} v(z)}}{\frac{v(y)}{\sum_{z \in A} v(z)} \frac{v(x)}{\sum_{z \in A-\{y\}} v(z)}} \\ &= \frac{1 - v(y)}{1 - v(x)} \end{aligned}$$

By Eqs 34 and 32,

$$\begin{aligned}
 \frac{p^*(\rho^*)}{p^*(\sigma^*)} &= \frac{p_A^*(\rho_1^*)p_{A-\{\rho_1^*\}}^*(\rho_2^*) \cdots p_{\{\rho_{(n-1)}^*, \rho_n^*\}}^*(\rho_{n-1}^*)}{p_A^*(\sigma_1^*)p_{A-\{\sigma_1^*\}}^*(\sigma_2^*) \cdots p_{\{\sigma_{(n-1)}^*, \sigma_n^*\}}^*(\sigma_{n-1}^*)} \\
 &= \frac{p_A^*(\rho_n)p_{A-\{\rho_n\}}^*(\rho_{n-1}) \cdots p^*(y, x)}{p_A^*(\sigma_n)p_{A-\{\sigma_n\}}^*(\sigma_{n-1}) \cdots p^*(x, y)} \\
 &= \frac{p^*(y, x)}{p^*(x, y)} \\
 &= \frac{p(x, y)}{p(y, x)} \\
 &= \frac{v(x)}{v(y)}
 \end{aligned}$$

By Eq. 31, $p^*(\rho^*) = p(\rho)$ and $p^*(\sigma^*) = p(\sigma)$, so

$$\frac{1 - v(y)}{1 - v(x)} = \frac{v(x)}{v(y)}.$$

Since x and y were arbitrary, $v(x) = \text{constant independent of } x$. Since $p^*(x, y) = p(y, x)$, it also follows that $v^*(x) = \text{constant independent of } x$. Thus

$$p_Y(x) - p_Y^*(x) = 1/|Y|.$$

This result is due to Block and Marschak (1960, p. 111); it generalizes to any n a result proved for $n = 3$ in Luce (1959, p. 69).

Clearly, Theorem 51 is an impossibility theorem. Its conclusion is unacceptable on empirical grounds; hence its hypotheses cannot all be correct assumptions in a theory of behavior. Block and Marschak, who did not make explicit in their formal statement of the theorem that they assumed a common random utility model and, in particular, that Eqs. 31 and 32 hold, interpreted Theorem 51 as powerful evidence against the strict utility model (or, what is the same, against Eqs. 33 and 34). There are, in our view, at least two alternative possibilities. First, one can question the assumption (embodied in Eqs. 31 and 32) that a common random utility model underlies the choice of the most preferred and of the least preferred outcomes and of the ranking from best to worst and from worst to best. As has been previously pointed out (Luce, 1959), the assumption $p(\rho) = p^*(\rho^*)$ is not obviously correct as a description of behavior. Second, one can question whether the whole problem makes any sense at all, that is, whether choices of least preferred outcomes have any operational meaning. Thus, although the conclusion of Theorem 51 is empirically unacceptable, it is not evident to us which of its hypotheses should be rejected.

7 PROBABILISTIC CHOICE THEORIES FOR UNCERTAIN OUTCOMES

By an experiment that has uncertain outcomes, we mean one in which knowledge of the stimulus presentation and of the response is insufficient to determine the actual "elementary" or "certain" outcome received by the subject. Rather, the stimulus and response determine a function ω from trials to the set A of certain outcomes, this was called a conditional outcome schedule in Sec 1.4 of Chapter 2, Vol I. Throughout this section we shall assume that the conditional outcome schedules are all simple random ones in the sense that for $x \in A$

$$\pi(x | \omega) = \Pr[\omega(n) = x]$$

is independent of the trial number n

Another way of characterizing uncertain outcomes is to say that the presentation plus the response determines a set $\{E(x)\}$ of mutually exclusive and exhaustive chance events that are in one to one correspondence with the set of certain outcomes A . A particular outcome x is delivered to the subject if and only if the event $E(x)$ occurs. The connection between these two ways of speaking is essentially that between random variables and sample spaces.

For most purposes of this section it is convenient to use the event language and to subscript symbols so that E_i is the event corresponding to outcome x_i , π_i is the probability of event E_i occurring, and if f is a function defined over A , $f_i = f(x_i)$.

In summary, then, an *uncertain outcome* is simply a probability distribution π over the set A of certain outcomes. Some authors speak of uncertain outcomes as wagers or gambles.

7.1 Expected Utility Models

The first idea that comes to mind to deal with uncertain outcomes is to combine the major algebraic concept of utility theory, namely, the expected utility hypothesis, with one or another of the general probability models described in Secs 5.2 and 5.3.

Let $\pi^j = (\pi_1^j, \pi_2^j, \dots, \pi_n^j)$, $j = 1, 2, \dots, m$, denote m uncertain outcomes defined over a set $A = \{x_i\}$ of n certain outcomes. Of course, some of the π_i^j may equal 0 meaning that that outcome x_i cannot be received by the subject when π^j is selected. Let $\mathcal{A} = \{\pi^j\}$. The defining properties of the several models are the following

WEAK EXPECTED UTILITY *There is a real valued function w over A such that*

$$p(\pi^j, \pi^k) \geq \frac{1}{2} \text{ if and only if } \sum_{i=1}^n \pi_i^j w_i \geq \sum_{i=1}^n \pi_i^k w_i$$

STRONG EXPECTED UTILITY *There is a real valued function u over A and a distribution function ϕ such that*

$$p(\pi^j, \pi^k) = \phi \left(\sum_{i=1}^n \pi_i^j u_i - \sum_{i=1}^n \pi_i^k u_i \right)$$

provided that $p(\pi^j, \pi^k) \neq 0$ or 1

STRICT EXPECTED UTILITY *There is a positive real valued function v over A such that*

$$p_{\mathcal{A}}(\pi^j) = \frac{\sum_{i=1}^n \pi_i^j v_i}{\sum_{k=1}^m \sum_{i=1}^n \pi_i^k v_i}$$

provided that $p_{\mathcal{A}}(\pi^j) \neq 0$ or 1

In their presentation of these models, Becker, DeGroot, and Marschak (1963a) state a weaker form of the strict and strong expected utility model that involves two functions, u and v , and assumes that the relevant scale values are

$$v \left(\sum_{i=1}^n \pi_i^j u_i \right)$$

This, however, does not seem to us to be the natural generalization of these utility models, so we have formulated them in terms of just one function.

Still another possibility was suggested by Suppes (1961). He considered a learning situation in which a subject must choose from a set \mathcal{A} of gambles all of the same form: if $\zeta \in \mathcal{A}$, then with probability $\pi(\zeta)$ the subject is reinforced (that is, given a desired outcome) and with probability $1 - \pi(\zeta)$ nothing happens. Using a one element stimulus sampling model (see Chapter 10, Vol II), he arrived at a Markov chain to describe the transition from one trial to the next. The asymptotic choice probability turns out to be of the form

$$p_{\mathcal{A}}(\zeta) = \frac{\iota(\zeta)}{\sum_{\eta \in \mathcal{A}} \iota(\eta)},$$

where

$$\iota(\zeta) = \frac{1}{\epsilon(\zeta)[1 - \pi(\zeta)]}$$

and $\epsilon(\zeta)$ is the probability that the stimulus element is conditioned to another response when ζ is not reinforced. We describe this model (for two responses) somewhat more carefully in Sec 7.3, here we merely point out its similarity in form to the strict utility model and the considerable

difference in the form of $v(x)$ from that assumed in the strict expected-utility model

RANDOM EXPECTED UTILITY *There is a random vector U whose components are in one-to-one correspondence with the elements of A such that*

$$p_{\mathcal{A}}(\pi^j) = \Pr \left(\sum_{i=1}^n \pi_i^j U_i \geq \sum_{i=1}^n \pi_i^k U_i, \quad k = 1, 2, \dots, m \right)$$

The first three models can readily be rephrased as subjective expected-utility models merely by replacing the objective probabilities π_i^j with subjective ones. The last one can be extended in at least two different ways by postulating a constant subjective probability function to replace π_i^j or by treating subjective probability as a random vector, just as utility is treated in this model.

At the moment, however, all such generalizations are really quite idle because next to nothing is known about any of these models, let alone their generalizations. An exception is the following special, but testable, result about the strict and random expected utility models (Becker, DeGroot, & Marschak, 1963a).

Theorem 52 *Let $\mathcal{A} = \langle \pi^j \rangle$ consist of $m + 1$ uncertain outcomes for which the $(m + 1)$ st is the average of the other m in the sense that*

$$\pi_i^{m+1} = \frac{1}{m} \sum_{j=1}^m \pi_i^j$$

If the strict expected utility model is satisfied, then

$$p_{\mathcal{A}}(\pi^{m+1}) = \frac{1}{m + 1}$$

If the random expected utility model is satisfied, then

$$p_{\mathcal{A}}(\pi^{m+1}) = 0$$

PROOF For the strict expected utility model, we have

$$\begin{aligned} p_{\mathcal{A}}(\pi^{m+1}) &= \frac{\sum_{i=1}^n \pi_i^{m+1} v_i}{\sum_{j=1}^m \sum_{i=1}^n \pi_i^j v_i + \sum_{i=1}^n \pi_i^{m+1} v_i} \\ &= \frac{\sum_{i=1}^n \pi_i^{m+1} v_i}{m \sum_{i=1}^n \pi_i^{m+1} v_i + \sum_{i=1}^n \pi_i^{m+1} v_i} \\ &= \frac{1}{m + 1} \end{aligned}$$

For the random expected utility model, it is easy to see that equalities among the random variables occur with probability 0, so

$$\begin{aligned} p_{\mathcal{A}}(\pi^{m+1}) &= \Pr\left(\sum_{i=1}^n \pi_i^{m+1} U_i > \sum_{i=1}^n \pi_i^k U_i, \quad k = 1, 2, \dots, m\right) \\ &= \Pr\left[\frac{1}{m} \sum_{j=1}^m \left(\sum_{i=1}^n \pi_i^j U_i\right) > \sum_{i=1}^n \pi_i^k U_i, \quad k = 1, 2, \dots, m\right] \\ &= 0, \end{aligned}$$

because the average of several quantities can never exceed each of them

Corollary *A strict expected utility model need not be a random expected utility model*

At first glance, this corollary may seem surprising in the light of Theorem 32 of Sec 5 3, but it should be clear that they are not inconsistent

It is easy to see that a proof paralleling that of Theorem 29 of Sec 5 2 establishes that a strong expected utility model is a weak expected utility model As stated, a strict expected utility model need not be a strong one, although this is true for the weaker definitions given by Becker, De Groot, and Marschak

Nothing much else seems to be known about these models, except for one experiment based on Theorem 52 (see Sec 8 4)

7 2 Decomposition Assumption

In this subsection we confine our attention to uncertain outcomes of the simplest possible form, namely, those with nonzero probability on only two certain outcomes It is convenient to introduce a slightly different notation from what we have been using By xEy we denote the uncertain outcome from which $x \in A$ is received by the subject if event E occurs and $y \in A$ is received if E does not occur If we let \mathcal{E} denote the set (actually, Boolean algebra) of events used to generate these binary uncertain outcomes, then $\mathcal{A} = A \times \mathcal{E} \times A$ is the set of possible uncertain outcomes Certain outcomes are included in \mathcal{A} by taking E to be the certain event

As we have been assuming all along, preference probabilities that the subject chooses the (uncertain or certain) outcome ζ when \mathcal{X} is presented, $p_{\mathcal{X}}(\zeta)$, $\zeta \in \mathcal{X} \subseteq \mathcal{A}$, are assumed to exist In addition, we postulate that the subject is able to make judgments about which of several events is most likely to occur—he must be able to do this, at least crudely, if he is to make any sense of selecting among uncertain outcomes Thus, when $E \in \mathcal{D} \subseteq \mathcal{E}$, we suppose that a probability $q_{\mathcal{D}}(E)$ exists that the subject selects E as the event in \mathcal{D} that is most likely to occur We speak of these as *judgment probabilities*

Suppose that xEy and xDy are presented and the subject must select one. It is not unreasonable that he should try to decompose his choice into two simpler ones by asking: Which outcome, x or y , would I rather receive and, independent of this, which event, E or D , do I believe is more likely to occur? It is clear that he should prefer xEy to xDy in just two of the four possible cases (ignoring indifferences): when x is preferred to y and E is judged more likely than D , and when y is preferred to x and D is judged more likely than E . If so and if the two decisions—preference of outcomes and judgments of events—are statistically independent then it is clear that the following condition must hold (Luce 1958, 1959)

Decomposition Assumption For all $x, y \in A$ and $E, D \in \mathcal{E}$,

$$p(xEy, xDy) = p(x, y)q(E, D) + p(y, x)q(D, E) \quad (35)$$

The following trivial result shows that if the decomposition assumption holds, it is possible to estimate the binary judgment probabilities from the binary preference probabilities

Theorem 53 If the decomposition assumption holds and if $x, y \in A$ are such that $p(x, y) \neq \frac{1}{2}$, then

$$q(E, D) = \frac{p(xEy, xDy)p(x, y) - p(xDy, xEy)p(y, x)}{p(x, y)^2 - p(y, x)^2}$$

The primary criticism of the decomposition assumption is that judgments of relative likelihood may not be independent of preference decisions. Certainly, this can be arranged by logically interlocking the outcomes with the events. L. J. Savage¹² suggested the following example. Let x and y be two greyhound dogs, E the event that x wins over y , and $D = \bar{E}$ the complementary event that y wins over x in a single race between them. Thus you will receive one of the dogs, but which one depends upon which gamble you choose and which dog wins the race. If you propose to race the dog received in the future, it seems clear that in the absence of other information about them you would prefer the dog that wins the race. Thus you would choose xEy with probability 1 over xDy , even though because of ignorance about the dogs $p(x, y) = q(E, \bar{E}) = \frac{1}{2}$. Clearly, this violates the decomposition assumption.

Usually in the laboratory, and often outside of it, events and outcomes are logically unrelated, and so examples of this kind are not particularly relevant except to show that independence is not a uniformly valid assumption. Nevertheless considerable experimental evidence shows that the outcomes of an uncertain choice may affect judgments of likelihood; however, one must be very careful about concluding that these studies disprove the decomposition assumption. Consider, for example, the work

¹² In correspondence with Luce

of Irwin and his students (Irwin, 1953, Marks, 1951; and unpublished data) In the simplest, and most relevant, version of these experiments, the subject predicts whether or not he will draw a marked card from a small deck of cards that has a known number of marked and unmarked ones The payoffs are as shown

		Event	
		Predicted	Not Predicted
Event	E	x	y
Occurs	\bar{E}	y	x

In our terms, the subject must choose between xEy and $x\bar{E}y$ F. W. Irwin and G. Snodgrass (unpublished) have shown that if, in addition to this payoff matrix, you reward (or punish) the subject by giving (or taking away) a fixed sum of money whenever E occurs, independent of what was predicted, then his frequency of predicting the event appears to be a monotonic increasing function of this "irrelevant" extra payment Although this finding is quite reasonably described as proving that an interaction exists between preferences and judgments of event likelihood, it does not really bear upon the decomposition assumption because the choice is between $(x + a)Ey$, and $x\bar{E}(y + a)$, where a is the amount paid when E occurs

The main theoretical results that are known about the decomposition assumption concern its conjunction with other probabilistic assumptions For example, we can easily prove the following result

Theorem 54 *If the decomposition assumption holds and there exist $x, y \in A$ such that $p(x, y) = 1$, then any property satisfied by the binary preference probabilities p is also satisfied by the binary judgment probabilities q*

PROOF Because $p(x, y) = 1$, the decomposition assumption implies that for any $E, D \in \mathcal{E}$, $q(E, D) = p(xEy, xDy)$, and so any proposition about the binary q probabilities can be immediately replaced by an equivalent statement about the p probabilities

In some cases, the transfer of a binary property from the p 's to the q 's can be proved without assuming the existence of x and y such that $p(x, y) = 1$ For example, we can prove the following result

Theorem 55 *Suppose that the decomposition assumption holds and that there exist $x, y \in A$ such that $p(x, y) \neq \frac{1}{2}$ If the binary preference probabilities satisfy strong stochastic transitivity, then the binary judgment probabilities do also*

PROOF Let $E, D \in \mathcal{E}$ be given. It is not difficult to show that strong stochastic transitivity implies that for all $\zeta \in \mathcal{A}$, $p(xEy, \zeta) - p(xDy, \zeta)$ has the same sign. For any $F \in \mathcal{E}$, set $\zeta = xFy$, apply the decomposition assumption, and collect terms

$$p(xEy, xFy) - p(xDy, xFy) = [p(x, y) - p(y, x)][q(E, F) - q(D, F)]$$

Because the left side has the same sign for all F and because $p(x, y) - p(y, x) \neq 0$, it follows that $q(E, F) - q(D, F)$ has the same sign for all F , thus proving strong stochastic transitivity for the judgment probabilities.

The other results in Luce (1958) are too intricate to be described here. Suffice it to say that they are concerned with conditions which, when added to the decomposition axiom, permit us to show that the preference probabilities satisfy a strong utility model. Here we prove a closely related, but simpler, theorem that shows how the distribution functions of the strong utility model are limited when both the decomposition assumption and the expected utility hypothesis are satisfied.

Theorem 56 Suppose that there exist a (utility) function u from $\mathcal{A} = A \times \mathcal{E} \times A$ into a bounded real interval, which we take to be $[0, 1]$ with no loss of generality, a (subjective probability) function s from \mathcal{E} into $[0, 1]$, and continuous distribution functions P and Q such that

- 1 if \bar{E} is the complement of E , where $E, \bar{E} \in \mathcal{E}$, then $s(\bar{E}) = 1 - s(E)$
- 2 for all $x, y \in A$, $E \in \mathcal{E}$, $u(xEy) = u(x)s(E) + u(y)[1 - s(E)]$
- 3 for all $\zeta, \eta \in \mathcal{A}$, $p(\zeta, \eta) = P[u(\zeta) - u(\eta)]$
- 4 for all $E, D \in \mathcal{E}$, $q(E, D) = Q[s(E) - s(D)]$ and
- 5 the decomposition assumption is satisfied

If A and \mathcal{E} are dense in the sense that for every $\alpha \in [-1, 1]$, there exist $x, y \in A$ and $E, D \in \mathcal{E}$ such that $u(x) - u(y) = \alpha = s(E) - s(D)$, and if $P(1) \neq \frac{1}{2}$, then there exist constants $\epsilon > 0$ and $k > 0$ such that

$$P(\alpha) = \begin{cases} \frac{1 + k\alpha^\epsilon}{2}, & \text{if } 0 \leq \alpha \leq 1, \\ \frac{1 - k|\alpha|^\epsilon}{2}, & \text{if } -1 \leq \alpha < 0 \end{cases}$$

$$Q(\alpha) = \begin{cases} \frac{1 + \alpha^\epsilon}{2}, & \text{if } 0 \leq \alpha \leq 1, \\ \frac{1 - |\alpha|^\epsilon}{2}, & \text{if } -1 \leq \alpha < 0 \end{cases}$$

PROOF Let $\alpha, \beta \in [-1, 1]$, and choose $x, y \in A$ and $E, D \in \mathcal{E}$ such that $\alpha = u(x) - u(y)$ and $\beta = s(E) - s(D)$. From the expected utility hypothesis, we see that

$$\alpha\beta = [u(x) - u(y)][s(E) - s(D)] = u(xEy) - u(xDy)$$

Thus, by the decomposition and strong utility assumptions,

$$\begin{aligned} P(\alpha\beta) &= P[u(xEy) - u(xDy)] \\ &= p(xEy, xDy) \\ &= p(x, y)q(E, D) + p(y, x)q(D, E) \\ &= P(\alpha)Q(\beta) + [1 - P(\alpha)][1 - Q(\beta)] \end{aligned} \quad (36)$$

Setting $\alpha = 1$ in Eq 36,

$$P(\beta) = Q(\beta)[2P(1) - 1] + 1 - P(1)$$

Because $P(1) \neq \frac{1}{2}$,

$$Q(\beta) = \frac{P(\beta) + P(1) - 1}{k},$$

where $k = 2P(1) - 1$. Substituting this in Eq 36 and simplifying yields

$$P(\alpha\beta) = \frac{2P(\alpha)P(\beta) - P(\alpha) - P(\beta) + P(1)}{k}$$

Define $f(\alpha) = [2P(\alpha) - 1]/k$. Substituting this and simplifying yields

$$f(\alpha\beta) = f(\alpha)f(\beta) \quad (37)$$

Note that $f(1) = 1$

Because P is continuous and monotonic increasing, so is f . It is well known that when $\alpha \geq 0$ and $f(1) = 1$ the only continuous monotonic solutions of Eq 37 are $f(\alpha) = \alpha^\epsilon$, where $\epsilon > 0$. Because $P(-\alpha) = 1 - P(\alpha)$, it follows that $f(-\alpha) = -f(\alpha)$, and so for $\alpha < 0$, the solution is $f(\alpha) = -f(-\alpha) = -|\alpha|^\epsilon$. Substituting these back into the expressions for P and Q , we obtain the forms stated earlier. Because P is monotonic increasing and $P(0) = \frac{1}{2}$, $k = 2P(1) - 1 > 0$.

Theorem 56 shows that the combination of the strong utility model with the expected utility hypothesis and the decomposition assumption is really very restrictive—the forms of the distribution functions are determined. It is clear that unless we introduce some *ad hoc* assumptions about 0 and 1 probabilities, this model does not apply to situations where the certain outcomes are simply ordered on a single dimension, for example, when they are sums of money or, more generally, amounts of something.

Completely different, but also very restrictive, conclusions arise when

we combine the decomposition assumption with the choice axiom (Def 19, p 336), even when we do not postulate the expected utility hypothesis

Theorem 57 *Suppose that the preference probabilities p satisfy the choice axiom, that the judgment probabilities q also satisfy the choice axiom, and that the binary p 's and q 's satisfy the decomposition assumption. For $E, D \in \mathcal{E}$, define $E \sim D$ if and only if $q(E, D) = \frac{1}{2}$. Then, either, for all $x, y \in A$, $p(x, y) = 0, \frac{1}{2}$, or 1, or \sim is an equivalence relation having at most three equivalence classes*

The proof of this result is too long to include here, see Luce (1959, pp 80-82)

Although experimental evidence is difficult to cite, everyone is confident from his experience and introspections that judgment probabilities permit more than three equivalence classes of events, and so we must conclude that when the assumptions of Theorem 57 are met, the preference probabilities over A actually form a simple algebraic structure of the type discussed in Secs 2 and 3. As was pointed out earlier, we expect this to happen with money outcomes. It is less clear that preferences are algebraic when the outcomes are different in kind, for example, a pony versus a bicycle. If empirical evidence is produced to show that preferences are truly probabilistic among some certain outcomes, then the assumptions of Theorem 57 are too strong to describe behavior in response to those outcomes. No matter what its descriptive status may ultimately be, Theorem 57 is of interest as an "existence proof" of a behavioral theory that leads to strong, discontinuous restrictions on some of the preference probabilities without these restrictions being particularly apparent in the initial assumptions.

Very little else is known about the consequences of the hypotheses of Theorem 57 except for one special result of some empirical interest that is appropriately described in the next section.

7.3 Several Theories Applied to a Special Case

Both learning and utility theories attempt to describe behavior when the outcomes are uncertain. In learning theory, the problem is approached by asking what remains invariant when the response probabilities change from trial to trial in a fixed choice situation. In utility theory, it is approached by assuming asymptotic behavior and asking what remains invariant in several related choice situations. Of course, learning models have the empirical advantage that they explicitly describe the sequential structure to be expected in the data. If we confine our attention to the asymptotic predictions of learning theories, then it is reasonable to expect

that we can find experiments where both classes of theories predict the behavior. Comparisons of this sort may help considerably in choosing among the various theories.

The special experiment that we shall consider involves two uncertain outcomes, both consisting of the same probability distribution over a pair of certain outcomes. It is most easily represented in tabular form

	Choice	
	1	2
Event	π	$\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$
Probability	$1 - \pi$	

Thus, if the subject chooses the uncertain alternative 2, he receives outcome x_{12} with probability π and outcome x_{22} with probability $1 - \pi$.

We confine our attention to one prediction, namely, the plot of $p = p(1, 2)$ versus π when the outcomes x_{ij} are held fixed. Experimentally, we develop such a plot by carrying out a series of runs at different values of π and estimating p from what appear to be asymptotic data. We shall examine what ten different theories, six learning and four utility, say about the nature of this plot.

A ONE-ELEMENT STIMULUS-SAMPLING MODEL. Suppes (1961) suggested the following stimulus-sampling model for this experiment under the added restriction that $x_{12} = x_{21} =$ no experimenter delivered outcome to the subject. We suppose that there is a single stimulus element that is conditioned at the beginning of any trial to one of the two responses. The conditioning of the element dictates which response is made. If the subject is reinforced, that is, if either x_{11} or x_{22} is the outcome, then it is assumed that the conditioning is not changed, if, however, he is not reinforced, it is assumed that the conditioning changes to the other response with a fixed probability ϵ_i , where i denotes the response that was made. The one-step transition trees for the conditioning of the element are shown in Fig. 6. These trees may be summarized in terms of the one step transition matrix for the Markov chain

$$\begin{array}{cc} & \text{Trial } n + 1 \\ & \begin{array}{cc} 1 & 2 \end{array} \\ \text{Trial } n & \begin{bmatrix} \pi + (1 - \pi)(1 - \epsilon_1) & (1 - \pi)\epsilon_1 \\ \pi\epsilon_2 & 1 - \pi + \pi(1 - \epsilon_2) \end{bmatrix} \end{array}$$

If we let p_n denote the probability of choosing response 1 on trial n , then the transition equation is easily seen to be

$$p_{n+1} = [\pi + (1 - \pi)(1 - \epsilon_1)]p_n + \pi\epsilon_2(1 - p_n)$$

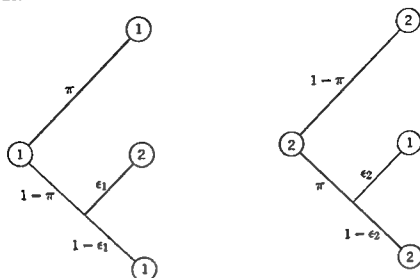


Fig 6 One step transition trees of the one element stimulus sampling model

At asymptote, $E(p_{n+1}) = E(p_n) = p_\infty$. So taking expectations and the limit as $n \rightarrow \infty$, we find

$$p_\infty = \frac{\pi}{\pi + (1 - \pi)\epsilon_1/\epsilon_2} \quad (38)$$

It is clear that if $\epsilon_1 = \epsilon_2$, which may hold when $x_{11} = x_{22}$, then $p_\infty = \pi$. This is the well publicized prediction of probability matching. More generally, we have $p_\infty \geq \pi$ if and only if $\epsilon_2 \geq \epsilon_1$. Thus the model predicts that the response probability either overshoots π for all values of π , matches it, or undershoots it for all values of π . (Several variants of this model are also considered in Suppes and Atkinson, 1960, Chapter 11.)

OBSERVING-RESPONSE MODEL A theoretical approach that also originates in the tradition of stimulus sampling theory but that adds a new concept is the one that postulates an implicit response by the organism prior to the observed response. This implicit response has been referred to as an observing, orienting, or approach avoidance response in the literature. The basic ideas are presented in Audley (1960), Bower (1959), and Estes (1960). A simplified version of the Estes model is formulated and tested in Suppes and Atkinson (1960, Chapter 11) and Atkinson (1961), their version is essentially defined by the following five assumptions, further simplified here to the case of two responses

- 1 On every trial each response has an approach value of 1 or 0
- 2 At the start of each trial one of the two responses is randomly observed
- 3 If the approach value of the observed response is 1, that response is made. If the approach value is 0, the other response is observed

4 If both responses have approach value 0, a randomly selected response is made after both responses have been observed

5 With probability π a rewarded response obtains the approach value 1 when it had the value 0, and with probability 1 an unrewarded response obtains the approach value 0 when it had the value 1

To apply these assumptions to the outcome matrix given at the beginning of this section, we impose the further restrictions that $x_{12} = x_{21}$ and $x_{11} = x_{22}$. From these restrictions and the five assumptions we can derive a 3 state Markov chain, which we do not produce here. The mean asymptotic result obtained from this chain is that

$$p_{\infty} = \frac{\pi^2 + \pi(1 - \pi)\phi}{\pi^2 + (1 - \pi)^2 + 2\pi(1 - \pi)\phi}, \quad (39)$$

where $\phi = 1/\pi$, that is, the ratio of the two approach parameters. The probability p_{∞} is a monotonically decreasing function of ϕ and it is bounded by the closed interval with end points $\frac{1}{2}$ and

$$\frac{\pi^2}{\pi^2 + (1 - \pi)^2}$$

STRONG AND WEAK CONDITIONING MODEL A still more promising modification of stimulus-sampling theory that is designed to account for the choice of uncertain outcomes with varying payoff was worked out by Atkinson (1962) and Myers and Atkinson (1964). The central idea is to assume not only that each stimulus element is conditioned to a response but that it is either strongly or weakly conditioned. We do not state the assumptions of the model, but they follow the intuitively obvious course. For example, if an element is weakly conditioned to a rewarded response, then with probability μ it becomes strongly conditioned. On the other hand, if the response is not rewarded, then with probability δ it becomes weakly conditioned to the other response. Similarly, if the element is strongly conditioned to an unrewarded response, then again with probability δ it becomes weakly conditioned to the same response.

It may be shown then that the asymptotic probability p_{∞} of choosing response 1 is

$$p_{\infty} = \frac{\pi^3 + \pi^2(1 - \pi)\phi}{\pi^3 + (1 - \pi)^3 + \pi(1 - \pi)\phi}, \quad (40)$$

where $\phi = \delta/\mu$. Note that p_{∞} is again a monotonically decreasing function of ϕ , and it is bounded by the closed interval with endpoints π and

$$\frac{\pi^3}{\pi^3 + (1 - \pi)^3}$$

EXPERIMENTER-CONTROLLED LINEAR LEARNING MODEL Bush and Mosteller (1955) and others have explored a learning model in which the response probabilities are assumed to be linearly transformed from trial to trial, the constants of the transformation depending on what happened on the trial. Little is known about the asymptotic properties of this model when the linear operators depend on the subject's response, so we are forced to assume that they only depend on the outcome received—the so-called experimenter-controlled case.

If we suppose that x_{11} is preferred to x_{12} and x_{22} to x_{21} , then it is reasonable that whenever the event that has probability π occurs the subject should increase his tendency to choose response 1 and that whenever the complementary event occurs he should decrease it. This leads to the model

$$p_{n+1} = \begin{cases} (1 - \theta_1)p_n + \theta_1, & \text{with probability } \pi \\ (1 - \theta_2)p_n, & \text{with probability } 1 - \pi \end{cases}$$

Calculating expected values,

$$E(p_{n+1}) = \pi[(1 - \theta_1)E(p_n) + \theta_1] + (1 - \pi)(1 - \theta_2)E(p_n)$$

Therefore at asymptote we have

$$p_\infty = \frac{\pi}{\pi + (1 - \pi)\theta_2/\theta_1} \quad (41)$$

Note that by substituting $\epsilon_2 = \theta_1$ and $\epsilon_1 = \theta_2$, this model predicts the same asymptotic mean probability as the one-element model (Eq. 38), even though that model required the assumption that $x_{12} = x_{21}$ and this one does not.

SUBJECT-EXPERIMENTER-CONTROLLED BETA MODEL The beta model is of the same general character as the linear model except that the stochastic operators have a simple nonlinear form. The main idea is that it is not p_n , but the quantity $v_n = p_n/(1 - p_n)$ that is transformed in a simple way (see Luce, 1959, and Sec. 2.5, Chapter 9 of Vol. II). Specifically, we assume that

$$v_{n+1} = \begin{cases} \beta_{11}v_n \\ \beta_{12}v_n \\ \beta_{21}v_n \\ \beta_{22}v_n \end{cases} \quad \text{with probability} \quad \begin{cases} \pi p_n \\ \pi(1 - p_n) \\ (1 - \pi)p_n \\ (1 - \pi)(1 - p_n) \end{cases},$$

where β_{ij} is the parameter corresponding to outcome i and response j . Taking logarithms, substituting $b_{ij} = \log \beta_{ij}$, and calculating expectations, we see that

$$E(\log v_{n+1}) - E(\log v_n) = K[E(p_n) - p_n],$$

where

$$K = \pi(b_{11} - b_{12}) + (1 - \pi)(b_{21} - b_{22})$$

and

$$p_{\infty} = \frac{b_{12}\pi + b_{22}(1 - \pi)}{K} \quad (42)$$

It has been shown that if $K \neq 0$ and if the limit of the expected value of v_n is not zero or infinity, then the asymptotic expected mean is given by Eq 42 (Luce, 1959, p 116). The condition $K \neq 0$ is seen to be violated when the events are experimenter-controlled, that is, when $b_{11} = b_{12}$ and $b_{21} = b_{22}$. The general situation is rather complicated. For example, if $\pi b_{12} + (1 - \pi)b_{22} < 0$ and $\pi b_{11} + (1 - \pi)b_{21} > 0$, then $p_{\infty} = 1$.

There is some formal resemblance between the asymptotic mean of the beta model, Eq 42, and of the experimenter-controlled linear model, Eq 41 (and so of the one element stimulus sampling model, Eq 38). The difference is that the beta model includes a $1 - \pi$ term in the numerator, whereas the others do not. Of course, it is not really fair to compare these two predictions, because the linear model has been worked out only for experimenter controlled events and so it is less general.

A THRESHOLD MODEL In Secs 2.3 and 5.2 of Chapter 3, Vol I, a threshold model was described for experiments with two presentations and two responses. The main idea is that as a result of the presentation the subject enters into one of two "detection" states, the probability depending on the stimulus presented. The response made is assumed to be governed by the state that the subject is in and by a response bias that he controls.

In the present experiment the presentations are the same on all trials, that is, there are no discriminative stimuli to indicate to the subject what is going to happen. Nevertheless, it is conceivable that some subjects behave as if they think that there are cues in the situation that can be used to tell which event is going to occur. If so, their behavior might be described by the threshold model with the added restriction that the two conditional probabilities of going into the detect state are, in fact, identical. If we let q denote this common probability, then a lower limb bias yields,

$$p_n = t_n q,$$

and an upper limb bias yields

$$p_n = q + u_n(1 - q)$$

Presumably, the biases change as a result of experience, and so they can be described by a learning process. Assuming the linear process of Sec 5.2 of Chapter 3, we find that

$$t_{\infty} = u_{\infty} = \frac{\pi}{\pi + (1 - \pi)\theta_2/\theta_1},$$

where θ_1 and θ_2 are learning rate parameters. If we assume that there is a value π_0 such that the subject uses a lower limb bias for $\pi < \pi_0$ and an upper limb bias otherwise, we have for the asymptotic mean probability

$$p_\infty = \begin{cases} \frac{\pi q}{\pi + (1 - \pi)\theta_2/\theta_1}, & \text{if } \pi < \pi_0 \\ q + \frac{\pi(1 - q)}{\pi + (1 - \pi)\theta_2/\theta_1}, & \text{if } \pi > \pi_0 \end{cases} \quad (43)$$

Note that when the learning rates are equal, $\theta_1 = \theta_2$, then both parts are linear in π , the lower limb starts at $(0, 0)$ and has slope q , the upper limb ends at $(1, 1)$ and has slope $1 - q$, and a discontinuity separates them at $\pi = \pi_0$. Of course, it is likely that π_0 must be treated as a random variable even for a single subject, and certainly it must be for a population of subjects. Thus in practice we cannot expect to see a discontinuous function, but rather an average of several. The difference is sketched in Fig 7.

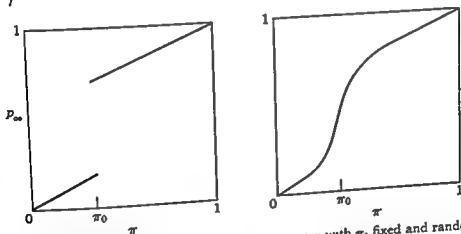


Fig 7 Threshold model prediction of p_∞ versus π with π_0 fixed and random

STRICT EXPECTED UTILITY MODEL The general strict expected utility model was stated in Sec 7.1, we simply specialize it to the present experiment. Let $v_{ij} = v(x_{ij})$, then

$$p_\infty = \frac{v_{11}\pi + v_{21}(1 - \pi)}{(v_{11} + v_{12})\pi + (v_{21} + v_{22})(1 - \pi)} \quad (44)$$

Superficially, this prediction seems to be the same as that of the beta model, Eq 42, however, there is an important difference. All the v 's in this model are positive, whereas some of the b 's in Eq 42 may be negative. Indeed, if we assume that x_{11} is preferred to x_{12} and x_{22} to x_{21} , then it is reasonable to suppose that $\beta_{11}, \beta_{12} > 1 > \beta_{21}, \beta_{22}$. Thus in particular, $b_{22} = \log \beta_{22} < 0$. Yet if we try to equate the coefficients of π and

$1 - \pi$ between Eqs 42 and 44, we find that $\iota_{21} = b_{22} < 0$, which is impossible in the strict utility model

Note that if we demand $p_{\infty} = 0$ when $\pi = 0$ and $p_{\infty} = 1$ when $\pi = 1$, then $v_{12} = \iota_{21} = 0$ and Eq 44 becomes

$$p_{\infty} = \frac{\pi}{\pi + (1 - \pi)v_{22}/v_{11}},$$

which is exactly the same form as Eq 41 of the experimenter-controlled linear model. If ι_{12} and $v_{21} \neq 0$, it is easy to see that p_{∞} is defined for all π , $0 \leq \pi \leq 1$, and that for $\pi = 0$, $p_{\infty} > 0$ and for $\pi = 1$, $p_{\infty} < 1$.

DECOMPOSITION AND CHOICE MODEL Luce (1959, pp 86-88) showed that if the assumptions of Theorem 57 of the previous Section are satisfied (namely, the decomposition assumption and the choice axiom for both the p 's and q 's), if for $x, y \in A$, $p(x, y) = 0, \frac{1}{2}$, or 1, and if in particular

$$p(x_{11}, x_{21}) = p(x_{12}, x_{22}) = 1,$$

then p must be a monotonic increasing step function of π . We do not reproduce the proof here. It should be noted that no results are known about the location or breadth of the steps, the theorem merely says that the function must be a step function.

UTILITY OF VARIABILITY MODEL Siegel (1959, 1961) suggested an asymptotic utility model that, in addition to considering the utilities of the outcomes, incorporates a term to represent the value to the subject of sheer variation in his responses. Siegel was led to consider this modification of the usual algebraic expected utility models because of their clear experimental inadequacy. He first worked it out for the two-response case, such as the probability prediction experiments in which the subject attempts to predict which of two mutually exclusive events will occur, and later Siegel and McMichael (1960) generalized it to n responses.

Siegel supposed that utility arises from three sources. u_{11} is contributed to the total whenever the event that has probability π of occurring is correctly predicted, u_{22} is contributed whenever the complementary event is correctly predicted, and u is a contribution due entirely to variability in the subject's responses, where it is assumed that variability is measured by $p(1 - p)$, p being the probability of response 1. Thus the expected utility is

$$E(u) = u_{11}\pi p + u_{22}(1 - \pi)(1 - p) + up(1 - p)$$

To find the probability p_{∞} that maximizes expected utility, we simply calculate the derivative of $E(u)$ with respect to p and set it equal to 0. This yields

$$p_{\infty} = \frac{\pi(a_1 + a_2) + (1 - a_2)}{2}, \quad (45)$$

where

$$a_1 = \frac{u_{11}}{u} \quad \text{and} \quad a_2 = \frac{u_{22}}{u}$$

It is not difficult to see that Eq 45 is a special case of Eq 42 that arose from the beta model. Specifically, if we choose the b 's so that $b_{11} - b_{12} = b_{21} - b_{22}$, then Eq 42 reduces to

$$p_\infty = \frac{\pi(b_{12} - b_{22}) + b_{22}}{b_{21} - b_{22}},$$

which is the same as Eq 45 with

$$a_1 = \frac{2b_{12}}{b_{21} - b_{22}} - 1 \quad \text{and} \quad a_2 = 1 - \frac{2b_{22}}{b_{21} - b_{22}}$$

Therefore any data supporting Siegel's asymptotic formula can equally well be interpreted as supporting the beta model formula.

RELATIVE EXPECTED LOSS MINIMIZATION (RELM) RULE In discussing his data (see Sec 8 3), Edwards (1956) introduced the following somewhat *ad hoc* proposal for the relation between p_∞ and π . He began with Savage's idea (see Sec 3 4) that decision rules for uncertain situations should be based not upon the payoff matrix but upon the matrix of the differences between what the outcome could have been had the subject known which event would occur and what was actually received. This is called the *loss matrix*. (This is the same as Savage's notion of a regret matrix, Sec 3 4.) Assuming numerical payoffs, then the loss matrix in our simple 2 by 2 situation is

$$\pi \begin{bmatrix} 1 & 2 \\ 0 & x_{11} - x_{12} \end{bmatrix}$$

$$1 - \pi \begin{bmatrix} x_{22} - x_{21} & 0 \end{bmatrix}$$

The expected loss for each response is therefore

$$EL_1 = \pi 0 + (1 - \pi)(x_{22} - x_{21})$$

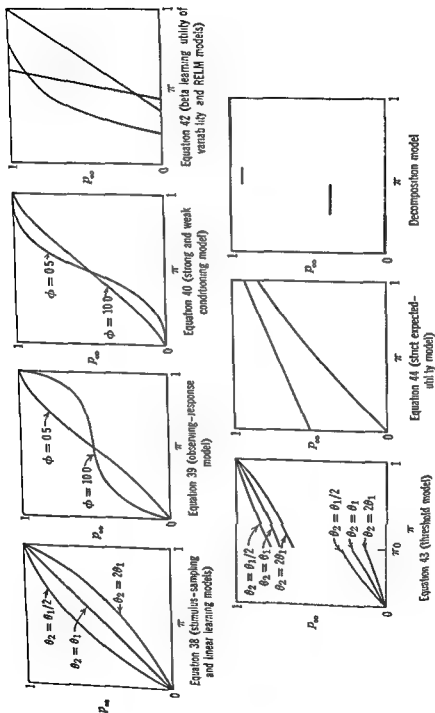
$$EL_2 = \pi(x_{11} - x_{12}) + (1 - \pi)0$$

The relative expected loss is

$$\frac{EL_2 - EL_1}{(EL_1 + EL_2)/2} = 2 \frac{\pi(x_{11} - x_{12}) - (1 - \pi)(x_{22} - x_{21})}{\pi(x_{11} - x_{12}) + (1 - \pi)(x_{22} - x_{21})}$$

Edwards' RELM rule is that the choice probability is a linear function of the relative expected loss, that is, there is a constant K such that

$$p = \frac{1}{2} + K \left(\frac{EL_2 - EL_1}{EL_1 + EL_2} \right)$$

Fig 8 Typical predictions of p_∞ versus π from ten different models.

It is not difficult to see that this is a special case of the beta model prediction. If the b 's are chosen so that the following four equations hold, then the predictions are identical

$$3(b_{22} + b_{12}) = b_{11} + b_{21}$$

$$b_{11} - b_{12} = x_{11} - x_{12}$$

$$b_{21} - b_{22} = x_{22} - x_{21}$$

$$\frac{b_{12}}{b_{11} - b_{12}} = K + \frac{1}{2}$$

SUMMARY Possibly the simplest way to see what these ten models predict is to plot specific examples of each equation. This we have done in Fig. 8. There are only seven plots because one pair and one triple of models do not lead to different predictions. The final seven are really quite different, and so we can hope to decide experimentally among them. The existing data are described in Sec. 8.3

8 EXPERIMENTAL TESTS OF PROBABILISTIC MODELS

8.1 Some Experimental Issues

The experiments intended to test probabilistic theories of preference are neither numerous, extensive, nor particularly conclusive. It is not certain that we know yet just which experiments to perform or how best to implement them, and it is quite certain that we do not know how to analyze them. It may be well to amplify somewhat these summary observations before examining specific studies.

Consider, first, the design of an experiment that is intended to test a probabilistic theory. Because these theories presume asymptotic behavior, learning theorists argue that the same set of outcomes should be presented for as many trials as are needed to demonstrate empirically that the behavior is "asymptotic"—judging by Edwards' (1961a) data, something of the order of 500 trials are needed to stabilize human responses under his experimental conditions. When comparing a subject's response probabilities in different but related choice situations, equally long runs must be carried out for each different presentation set.

Aside from impracticality when one wishes to estimate response probabilities for very many different presentation sets, this approach has been criticized because it permits serious memory effects to intrude, and these are not incorporated into the theories. These critics argue that

many presentation sets should be interleaved in some more or less random fashion, only later to be regrouped when estimating the response probabilities. Some have gone so far as to say that each presentation set should occur only once, but this makes it most difficult to test adequately any probabilistic theory—single observations do not lead to refined estimates of nonzero and one probabilities.

The primary objection to interleaving a number of different presentations is that there may exist appreciable, but not easily observed, sequential effects among the responses. If the general view embodied in the various mathematical theories of learning is approximately correct, we can hardly hope for asymptotic behavior when we continually alter the choice situation. We know of no demonstration of this *a priori* fear nor of any answer to it except the fond counterhope that people actually are better able to uncouple their responses to different presentations than current theories of learning lead us to believe.

Another difference of opinion exists concerning how much a subject should know about the chance events that are used to generate the uncertain outcomes. In experiments arising from the learning tradition, especially the probability prediction experiments, the subject is told little or nothing about the mechanism used to generate the conditional outcome schedules; indeed, his task is to discover the probability structure over the outcomes. It is in this context that hundreds of trials are needed before the behavior seems to stabilize. If, however, one is interested only in asymptotic behavior rather than in the transients of learning, it hardly seems efficient to force the subject to discover by his own crude, empirical means much of what can be conveyed accurately to him in words. So, other experimenters have gone to the other extreme of telling subjects everything they can about the generation of the outcome schedule except what will actually occur trial by trial. In some experiments the mechanism for generating the events, for example, dice, is part of the visible apparatus and the subject actually sees each chance event run off; in addition, he is sometimes given complete information about the probabilities of the several relevant events.

In general, experimenters either have coupled long experimental runs of a single presentation set with no *a priori* information about the conditional outcome schedule or they have coupled interleaved presentation sets with maximal *a priori* information about the generation of the schedule. The other two combinations do not seem to have been much used (an exception is Suppes and Atkinson, 1960), even though there are good reasons for trying a single presentation set with complete information: asymptotic behavior might occur quickly and the problem of interaction between different presentation sets would not exist.

In addition to these two broad issues of experimental design, numerous minor differences exist in the way studies are run, with as yet little or no agreement about how they might best be done. Thus, existing preference experiments are far less comparable to one another than are, say, many experiments in psychophysics or in parts of animal learning.

However we may decide to run such an experiment, once it is completed we face nasty statistical problems that have hardly begun to be formulated, let alone solved. Suppose that we are willing to assume that there is no variability other than binomial in our estimates of the response probabilities and that we wish to test whether the hypothesis of strong stochastic transitivity is met, that is, whenever $p(x, y) \geq \frac{1}{2}$ and $p(y, z) \geq \frac{1}{2}$, then $p(x, z) \geq \max [p(x, y), p(y, z)]$. It is clear that when we simply substitute estimates \hat{p} for p in this or any other similar condition, we are going to make three kinds of errors. It may be that $\hat{p}(x, y) \geq \frac{1}{2}$ when, in fact, $p(x, y) < \frac{1}{2}$, causing us to check the condition when its hypotheses are not really satisfied, it may be that $\hat{p}(x, z) < \hat{p}(x, y)$ when, in fact, $p(x, z) \geq p(x, y)$, causing us to reject the condition when it is true, and it may be that $\hat{p}(x, z) \geq p(x, y)$ when, in fact, $p(x, z) < p(x, y)$, causing us to accept it when it is false. Thus, we have the usual problem of trying to avoid both types of errors. Nothing much seems to be known about testing this hypothesis except under rather implausible assumptions (see Sec 8 2), and no work has been done in those apparently simpler cases when a theory prescribes an explicit dependence of $p(x, z)$ upon $p(x, y)$ and $p(y, z)$, such as the product rule

$$p(x, z) = \frac{p(x, y)p(y, z)}{p(x, y)p(y, z) + p(z, y)p(y, x)} \quad (46)$$

which derives from the strict utility model.

Lacking satisfactory statistical methods, authors often simply report, for example, the percentage of violations of strong transitivity and, on some intuitive basis, they conclude whether the failures are sufficiently numerous to reject the hypothesis. In other situations, tables or plots of data are reported and the reader is left pretty much on his own to reach a conclusion. Because the results are almost never clear cut, one is left with a distinct feeling of inconclusiveness.

Let us turn now from the broad issues of experimentation in this area to the studies that have actually been performed. We have organized the experiments according to the theoretical proposition under test. In Sec 8 2 we report those studies bearing on the stochastic transitivity properties, which are the only binary properties that experimenters have so far explored. In Sec 8 3 we present data designed to get at the plot of P_∞ versus π in a two-response situation. Finally, in Sec 8 4 are reported

three studies that are concerned with probabilistic expected utility models. As far as we are aware, except for two minor results that are mentioned in footnotes, these are the only theoretical ideas that have been studied empirically.

8.2 Stochastic Transitivity

Of the various ideas about probabilistic choices, the most thoroughly examined are the three stochastic transitivity properties. The experimental results are none too consistent, except that weak transitivity seems to have stood up whenever it was examined, in several papers nothing was said about it, but we suspect that this means that it was satisfied. Edwards (1961b, p. 483) makes this comment:

No experiment yet reported has created conditions deliberately designed to be unfavorable to transitivity, strong or weak, and ended up accepting even weak stochastic transitivity.¹³ In short, as a basis for psychological theorizing, algebraic transitivity is dead, and stochastic transitivity, strong or weak, has yet to be exposed to the adverse climate of hostile experiments. It seems likely that conditions can be designed in which subjects choose intransitively most of the time (unpublished research so indicates), it is even possible that the direction of the intransitive cycles can be controlled by experimental manipulation.

We group the following experiments into three categories according to the number of observations that were used to estimate the response probabilities.

ONE OBSERVATION Davidson and Marschak (1959) performed the first study in which each choice was made just once. A typical presentation of the experiment was

$$\begin{array}{cc} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{l} ZOJ \\ Z EJ \end{array} & \left[\begin{array}{cc} -5\phi & +36\phi \\ -21\phi & -38\phi \end{array} \right], \end{array}$$

where each nonsense syllable refers to the labels on three faces of a die. Three dice with different pairs of syllables were used to generate the chance events. A total of 319 different choices were presented to each subject, of which 15 served as pretraining and to test whether or not the events had subjective probability of $\frac{1}{2}$ (a point that does not concern us here). The payoffs were actually carried out in 107 randomly selected cases.

¹³ The wording of this sentence suggests that experiments have been performed that provide grounds for rejecting weak transitivity, we are not aware of which they are and Edwards does not give any references.

Each of the 17 student subjects, 11 men and 6 women, were run individually in three sessions of 35 to 55 minutes each that were spaced from one to five days apart

There were 76 triples of presentations for which any of the transitivity conditions could be checked (In addition, transitivity of "intervals" was examined, but we will not go into that) Consider any triple of presentations (x, y) , (y, z) , and (z, x) and one response to each Of the eight possible observations, six satisfy strict algebraic transitivity and two do not The problem is to use the observations from a number of triples to infer whether or not it is reasonable to suppose that the underlying probability vectors $\langle p(x, y), p(y, z), p(z, x) \rangle$ satisfy one or another of the stochastic transitivity conditions It is clear that any such vector must lie in the unit cube U , that those vectors satisfying weak transitivity lie in some subset W of U , and that those satisfying strong transitivity lie in another subset S , where, of course, $S \subset W \subset U$ We consider three hypotheses

- H_0 the distribution of all vectors is uniform over U ,
 H_w the distribution of all vectors is uniform over W ,
 H_s the distribution of all vectors is uniform over S

Davidson and Marschak calculated the probabilities of an intransitive observation on the assumption that each hypothesis is true, the numbers are the following

Hypothesis	Probability of an Intransitive Observation
H_0	0.2500
H_w	0.1875
H_s	0.1375

The decision rule that they investigated places the error probability of the hypothesis under test, H_w or H_s , equal to the error probability of the alternative hypothesis H_0 Thus, for example, H_w is accepted if the number r of intransitive cycles $\leq c$, where c is the number such that

$$Pr(r < c | H_0 \text{ is true}) = Pr(r \geq c | H_w \text{ is true}),$$

and this common probability is the significance level With a sample size of 76, one accepts H_w if $r \leq 17$ and the significance level is 0.24, and one accepts H_s if $r \leq 15$ and the significance level is 0.10

A second way to evaluate the results is to look at the likelihood ratio

$$L_w(r, n) = \frac{P_w(r, n)}{P_0(r, n)}.$$

where $P_w(r, n)$ is the probability that, when H_w is true, exactly r intransitive observations will occur if n observations are made, and $P_0(r, n)$ is the same thing when H_0 is true. A similar quantity can be calculated for hypothesis H_s .

The data and the two likelihood ratios for the 17 subjects are shown in Table 4. It is seen that H_w is acceptable according to the decision criterion for all subjects and that H_s is rejected for two, J and N .

Table 4 Number of Intransitive Observations in 76 Triples and the Likelihood Ratios for H_w and H_s against H_0 as Reported by Davidson and Marschak (1959)

Subject	Number of Intransitive Observations	Likelihood Ratios	
		H_w against H_0	H_s against H_0
A	4	100	2,100
B	10	11	26
C	11	7.6	12
D	11	7.6	12
E	1	300	20,000
F	9	16	54
G	5	69	1,000
H	4	100	2,100
I	4	100	2,100
J	16	1.2	0.3
K	8	23	110
L	2	210	9,300
M	5	69	1,000
N	16	1.2	0.3
O	7	33	240
P	7	33	240
Q	14	2.5	1.3

It is noteworthy that the number of intransitive observations dropped over the three sessions: the group percentages were 13.9, 10.9, and 6.8, with an over all average of 10.4%.

Davidson and Marschak concluded that the experiment provides no evidence for rejecting weak transitivity and that in only two cases is strong transitivity in serious doubt. The difficulty with this conclusion, as they recognized, is that the experimental design was such that these tests are not very powerful. As Morrison (1963) has pointed out, if each pair of n stimuli is presented, the maximum proportion of intransitive triples is

$(n + 1)/4(n - 2)$, which for large n approaches $\frac{1}{4}$. Thus, unless the subject is trying very hard to be intransitive—is *diabolic* in Morrison's terms—we have little hope of rejecting either null hypothesis, H_{00} or H_{01} , whether or not weak and strong stochastic transitivity are false. Moreover, the formulation of the hypotheses in terms of uniform distributions over the relevant regions of the unit cube is most questionable, however, without a far more complete theory of behavior, it is not clear what distribution we should assume.

Table 5 Scale Values Obtained by Applying Thurstone and Linear Programming Models to the Pair Comparisons of 282 Men and Women Expressing Preference for Ten Categories of Newspaper Content (Sanders, 1961, p. 27)

Content Category	Thurstone Model		Linear Programming Model	
	Scale Value	Interval	Scale Value	Interval
Government News	1000	242	1000	276
Medical and Health News	758	43	724	69
Feature Columns	715	22	655	69
Editorial Comment	693	121	586	69
Economic News	572	20	517	69
Science News	552	241	448	138
Crime and Accidents	311	1	310	69
Sports	310	135	241	103
Personal and Social News	175	175	138	138
Comics	0		0	

For further discussion of the statistical problems when only one observation is made per presentation, see Block and Marshak (1960) and Morrison (1963)

A large-scale study of newspaper content preference was made by Sanders (1961) who utilized the Thurstone pair comparison model and the linear programming model of Davidson, Suppes, and Siegel (1957). A sample of 282 men and women were asked to state their preferences for ten categories of newspaper content presented in pair comparisons. The results of applying the two models to the data are shown in Table 5

Table 6 Number of Circular Triads Observed in 45 Pair Comparison Choices by 282 Men and Women (Sanders, 1961, p. 23)

Number of Circular Triads	Number of Subjects	Number of Circular Triads	Number of Subjects
0	58	11	9
1	35	12	6
2	30	13	8
3	24	14	4
4	20	15	2
5	16	16	2
6	17	17	1
7	14	18	1
8	13	20	1
9	12	21	1
10	7	30	1

To avoid a possible confusion we remark that the linear programming model was applied to the proportions of group preferences, and not to the individual choices, as described in Sec. 4.2. With the scale values transformed to a standard $[0, 1000]$ range it is clear from the table that the two models yield very similar preference scales.

Although Sanders did not analyze his data for violations of weak or strong stochastic transitivity, he did count the number of circular triads observed in the 45 pair comparisons made by each subject. These results are shown in Table 6. If choices were made on a completely random basis, the expected number of circular triads is 30. As Table 6 shows, almost all the 282 subjects produced a far smaller number of circular triads. In fact, the majority produced not more than three such triads.

A FEW OBSERVATIONS Two studies fall into this small sample range. Papandreou et al. (1957) prepared alternatives of the following type

A triple $c = (c_1, c_2, c_3)$ consists of imaginary alternatives

$c_1 = 3$ tickets to x and 1 to y ,

$c_2 = 2$ tickets to x and 2 to y ,

$c_3 = 1$ ticket to x and 3 to y ,

where each ticket was valued at \$3 60 and x and y were chosen from ten activities, five of which were "esthetic" such as a play, opera, etc., and five of which were "athletic" such as a baseball game, a tennis match, etc. These outcomes were described in some detail to the subjects. Each of the three binary choices from each triple was offered six times, so that a total of $6 \times 3 \times 45 = 810$ choices were made. The various orders of things were randomized in the usual ways. A total of 18 student subjects participated, their responses being obtained over a period of $2\frac{1}{2}$ weeks.

After these data were collected, the experiment was criticized on the grounds that if a person were to prefer strongly one kind of activity to the other, then he would only consider the preferred one and so the numbers of tickets of that activity would be controlling and would automatically ensure transitivity. This quite reasonable objection led the authors to replicate the study with five new subjects using triples in which each c_i was constructed from two basic commodities, but no commodity appeared in more than one c_i of the triple. Thus each triple involved six basic commodities. Twenty one triples of this sort were constructed, to which 11 irrelevant ones were added forming a set of 32 that was used. Each subject responded eight times to each pair during a period of from 1 to 2 days.

A statistical model, which we do not reproduce, was proposed in their paper, from it strong stochastic transitivity follows. The hypothesis tested was whether or not the observations were consistent with the model, and therefore with strong transitivity. The test used was the likelihood ratio λ of the null hypothesis that the model is true against the alternative hypothesis that the model is not true, with critical regions defined in terms of the parameter space of the model. The frequency distributions of λ are given in Table 7. The authors point out that the statistical properties of this λ test are not known.

However, the whole problem of statistical methodology has not been pushed further because the evidence provided by the data is so overwhelming by any criterion [p. 7].

The second small sample study is by Chipman (1960b). The events in this experiment were generated by drawings from match boxes that contained a total of 100 heads and stems of broken matches. In one class of events, the only ones we consider here, the subjects knew the composition

of stems and heads, in the other, they knew only the composition of a rigged sample of 10 that they saw. Three pairs of payoffs were associated with the events $(25¢, -25¢)$, $(25¢, 0)$, and $(0, -25¢)$. Because a pair of payoffs could be assigned to heads and stems in either order, the same choice was generated in two ways. With three pairs of payoffs, this means that there were six uncertain outcomes associated with each pair of events. Assuming that the decomposition assumption (Sec 7.2) is correct, as Chipman did, this yields six independent observations from which to estimate the probabilities.

Table 7 The Frequency Distributions of Likelihood Ratios Reported by Papandreou et al (1957)

Study	$\lambda = 1$	$\lambda < 1$	Unclassifiable
1	704	101	5
2	89	13	2

A total of ten male students participated in the study.

There are only two triples of uncertain outcomes in this study for which stochastic transitivity can be tested.¹⁴ For one triple, all subjects satisfied weak and moderate transitivity and four appeared not to satisfy strong transitivity. For the other triple, weak and so moderate and strong transitivity were violated by one subject and strong by five others. Chipman did not express an opinion about how these results should be interpreted, but he cautioned the reader about the smallness of the sample size. To get some notion of what we might expect with small samples, we carried out Monte Carlo runs for three sets of three probabilities satisfying strong stochastic transitivity (in fact, satisfying Eq 46, p 379 that derives from the strict utility model), the percentages of violations of the several transitivity properties are shown in Table 8.¹⁵

A MODERATE NUMBER OF OBSERVATIONS Again, two studies fall into this category. In the first, Coombs (1958, 1959) obtained preference reports on shades of gray from four subjects, two men and two women.

¹⁴ Chipman also looked at several other things, among them a single direct test of the equation

$$p_X(x) = p_T(x)p_X(Y)$$

that arises from the strict utility model and from the choice axiom when the several probabilities are different from 0 and 1. The test seems inappropriate however, because one of the probabilities was estimated to be 0 for all ten subjects, in which case the equation is not necessarily expected to hold. It does not.

¹⁵ We wish to thank Richard Willens for carrying out these calculations.

The stimuli were 12 gray chips, ranging from almost white to almost black. The presentation sets consisted of four chips, and the subjects were instructed to rank them according to their preferences—preferences for what purpose was not specified. Each of the 495 different presentation sets was presented twice. Each pair of stimuli occurred in 45 of the sets, and the number of times that one was ranked higher than the other in the 90 opportunities for comparison was accepted as an estimate of the binary choice probability. As we pointed out in Sec. 6.1, this involves a

Table 8 Percentage of Failures of Weak, Moderate, and Strong Stochastic Transitivity Found in Estimates of Probabilities from Monte Carlo Runs Subject to True Probabilities that Satisfy Strong Stochastic Transitivity.

For each of the three sets of true probabilities that satisfy the product rule (Eq. 46, p. 379) estimated probabilities were calculated for sample sizes (n) of 5, 9, and 19. The percentage of failures of the several transitivity conditions were calculated from, respectively, 200, 100, and 50 Monte Carlo runs. The "circular" cases, for which there is an ordering x, y, z such that $\hat{p}(x, y)$, $\hat{p}(y, z)$, and $\hat{p}(z, x)$ are all $> \frac{1}{2}$, can be counted either as a single or triple failure of all three conditions; both tabulations are given.

Probabilities	n	N	Per Cent Circular Triples	Per Cent Failures When Circular Cases Are Counted Once			Per Cent Failures When Circular Cases Are Counted Three Times		
				Weak	Moderate	Strong	Weak	Moderate	Strong
$p(x, y) = 0.60$	5	200	22.5	22.5	32.0	51.4	46.5	53.1	66.6
$p(y, z) = 0.60$	9	100	17.0	17.0	25.0	57.0	38.0	44.1	67.8
$p(x, z) = 0.69$	19	50	6.0	6.0	26.0	58.0	16.1	33.9	62.5
$p(x, y) = 0.60$	5	200	0.0	0.0	2.0	17.0			
$p(y, z) = 0.90$	9	100	0.0	0.0	2.0	26.0			
$p(x, z) = 0.93$	19	50	0.0	0.0	0.0	20.0			
$p(x, y) = 0.80$	5	200	0.5	0.5	3.0	16.0	1.5	4.0	16.8
$p(y, z) = 0.80$	9	100	0.0	0.0	1.0	14.0			
$p(x, z) = 0.94$	19	50	0.0	0.0	2.0	4.0			

strong assumption that need not be true. Aside from this untested assumption and from the question whether it is really meaningful to ask in the abstract for preferences of shades of gray, this study offers the most convincing data about stochastic transitivity in the literature.

The first question is whether weak transitivity is satisfied or, put another way, whether we can order the stimuli for each subject in such a way that the probability that a stimulus with higher rank was preferred to one with lower rank is always at least $\frac{1}{2}$. Surprisingly, this was possible without exception for all four subjects. Thus, since there is no reason to question weak transitivity, we turn to strong transitivity.

The motivation for the remainder of the analysis depends on a particular random utility model suggested by Coombs (1958) as a probabilistic generalization of his unfolding technique (see end of Sec. 2.4). For a

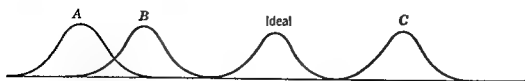


Fig 9 Distributions of stimuli and an ideal point in Coombs' random utility model

formal statement of the model, see Coombs, Greenberg, and Zinnes (1961). In words, the main idea is that random utilities can be assigned not only to the stimuli but also to the subject, representing his ideal outcome, and that preferences are determined by the absolute values of the deviations from the subject's ideal. Specifically, if U is the random vector over stimuli and I the random variable associated with the subject, then $-|U(x) - I|$ is the random utility assigned to stimulus x in the usual random utility model (Def 20, p 338) and so

$$p_T(x) = \text{Pr}[|U(x) - I| \leq |U(y) - I|, y \in Y]$$

Let us suppose, for the sake of argument, that each of the several random variables is independently distributed according to some unimodal distribution, just as is usually assumed in the Thurstonian models. Now, consider the preferences between pairs of stimuli that are generated by the model. If both stimuli have distributions that lie on the same side of the distribution of the ideal point, as A and B do in Fig 9, then the particular value of the ideal is irrelevant in the comparison. Such a pair Coombs calls *unilateral*. If they lie on opposite sides, as A and C do, then the particular value of the ideal random variable makes a great deal of difference in the response made by the subject. Such a pair he calls *bilateral*.

Next, consider triples of stimuli, as we do when testing strong stochastic transitivity. If we think of the continuum as being folded 180° about the mean ideal point, as in Fig 10, then we can distinguish three possibilities. All three of the stimuli may lie on one side of the ideal, in which case they are said to form a *unilateral triple*. Examples are A, B , and C and D, E , and F in Fig 10. Or two may lie on one side and one on the other side. If the isolated stimulus falls between the other two on the folded scale, as

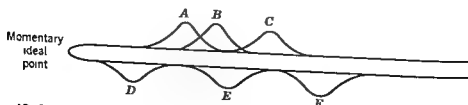


Fig 10 Random utility distributions when the scale is folded about the momentary ideal point

E falls between A and C , then the triple is called *bilateral split*, otherwise, when the isolated one either is nearer the ideal than the other two (D, B, C) or is further than the other two from the ideal (A, B, F), then the triple is called *bilateral adjacent*.

Note that the regions of overlap among the three distributions of a unilateral triple are unaffected by the fact that the ideal is a random variable. Thus, making the Case V Thurstone or weaker assumptions (see Theorem 44, p 348), the general prediction of strong stochastic transitivity holds for unilateral triples. For bilateral triples, variability in

Table 9 Number of Violations of Strong Stochastic Transitivity Reported by Coombs (1958)

Subject	Bilateral Split		Unilateral		Bilateral Cases	Adjacent Violations
	Cases	Violations	Cases	Violations		
1	34	0	20	1	66	18
2	24	0	56	2	40	24
3	38	0	20	2	62	29
4	38	0	20	10	62	48
Totals	134	0	116	15	230	119

the ideal point in effect increases the variability of the distributions when bilateral pairs are compared, but leaves it unchanged when unilateral ones are compared. An increase in the variability of the distributions of a given pair simply drives the corresponding choice probability towards $\frac{1}{2}$. Thus variability in the ideal point enhances the chance of finding strong transitivity satisfied by bilateral split triples and reduces the chance for bilateral adjacent ones. So, according to this model, we expect strong transitivity to be satisfied most often by bilateral split ones.

To test this hypothesis, we must classify each triple into one of the three types. Coombs did this by assuming that the underlying ordering of the stimuli was by brightness, and he then asked whether a mean ideal point could be chosen for each subject so that the folded scale predicted the observed preference ordering. It turned out that this was possible. Once this was done, then each triple was classified and violations of strong stochastic transitivity were counted. The results are shown in Table 9. The confirmation of his prediction is most impressive, and one cannot help but take the model seriously in spite of the questionable assumption underlying the estimates of binary choice probabilities from ranking data. Perhaps the most important unresolved problems about the Coombsian

model are a characterization of those preference situations for which the concept of an ideal is appropriate (it seems to be for this experiment, but it is much more questionable for money outcomes) and derivations of more of its mathematical properties. An interesting theoretical discussion of Coombs' experiment from a somewhat different viewpoint is to be found in Restle (1961, Chapter 4).

The second study having a moderate number of observations per presentation set was reported by Griswold and Luce (1962). Five subjects made choices among 74 different binary uncertain alternatives that were each presented, depending on the subject, from 32 to 50 times in random order. In 34 of the presentations the outcomes were sums of money ranging from 1 cent to 50 cents, and in the remainder they were packages of cigarettes of different brands. The chance events were generated by a simple pinball machine, and the events were run off after each choice were made. No payoffs were made during the course of a session, but at the end of each session one of the money and one of the cigarette pairs were chosen at random and the subject was paid off according to the outcome that had previously been determined.

With the cigarette outcomes, it was found that strong stochastic transitivity appeared to be violated in 18 out of 67 possible cases. Arguments based upon apparent shifts in preference among the cigarette brands were adduced to suggest that 27% violations may not be as unfavorable as it seems. For money outcomes, only cases of pure transitivity—where $p(x, y)$ and $p(y, z)$ are both estimated to be one—are reported, there was only one violation of transitivity in 56 cases, which seems impressive. The difference between the money and cigarette brand outcomes again hints that preferences among well-ordered outcomes may not satisfy the same model as those among more complex outcomes.¹⁶

8.3 Plots of p_{∞} versus π

A fairly large number of two-response studies have been concerned with the dependence of the asymptotic choice probability p_{∞} on the outcome probability π . The vast majority of these experiments were performed as tests of learning theory ideas, and as a result they are not as well suited to our purposes as they might first seem. We know of only two experimental studies of this plot that were prompted by preference ideas.

PROBABILITY PREDICTION The learning studies that we examine have come to be known as probability prediction experiments. They vary

¹⁶ The other main hypothesis that was tested in this study was the decomposition assumption (Sec. 7.2) and the data were interpreted as giving it modest support.

considerably in their details, and a number of them were devoted to issues of interest to learning theorists which, however, are not of immediate concern here. It does not seem appropriate to try to describe each study in detail, especially since most of these details seem to have only second order effects on the plot of p_∞ versus π , so we simply outline the main features that are common to all designs.

The subject is repeatedly required to predict which of two events, for example, which of two lights, will occur on a trial. These events are programmed according to a simple random schedule¹⁷ with probabilities π and $1 - \pi$, these probabilities are not known to the subjects at the beginning of the experiment. In most cases, one and only one of the two events occurred on a given trial, but in some the two conditional outcome schedules were independent. In these cases, the experimenter had the option of informing the subject what would have happened if he had made the other response. Anywhere from 60 to 1000 trials have been run, with the more recent studies tending toward longer runs. It is generally conceded today that the behavior is not likely to be asymptotic before two or three hundred trials, and possibly not even then.

The data reported in the literature are always averages for groups of subjects, the smallest group size being four and the largest well over 100. Considerable evidence (not much of it published) indicates that these group results can be quite misleading. The distribution of estimated p_∞ over subjects often seems not to be binomial, and sometimes there is little doubt but that it is bimodal, with the valley of the distribution coinciding roughly with the group mean probability.

In 1956, Edwards presented a table of all the asymptotic results that had been reported up to 1955. He gave "... an estimate of the asymptote (subjective, but as unbiased as I can make it)." If one compares his estimates with the original data, it appears that when the data curves had not leveled off to their asymptotic values Edwards extrapolated reasonable, smooth curves to determine his estimates of the asymptotes. As this is so highly subjective, we prefer simply to report the observed values averaged over the last few blocks of trials and to note whether or not, in our opinion, these are actually asymptotic values. In all nonasymptotic cases, the curves are increasing for $\pi > \frac{1}{2}$ and decreasing for $\pi < \frac{1}{2}$. Our estimates are given in Tables 10 and 11. The studies have been separated according to whether no explicit payoff matrix was used (Table 10) or whether it was part of the design (Table 11). The payoffs were cents except in two cases where they were points. The reader may work out for

¹⁷ Schedules other than simple random have been used, but we confine our attention to the simple random ones.

Table 10 "Asymptotic" Group Mean Probabilities from Two-Response Probability Prediction Experiments with Simple Random Schedules and No Payoff Matrices

(The number of trials shown is the total run for each condition)

Experimenter	Size of Group	No of Trials	π	Est p_{∞}	Comments
Grant Hake & Hornseth (1951)	37	60	0 00	0 00	
			0 25	0 24*	
			0 50	0 55	
			0 75	0 78*	
			1 00	1 00	
Jarvik (1951)	29	87	0 60	0 57	
			0 67	0 67	
			0 75	0 76	
Hake & Hyman (1953)	10	240	0 50	0 50	
			0 75	0 77	
Burke Estes & Hellyer (1954)	72	120	0 90	0 87	
Estes & Straughan (1954)	16	240	0 30	0 28*	
		120	0 50	0 48	
		120	0 85	0 87*	
Neimark (1956)	20	100	0 66	0 62	When outcome of other choice is unknown 0 60 and 0 99
		66	1 00	1 00	
Gardner (1957)	24	450	0 60	0 62	
			0 70	0 72	
Engler (1958)	20	120	0 25	0 29*	
			0 75	0 71*	
Cotton & Rechtschaffen (1958)	24	450	0 60	0 64	
			0 70	0 74	
Neimark & Shuford (1959)	36 x 3 runs	100	0 67	0 63*	

Table 10 (continued)

Experimenter	Size of Group	No of Trials	π	Est p_{∞}	Comments
Rubinstein (1959)	44	126	0 67	0 65	The three conditions differed in the representation of the event
	41		0 67	0 69*	
	37		0 67	0 78	
Anderson & Whalen (1960)	18	300	0 50	0 52	
			0 65	0 67	
			0 80	0 82	
Suppes & Atkinson (1960)	30	240	0 60	0 59	
Edwards (1961a)	10	1000	0 30	0 11	
			0 40	0 31	
			0 50	0 40	
			0 60	0 69	
			0 70	0 83	
Myers et al (1963)	20	400	0 60	0 62	
			0 70	0 75	
			0 80	0 87	
Friedman et al (1964)	80	288	0 80	0 81	The third of three experimental sessions

* Definitely appears not to be asymptotic

himself the general comparisons between these numerous experimental results and the models discussed in Sec 7 3

With the exception of Edwards' (1961a) 1000 trial study, the experiments without payoff matrices appear to confirm the probability matching hypothesis, at least for group averages. When, however, payoffs are included, it is clear from Table 11 that probability matching is not confirmed. This can also be seen in Fig 11, where we have plotted the data from those studies in which two or more values of π were used and in which the payoff matrix was symmetric in the sense that $x_{11} = x_{22}$ and $x_{12} = x_{21}$. As Edwards pointed out in his 1956 paper, it seems as if the curve is approximately a straight line that crosses the 45° line of the matching prediction somewhere in the neighborhood of $\pi = \frac{1}{2}$ and

Table 11 "Asymptotic" Group Mean Probabilities from Two-Response Probability Prediction Experiments with Simple Random Schedules and Payoff Matrices

Except where noted, the payoffs are in cents (The number of trials shown is the total run for each condition)

Experimenter	Size of Group	No of Trials	Payoff Matrix	π	Est. p_{∞}	Comments
Goodnow (1955)	10	120	$\begin{pmatrix} 1, & -1 \\ -1, & 1 \end{pmatrix}$	0.50	0.52	
	14			0.70	0.82*	
	14			0.90	0.96	
Edwards (1956)	24	150	$\begin{pmatrix} 10, & -5 \\ -5, & 10 \end{pmatrix}$	0.30	0.18*	
				0.50	0.48	
				0.60	0.62	
				0.80	0.96*	
Edwards (1956)	6	150	$\begin{pmatrix} 4, & -2 \\ -2, & 4 \end{pmatrix}$	0.50	0.59	When outcome of other choice is unknown, 0.59, 0.86, 0.94
				0.70	0.85	
				0.80	0.98	
Edwards (1956)	6	150	$\begin{pmatrix} 4, & -2 \\ -2, & 12 \end{pmatrix}$	0.50	0.30	When outcome of other choice is unknown, 0.42, 0.63, 0.84, 0.93*.
				0.70	0.46*	
				0.80	0.80*	
				0.90	0.95*	
Galanter & Smith (1958)	24	100	$\begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix}$	0.75	0.72*	Chips
	30	200		0.75	0.78	Payoffs were known step-functions of the number of correct responses
				0.75	0.90	
Nicks (1959)	72	380	$\begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix}$	0.50	0.58	Points
	144			0.67	0.71*	
	72			0.75	0.79*	
Siegel & Goldstein (1959)	4	300	$\begin{pmatrix} 5, & 0 \\ 0, & 5 \end{pmatrix}$	0.75	0.75	
			$\begin{pmatrix} 5, & 0 \\ 0, & 5 \end{pmatrix}$	0.75	0.86	
			$\begin{pmatrix} 5, & -5 \\ -5, & 5 \end{pmatrix}$	0.75	0.95	
Suppes & Atkinson (1960)	24	60	$\begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix}$	0.60	0.63*	Six pairs of alternatives were interleaved for a total of 360 trials.
	30	240	$\begin{pmatrix} 5, & -5 \\ -5, & 5 \end{pmatrix}$	0.80	0.73*	
			0.60	0.64		
	30	240	$\begin{pmatrix} 10, & -10 \\ -10, & 10 \end{pmatrix}$	0.60	0.69	
			0.60	0.69		
Siegel & Abelson (in Siegel, 1961)	20	300	$\begin{pmatrix} 5, & -5 \\ -5, & 5 \end{pmatrix}$	0.65	0.75	
			$\begin{pmatrix} 5, & -5 \\ -5, & 5 \end{pmatrix}$	0.75	0.93	

Table 11 (continued)

Experimenter	Size of Group	No of Trials	Payoff Matrix	π	Est P_{∞}	Comments
Myers et al (1963)	20	400	$\begin{pmatrix} 1, & -1 \\ -1, & 1 \end{pmatrix}$	0.60	0.65	
			$\begin{pmatrix} 1, & -1 \\ -1, & 1 \end{pmatrix}$	0.70	0.87	
			$\begin{pmatrix} 1, & -1 \\ -1, & 1 \end{pmatrix}$	0.80	0.93	
			$\begin{pmatrix} 10 & -10 \\ -10 & 10 \end{pmatrix}$	0.60	0.71	
			$\begin{pmatrix} 10, & -10 \\ -10 & 10 \end{pmatrix}$	0.70	0.87	
			$\begin{pmatrix} 10 & -10 \\ -10, & 10 \end{pmatrix}$	0.80	0.95	

* Definitely appears not to be asymptotic

intersects the 0 and 1 lines somewhat before π reaches 0 and 1. Whether it really is straight cannot be decided from the meager parametric data now available. It is also clear that both the exact slope of the line and its crossing point are functions of the payoff matrix (see Edwards, 1956, Galanter & Smith, 1958, and Siegel & Goldstein 1959).

One of the Suppes and Atkinson (1960) experiments bears a few words. It was explicitly designed to test the one-element stimulus-sampling model

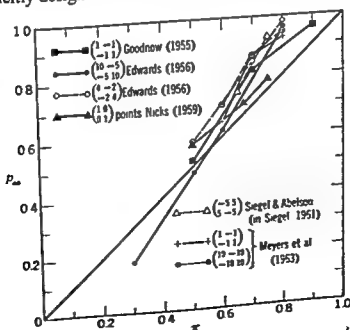


Fig 11 'Asymptotic' values of p_{∞} versus π from seven probability prediction experiments in which symmetric payoff matrices were used (See Table 11 for more details)

described in Sec 7.3. There were four levers, each having a fixed probability of a one cent reward (the probabilities were 0.2, 0.4, 0.6, and 0.8). On each trial, the subject was told which two to select from. Each of the six pairs was presented sixty times, and in the analysis these sets of trials were dealt with separately as if they were independent of the others among which they had been interleaved. Thus, in contrast to the other experiments of this section, this one involved the interleaving of several presentation sets. These data, including some of their sequential features, were analyzed in considerable detail by Suppes and Atkinson in terms of the stimulus sampling model, and they concluded that the correspondence between theory and data is not good. As far as the asymptotic results are concerned, the mean learning curves suggest that the subjects were not stabilized at the end of 60 trials. On the other hand, the fit of the observing-response model described in Sec 7.3 to these data is much better.

A PREFERENCE EXPERIMENT Luce and Shipley (1962) attempted to test the prediction of the decomposition assumption and the choice axiom that, under certain conditions on the payoff matrix, the plot of p_∞ versus π is a monotone increasing step function. They employed a design in which several different presentations were interleaved over trials and in which the subjects had complete information about the mechanism generating the chance events.

On each trial, the subject was presented with a card of the form

	Option	
	I	II
Event	$E \begin{bmatrix} x_{11} & x_{12} \end{bmatrix}$	
	$\bar{E} \begin{bmatrix} x_{21} & x_{22} \end{bmatrix}$	

After he selected one of the options, the chance event E was run off to determine which outcome he was to receive. The outcomes were points, and the total accumulated during the experiment determined the subject's share of a \$200 bonus. Two different sets of three payoff matrices were used. Within each set, the matrices differed only by a linear transformation. For one set, the entries satisfied the inequalities needed to prove that p_∞ versus π is a step function, for the other set, these inequalities were not satisfied.

The chance events were generated by tumbling five dice in a wire cage, as in the Mosteller and Nogee (1951) experiment. The possible outcomes were ranked into hands, much as in poker, and each subject was given an ordered list of the 252 hands and the probabilities of beating each. Each outcome matrix was coupled with 15 different events that spanned a 0.2 probability range, which was selected on the basis of some preliminary

runs Each of the 90 event-matrix combinations occurred once during successive blocks of 90 trials for a total of 50 presentations each during the experiment Within each 90 trial block, the order was randomized

The five student subjects were run as a group by a single experimenter, but they received independent presentations

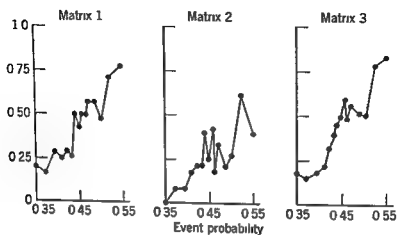
For two of the subjects, the plots of \hat{p}_∞ versus π are simple discontinuous functions from 0 to 1 The discontinuity for one subject was at $\pi = 0.4$, the rational breakpoint according to a maximization of expected money criterion, and for the other it was at 0.5 The data for the other three subjects are shown in Fig 12 Although it is not easy to say just what mathematical function might underly these observations, when we compare them with the seven types of plots that arose in Sec 7.3 (see Fig 8, p 376), they seem most consistent with the step functions of the decomposition choice model Luce and Shipley used Monte Carlo techniques to decide whether the observed "plateaus" might have resulted from binomial variability and a continuous ogive, and they concluded that this was most unlikely

It is evident that these results and those from the probability prediction experiments are not particularly compatible Among the differences that might contribute to the inconsistency are long runs of the same presentation versus interleaved presentations averages over groups of subjects versus individual plots, and a possible lack of asymptotic stability in the preference experiment

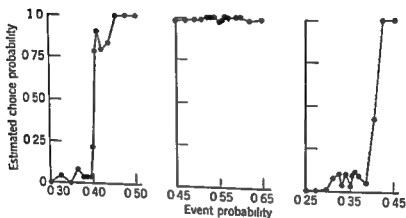
8.4 Probabilistic Expected Utility Models

RANDOM VERSUS STRICT EXPECTED UTILITY It will be recalled that Becker, DeGroot, and Marschak (1963a) showed that if a choice is made from a set consisting of m uncertain alternatives plus the average of these m , then the average alternative is selected with probability 0 when the choice probabilities satisfy the random expected utility model and with probability $1/(m+1)$ when they satisfy the strict expected utility model (see Theorem 52, p 361) In a later paper (1963b) these same authors performed an experiment to test this differential prediction Each choice set consisted of three uncertain alternatives a, b, c , and the average of these two (that is each outcome had probability $1/2$ of occurring) where the pure outcomes were sums of money ordered $a < b < c < d$ A total of 25 such sets were presented to each of 62 subjects, half of whom received no money payments and the other half were paid off in pennies Since there were no obvious differences in behavior, the results were combined

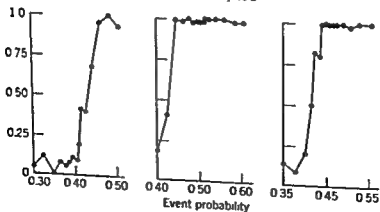
Subject 1



Subject 2



Subject 3



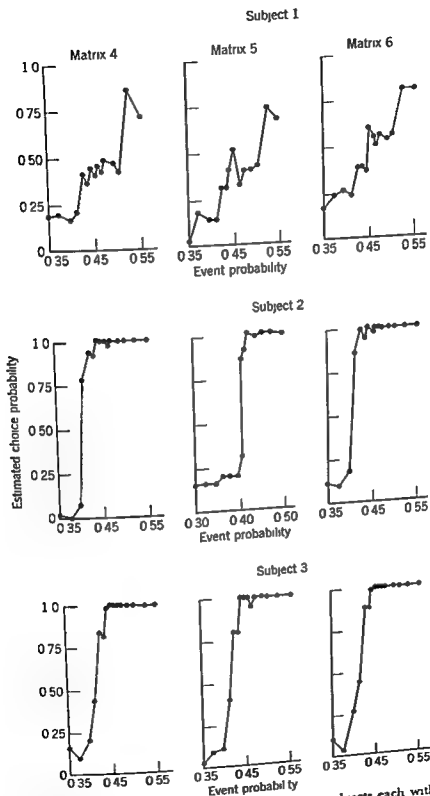


Fig 12 "Asymptotic" values of p_{∞} versus π for three subjects each with six different payoff matrices in a gambling experiment. Adapted by permission from Luce & Shipley (1962, p. 45)

There is little doubt that the random expected utility model is wrong. 60 of the 62 subjects chose the average alternative at least once. Moreover, the data also reject the strict expected utility model, at least for some subjects. That model predicts a binomial distribution with $p = \frac{1}{2}$. Table 12 compares the observed frequency distribution of the 62 subjects with the predicted one (using the normal approximation to the binomial distribution). It is evident that the discrepancy is larger than can be expected by chance. Becker et al. argue that one must conclude that at least 18% of the population fail to satisfy this model.

Table 12 Numbers of Subjects Selecting the Average Alternative (see text)

Number of Choices in 25 Trials of Average Alternative	Number of Subjects	
	Observed	Predicted
≤ 4	18	2.04
5	5	2.88
6	2	5.06
7	11	7.75
8	7	9.80
9	2	10.38
10	8	9.26
11	2	6.86
12	1	4.33
≥ 13	6	3.68

REGULARITY Debreu and Savage (see Sec. 5.2) have both suggested that $p(x, y) = p(x, y')$ whenever y' is in some sense very similar to y and in particular when y' and y denote the same thing. This, it will be recalled, is not what we would expect if the strict utility model, among others, were true. Becker, DeGroot, and Marschak (1963c) reported an experimental test of this hypothesis. As in the previous experiment, let $a < b < c < d$ be money outcomes and let $x = \langle a, a, d, d \rangle$, $y = \langle c, c, b, b \rangle$, and $y' = \langle b, b, c, c \rangle$ be the uncertain outcomes in which each component has probability $\frac{1}{2}$ of occurring. Note that y and y' are similar in the sense that the pure outcomes occur with exactly the same probabilities, although they are associated with different events. Four choice sets were studied:

(x, y) , (x, y) , (x, y, y) , and (x, y, y')

Focusing on the choice of x , there are four interesting comparisons

- 1 (x, y) and (x, y, y) ,
- 2 (x, y) and (x, y, y') ,
- 3 (x, y') and (x, y, y) ,
- 4 (x, y') and (x, y, y')

Each of 62 subjects was presented with 25 choice sets of each of the four types, these were generated by different selections for a , b , c , and d . Let $n_2(i, j)$ denote the number of times in the j th comparison that subject i chose x from the two-element set and not from the three-element set, and let $n_3(i, j)$ denote the number of times that he chose x from the three-element set and not from the two-element one. If the Debreu-Savage null hypothesis is correct, the random variable $n_2(i, j)/[n_2(i, j) + n_3(i, j)]$ is binomially distributed with $p = \frac{1}{2}$. From an examination of each (i, j) pair separately, Becker et al. concluded that "the observed data provide no reason to doubt that Debreu's comments are valid for most of the population." However, it should be kept in mind that the sample sizes are very small (none exceeded 19, and over half were less than 10), and so the tests are not very powerful.

To increase the power somewhat, we can sum either over i (subjects) or over j (comparison sets). When we do the latter and look at the statistic t , in this case $= [n_2(i) - n_3(i)]/\sqrt{n_2(i) + n_3(i)}$, where $n_k(i) = \sum_{j=1}^4 n_k(i, j)$, we find that t exceeds 2 for 11 subjects, is between -2 and 2 for 34, and is less than -2 for 7. The remaining 10 subjects continue to have sample sizes less than 10. Thus regularity seems to be violated by 7 of 52 subjects, and the Debreu-Savage hypothesis cannot be rejected for 34.

When we sum over subjects, the resulting t values are -0.75, +1.78, +2.55, and +0.54, respectively, for the four conditions. This suggests that comparisons 1 and 4 may be somewhat different from 2 and 3. Note that the identical alternative appeared twice in the three-element sets of comparisons 1 and 4, evidently there is some tendency to distinguish identity of the presentation from mere similarity, as in 2 and 3.

STRONG EXPECTED UTILITY By extending the linear programming methods of Davidson, Suppes, and Siegel (1957), Dolbear (1963) applied to data a strong or Fechnerian expected utility model in the sense of Sec 7.1. In order to develop a computationally feasible maximum-likelihood estimation procedure to determine the utility function, he assumed the following specific form of the cumulative distribution ϕ

$$\phi[u(x) - u(y)] = \begin{cases} \frac{1}{2} \exp [u(x) - u(y)] & \text{if } u(x) \leq u(y), \\ 1 - \frac{1}{2} \exp [u(y) - u(x)] & \text{if } u(x) \geq u(y) \end{cases}$$

This choice for ϕ was also motivated by its ability to fit fairly well Mosteller and Nogee's (1951) curves of the estimated choice probabilities plotted as a function of estimated expected utility differences.

We do not attempt to summarize Dolbear's method of using a linear programming method to approximate the maximum-likelihood function of the utilities, readers are referred to his dissertation for details.

He applied the strong expected utility model to data from an experiment in which ten subjects were run, five of whom were graduate students and five of whom were at an Air Force language school at Yale. Each subject participated in three sessions. During each session the subject was presented with 100 pairs of options, each of which had two outcomes, and he was required to choose one from each pair. One of these pairs was selected at random from each session and they were run off at the end of the experiment. The smaller outcome probability was either $\frac{1}{10}$, $\frac{1}{5}$, or $\frac{1}{2}$. The outcomes ranged from losing \$1.50 to winning \$9.75, which is a considerably wider range than has been used in most experiments with monetary outcomes.

For 250 choices made in three sessions Dolbear compared the number of correct predictions made by his strong expected utility model with the number made by the actuarial model and by the minimax decision rule. The remaining 50 choices were used to estimate the utility function. Averaging over subjects, the strong utility model correctly predicted 79.3% of the choices, the actuarial model 68.1%, and the minimax model 57.2%. Using a one-tail test of the null hypothesis that the strong utility and actuarial models have the same predictive ability, he found that the former was better at the 0.01 significance level for four subjects, at the 0.05 level for a fifth subject, at the 0.06 level for a sixth, and at 0.17 level for a seventh. Using the same test, the strong utility model was better than the minimax model at the 0.01 level for 8 of the 10 subjects.

References

- Abelson, R. M., & Bradley, R. A. A two by two factorial with paired comparisons. *Biometrics*, 1954, 10, 487-502.
- Aczel, J. Quasigroups nets nomograms. *Advances in mathematics*, 1964, 2, in press.
- Adams, E. W. Survey of Bernoullian utility theory. In H. Solomon (Ed.), *Mathematical thinking in the measurement of behavior*. Glencoe, Illinois: The Free Press, 1960.
- Adams, E. W. Remarks on inexact additive measurement. Unpublished manuscript, 1963.
- Adams, E. W., & Fagot, R. F. A model of riskless choice. *Behavioral Sci.*, 1959, 4, 1-10.
- Allais, M. Le comportement de l'homme rationnel devant le risque: critique des postulats et axiomes de l'école américaine. *Econometrica*, 1953, 21, 503-546.
- Anderson, N. H., & Whalen, R. E. Likelihood judgments and sequential effects in a two-choice probability learning situation. *J. exp. Psychol.*, 1960, 60, 111-120.
- Armstrong, W. E. The determinateness of the utility function. *Econ. J.*, 1939, 49, 453-467.
- Armstrong, W. E. Uncertainty and the utility function. *Econ. J.*, 1948, 58, 1-10.
- Armstrong, W. E. Utility and the theory of welfare. *Oxford Economics Papers*, 1951, 3, 259-271.

REFERENCES

- Arrow, K J Alternative approaches to the theory of choice in risk taking situations *Econometrica*, 1951, 19, 404-437 (a)
- Arrow, K J *Social choice and individual values* New York Wiley, 1951(b), 2nd ed, 1964
- Arrow, K J Utilities, attitudes, choices a review note *Econometrica*, 1958, 26, 1-23
- Atkinson, R C The observing response in discrimination learning *J exp Psychol*, 1961, 62, 253-262
- Atkinson, R C Choice behavior and monetary payoff strong and weak conditioning In Joan H Criswell, H Solomon, and P Suppes (Eds), *Mathematical methods in small group processes* Stanford Stanford Univer Press, 1962 Pp 23-34
- Audley, R J A stochastic model for individual choice behavior *Psychol Rev*, 1960, 67, 1-15
- Aumann, R J Utility theory without the completeness axiom *Econometrica*, 1962, 30, 445-462
- Becker, G M Decision making objective measures of subjective probability and utility *Psychol Rev*, 1962, 69, 136-148
- Becker, G M, DeGroot, M H, & Marschak, J Stochastic models of choice behavior *Behavioral Sci*, 1963, 8, 41-55 (a)
- Becker, G M, DeGroot, M H, & Marschak, J An experimental study of some stochastic models for wagers *Behavioral Sci*, 1963, 8, 199-202 (b)
- Becker, G M, DeGroot, M H, & Marschak, J Probabilities of choices among very similar objects *Behavioral Sci*, 1963, 8, 306-311 (c)
- Becker, G M, DeGroot, M H, & Marschak, J Measuring utility by a single response sequential method *Behavioral Sci*, 1964, 9, 226-232
- Bentham, J *The principles of morals and legislation* London, 1789
- Bernoulli, D Specimen theoriae novae de mensura sortis *Comentarii academiae scientiarum imperiales petropolitanae*, 1738, 5, 175-192 (Trans by L. Sommer in *Econometrica*, 1954, 22, 23-36)
- Birkhoff, G *Lattice theory* (Rev Ed) Providence American Math Society, 1948
- Blackwell, D, & Girshick, M A *Theory of games and statistical decisions* New York Wiley, 1954
- Block, H D, & Marschak, J Random orderings and stochastic theories of responses In I Olkin, S Ghurye, W Hoeffding, W Madow, & H Mann (Eds), *Contributions to probability and statistics* Stanford Stanford Univer Press, 1960 Pp 97-132
- Bower, G H Choice-point behavior In R R Bush & W K Estes (Eds), *Studies in mathematical learning theory* Stanford Stanford Univer Press, 1959 Pp 109-124
- Bradley, R A Rank analysis of incomplete block designs II Additional tables for the method of paired comparisons *Biometrika*, 1954, 41, 502-537 (a)
- Bradley, R A Incomplete block rank analysis on the appropriateness of the model for a method of paired comparisons *Biometrics*, 1954, 10, 375-390 (b)
- Bradley, R A Rank analysis of incomplete block designs III Some large-sample results on estimation and power for a method of paired comparisons *Biometrika*, 1955, 42, 450-470
- Bradley, R A, & Terry, M E Rank analysis of incomplete block designs I The method of paired comparisons *Biometrika*, 1952, 39, 324-345
- Burke, C J, Estes, W K, & Hellyer, S Rate of verbal conditioning in relation to stimulus variability *J exp Psychol*, 1954, 48, 153-161.
- Bush, R R, & Mosteller, F *Stochastic models for learning* New York Wiley, 1955
- Cantor, G Beiträge zur Begründung der transfiniten Mengenlehre *Math Annalen*, 1895, 46, 481-512

- Chipman, J S Stochastic choice and subjective probability, (abstract), *Econometrica*, 1958, 26, 613
- Chipman, J S The foundations of utility *Econometrica*, 1960, 28, 193-224 (a) Also in R D Luce, R R Bush, & E Galanter (Eds), *Readings in mathematical psychology*, Vol II New York Wiley, 1964 Pp 419-450
- Chipman, J S Stochastic choice and subjective probability In Dorothy Willner (Ed), *Decisions, values and groups*, Vol I New York Pergamon Press, 1960 Pp 70-95 (b)
- Cohen, J *Chance, skill and luck* London Penguin, 1960
- Cohen, J, & Hansel, C E M *Risk and gambling. the study of subjective probability* New York Philosophical Library, 1956
- Coombs, C H Psychological scaling without a unit of measurement *Psychol Rev*, 1950, 57, 145-158 Also in R D Luce, R R Bush, & E Galanter (Eds), *Readings in mathematical psychology*, Vol II New York Wiley, 1964 Pp 451-464
- Coombs, C H *A theory of psychological scaling* Engr Res Instit Bull No 34 Ann Arbor Univer of Mich Press, 1952
- Coombs, C H Social choice and strength of preference In R M Thrall, C H Coombs, & R L Davis (Eds), *Decision Processes* New York: Wiley, 1954 Pp 69-86
- Coombs, C H On the use of inconsistency of preferences in psychological measurement *J exp Psychol*, 1958, 55, 1-7
- Coombs, C H Inconsistency of preferences as a measure of psychological distance In C W Churchman & P Ratoosh (Eds), *Measurement definitions and theories* New York Wiley, 1959 Pp 221-232
- Coombs, C H, & Beardslee, D C On decision making under uncertainty In R M Thrall, C H Coombs, & R L Davis (Eds), *Decision processes* New York: Wiley, 1954 Pp 255-286
- Coombs, C H, Greenberg, M, & Zinnes, J L A double law of comparative judgment for the analysis of preferential choice and similarities data *Psychometrika*, 1961, 26, 165-171
- Coombs, C H, & Komorita, S S Measuring utility of money through decisions *Amer J Psychol*, 1958, 71, 383-389
- Coombs, C H, & Pruitt, D G Components of risk and decision making probability and variance preferences *J exp Psychol*, 1960, 60, 265-277
- Cotton, J W, & Rechtschaffen, A Replication report Two- and three-choice verbal-conditioning phenomena *J exp Psychol*, 1958, 56, 96
- Davidson, D, & Marschak, J Experimental tests of a stochastic decision theory In C W Churchman & P Ratoosh (Eds), *Measurement definitions and theories* New York Wiley, 1959 Pp 233-269
- Davidson, D, & Suppes, P A finitistic axiomatization of subjective probability and utility *Econometrica*, 1956, 24, 264-275
- Davidson, D, Suppes, P, & Siegel, S *Decision making an experimental approach* Stanford Stanford Univer Press, 1957
- Debreu, G Representation of a preference ordering by a numerical function In R M Thrall, C H Coombs, & R L Davis (Eds) *Decision Processes* New York Wiley, 1954 Pp 159-166
- Debreu, G Stochastic choice and cardinal utility *Econometrica*, 1958, 26, 440-444
- Debreu, G Topological methods in cardinal utility theory In K J Arrow, S Karlin, & P Suppes (Eds) *Mathematical methods in the social sciences*, 1959 Stanford: Stanford Univer Press, 1960 (a) Pp 16-26

REFERENCES

- Debreu, G. Review of R. D. Luce, *Individual choice behavior: a theoretical analysis*. *Amer Econ Rev*, 1960, 50, 186-188 (b)
- Debreu, G. *On the continuity properties of Paretian utility*. Tech Rept No 16, Center for Research in Management Science, Univer of California, Berkeley, 1963
- de Finetti, B. La prevision ses lois logiques, ses sources subjectives. *Ann Inst Poincare*, 1937, 7, 1-68. English translation in H. E. Kyburg, Jr., & H. E. Smokler (Eds.) *Studies in subjective probability*. New York: Wiley, 1964. Pp 93-158
- de Finetti, B. Recent suggestions for the reconciliation of theories of probability. In J. Neyman (Ed.), *Proceedings of the second Berkeley symposium on mathematical statistics and probability*. Berkeley: Univer of Calif Press, 1951
- DeGroot, M. H. Some comments on the experimental measurement of utility. *Behavioral Sci*, 1963, 8, 146-149
- Dembo, T. Der Ärger als dynamisches Problem. *Psychologische Forschung*, 1931, 15, 1-144
- Deutsch, M. Trust and suspicion. *J Conflict Resolution*, 1958, 2, 267-279
- Deutsch, M. Trust, trustworthiness, and the F scale. *J abnorm soc Psychol*, 1960, 61, 138-140
- Dolbear, F. T., Jr. Individual choice under uncertainty—an experimental study. *Yale Economic Essays*, 1963, 3, 419-470
- Edgeworth, F. Y. *Mathematical Psychics*. London: Kegan Paul, 1881
- Edwards, W. Probability-preferences in gambling. *Amer J Psychol*, 1953, 66, 349-364
- Edwards, W. Probability preferences among bets with differing expected values. *Amer J Psychol*, 1954, 67, 56-67 (a)
- Edwards, W. The reliability of probability preferences. *Amer J Psychol*, 1954, 67, 68-95 (b)
- Edwards, W. Variance preferences in gambling. *Amer J Psychol*, 1954, 67, 441-452 (c)
- Edwards, W. The theory of decision making. *Psychol Bull*, 1954, 51, 380-417 (d)
- Edwards, W. The prediction of decisions among bets. *J exp Psychol*, 1955, 51, 201-214
- Edwards, W. Reward probability, amount, and information as determiners of sequential two-alternative decisions. *J exp Psychol*, 1956, 52, 177-188
- Edwards, W. Probability learning in 1000 trials. *J exp Psychol*, 1961, 62, 385-394 (a)
- Edwards, W. Behavioral decision theory. In P. R. Farnsworth, Olga McNemar, Q. McNemar (Eds.), *Ann Rev of Psychol*. Palo Alto: Annual Reviews, Inc., 1961. Pp 473-498 (b)
- Edwards, W. Subjective probabilities inferred from decisions. *Psychol Rev*, 1962, 69, 109-135. Also in R. D. Luce, R. R. Bush, & E. Galanter (Eds.) *Readings in mathematical psychology*, Vol II. New York: Wiley, 1964. Pp 465-491
- Engler, Jean. Marginal and conditional stimulus and response probabilities in verbal conditioning. *J exp Psychol*, 1958, 55, 303-317
- Estes, W. K. A random walk model for choice behavior. In K. J. Arrow, S. Karlin, & P. Suppes (Eds.), *Mathematical methods in the social sciences, 1959*. Stanford: Stanford Univer Press, 1960. Pp 265-276
- Estes, W. K., & Straughan, J. H. Analysis of a verbal conditioning situation in terms of statistical learning theory. *J exp Psychol*, 1954, 47, 225-234
- Fagot, R. F. *A theory of decision making*. Ph.D. dissertation, Stanford. Stanford Univer, 1956
- Fagot, R. F. A model for ordered metric scaling by comparison of intervals. *Psychometrika*, 1959, 24, 157-168

- Fisher, I *The nature of capital and income* New York Macmillan Co, 1906
- Ford, L R, Jr Solution of a ranking problem from binary comparisons *Amer. Math Mon*, Herbert Ellsworth Slaughter Memorial Papers, 1957, 64, 28-33
- Friedman, M P, Burke, C J, Cole, M, Keller, L, Millward, R B, & Estes, W K Two choice behavior under extended training with shifting probabilities of reinforcement In R C Atkinson (Ed), *Studies in mathematical psychology* Stanford: Stanford Univer Press, 1964 Pp 250-316
- Galanter, E H, & Smith, W A S Some experiments on a simple thought problem *Amer J Psychol*, 1958, 71, 359-366
- Gardner, R A Probability-learning with two and three choices *Amer J Psychol*, 1957, 70, 174-185
- Georgescu-Roegen, N Threshold in choice and the theory of demand *Econometrica*, 1958, 26, 157-168
- Gerlach, Muriel W *Interval measurement of subjective magnitudes with subliminal differences* Ph D dissertation Stanford Stanford Univer, 1957
- Glaze, J A The association value of nonsense syllables *J Genetic Psychol*, 1928, 35, 255-267
- Goodnow, Jacqueline J Determinants of choice distribution in two-choice situations *Amer J Psychol*, 1955, 68, 106-116
- Grant, D A, Hake, H W, & Hornseth, J P Acquisition and extinction of verbal expectations in a situation analogous to conditioning *J exp Psychol*, 1951, 42, 1-5
- Griffith, R M Odds adjustments by American horse race bettors *Amer J Psychol*, 1949, 62, 290-294
- Griswold, Betty J, & Luce, R D Choices among uncertain outcomes a test of a decomposition and two assumptions of transitivity *Amer J Psychol*, 1962, 75, 35-44
- Guilbaud, G Sur une difficulté de la théorie du risque *Colloques internationaux du centre national de la recherche scientifique (Econometric)*, 1953, 40, 19-25
- Gulliksen, H A generalization of Thurstone's learning function *Psychometrika*, 1953, 18, 297-307
- Hake, H W, & Hyman, R Perception of the statistical structures of a random series of binary symbols *J exp Psychol*, 1953, 45, 64-74
- Hausner, M Multidimensional utilities In R M Thrall, C H Coombs, & R L Davis (Eds), *Decision processes* New York Wiley, 1954
- Herstein, I N, & Milnor, J An axiomatic approach to measurable utility *Econometrica*, 1953, 21, 291-297
- Hurst, P M, & Siegel, S Prediction of decision from a higher-ordered metric scale of utility *J exp Psychol*, 1956, 52, 138-144
- Irwin F W Stated expectations as functions of probability and desirability of outcomes *J Personality*, 1953, 21, 329-335
- Irwin F W An analysis of the concepts of discrimination and preference *Amer J Psychol*, 1958, 71, 152-163
- Jarvik, M E Probability learning and a negative recency effect in the serial anticipation of alternative symbols *J exp Psychol*, 1951, 41, 291-297
- Koopman, B O The bases of probability *Bull Amer Math Soc*, 1940, 46, 763-774 (a)
- Koopman, B O The axioms and algebra of intuitive probability *Annals of Math*, 1940, 41, 269-292 (b)
- Koopman, B O Intuitive probabilities and sequences *Annals of Math*, 1941, 42, 169-187

REFERENCES

- Kraft, C H, Pratt, J W, & Seidenberg, A Intuitive probability on finite sets *Ann Math Statistics*, 1959, 30, 408-419
- Kyburg, H E, Jr, & Smokler, H E (Eds) *Studies in subjective probability* New York Wiley, 1964
- Lindman, H, & Edwards, W Supplementary report unlearning the gambler's fallacy *J exp Psychol*, 1961, 62, 630
- Luce, R D Semiorders and a theory of utility discrimination *Econometrica*, 1956, 24, 178-91
- Luce, R D A probabilistic theory of utility *Econometrica*, 1958, 26, 193-224
- Luce, R D *Individual choice behavior a theoretical analysis* New York Wiley, 1959
- Luce, R D Response latencies and probabilities In K J Arrow, S Karlin, & P Suppes (Eds), *Mathematical methods in the social sciences, 1959* Stanford Stanford Univer Press, 1960 Pp 298-311
- Luce, R D, & Raiffa, H *Games and decisions introduction and critical survey* New York Wiley, 1957
- Luce, R D, & Shipley, Elizabeth F Preference probability between gambles as a step function of event probability *J exp Psychol*, 1962, 63, 42-49
- Luce, R D, & Tukey, J W Simultaneous conjoint measurement a new type of fundamental measurement *J Math Psychol*, 1964, 1, 1-27
- Lutzker, D R Internationalism as a predictor of cooperative behavior *J Conflict Resolution*, 1960, 4, 426-430
- Majumdar, T *The measurement of utility* London Macmillan, 1958
- Marks, Rose W The effect of probability, desirability, and 'privilege' on the stated expectations of children *J Personality*, 1951, 19, 332-351
- Marschak, J Rational behavior, uncertain prospects and measurable utility *Econometrica*, 1950, 18, 111-141
- Marschak, J Binary-choice constraints and random utility indicators In K J Arrow, S Karlin, & P Suppes (Eds), *Mathematical methods in the social sciences, 1959* Stanford Stanford Univer Press, 1960 Pp 312-329
- McGlothlin, W H Stability of choices among uncertain alternatives *Am J Psychol*, 1956, 69, 604-615
- Menger, K Probabilistic theories of relations *Proc Nat Acad Sci*, 1951, 37, 178-180
- Milnor, J Games against nature In R M Thrall, C H Coombs, & R L Davis (Eds), *Decision processes* New York Wiley, 1954 Pp 49-59
- Morrison, H W Testable conditions for triads of paired comparison choices *Psychometrika*, 1963, 28, 369-390
- Mosteller, F, & Nogee, P An experimental measurement of utility *J polit Econ*, 1951, 59, 371-404
- Murakami, Y A further note on Chipman's representative theorem *J Econ Behavior*, 1961, 1, 171-172
- Myers, J L, & Atkinson, R C Choice behavior and reward structure *J Math Psychol*, 1964, 1, 170-203
- Myers, J L, Fort, J G, Katz, L, & Suydam, M M Differential monetary gains and losses and event probability in a two-choice situation *J exp Psychol*, 1963, 66, 521-522
- Nash, J F Equilibrium points in n person games *Proc Nat Acad Sci, U.S.A.*, 1950, 36, 48-49
- Neimark, Edith D Effect of type of nonreinforcement and number of alternative responses in two verbal conditioning situations *J exp Psychol*, 1956, 52, 209-220

- Neimark, Edith D., & Shuford, E. Comparison of predictions and estimates in a probability learning situation *J exp Psychol*, 1959, 57, 294-298
- Newman, P., & Read, R. Representation problems for preference orderings *J Econ Behavior*, 1961, 1, 149-169
- Nicks, D. C. Prediction of sequential two-choice decisions from event runs *J exp Psychol*, 1959, 57, 105-114
- Papandreou, A. G., Sauerlender, O. H., Brownlee, O. H., Hurwicz, L., & Franklin, W. A test of a stochastic theory of choice *Univ. of California Publ. in Economics*, 1957, 16, 1-18
- Pareto, V. *Manuale di economia politica, con una introduzione alla scienza sociale* Milan, Italy: Societa Editrice Libreria, 1906
- Pfanzagl, J. *Die axiomatischen Grundlagen einer allgemeinen Theorie des Messens* Schriftenreihe des Statistischen Instituts der Universität Wien New Folge Nr. 1, Physica Verlag Würzburg 1959 (a)
- Pfanzagl, J. A general theory of measurement—applications to utility *Naval Research Logistics Quarterly*, 1959, 6, 283-294 (b) Also in R. D. Luce, R. R. Bush, & E. Galanter (Eds.), *Readings in mathematical psychology*, Vol. II New York: Wiley, 1964 Pp. 492-502
- Philip, B. R. Generalization and central tendency in the discrimination of a series of stimuli *Can. J. Psychol.*, 1947, 1, 196-204
- Preston, M. G., & Baratta, P. An experimental study of the auction value of an uncertain outcome *Amer. J. Psychol.*, 1948, 61, 183-193
- Pruitt, D. G. Pattern and level of risk in gambling decisions *Psychol. Rev.*, 1962, 69, 187-201
- Raiffa, H., & Schlaifer, R. *Applied statistical decision theory* Boston: Harvard Univ., 1961
- Ramsey, F. P. Truth and probability In F. P. Ramsey, *The foundations of mathematics and other logical essays* New York: Harcourt, Brace, 1931 Pp. 156-198
- Rapoport, A. Mathematical models of social interaction In R. D. Luce, R. R. Bush, & E. Galanter (Eds.) *Handbook of mathematical psychology, II* New York: Wiley, 1963 Pp. 493-579
- Restle, F. *Psychology of judgment and choice* New York: Wiley, 1961
- Royden, H. L., Suppes, P., & Walsh, K. A model for the experimental measurement of the utility of gambling *Behavioral Sci.*, 1959, 4, 11-18
- Rubinstein, I. Some factors in probability matching *J exp Psychol*, 1959, 57, 413-416
- Sanders, D. L. *The intensity of newspaper content preferences* Ph.D. dissertation Stanford: Stanford Univ., 1961
- Savage, L. J. The theory of statistical decision *J Amer Stat Assoc.*, 1951, 46, 55-67
- Savage, L. J. *The foundations of statistics* New York: Wiley, 1954
- Scodel, A., Minas, J. S., Ratoosh, P., & Lipetz, M. Some descriptive aspects of two-person non zero sum games *J Conflict Resolution* 1959, 3, 114-119
- Scott, D. Measurement structures and linear inequalities *J Math Psychol*, 1964, 1, in press
- Scott, D., & Suppes, P. Foundational aspects of theories of measurement *J Symbolic Logic*, 1958, 23, 113-128
- Shackle, G. L. S. *Expectation in economics* Cambridge: Cambridge Univ. Press, 1949
- Shackle, G. L. S. *Uncertainty in economics* Cambridge: Cambridge Univ. Press, 1955

REFERENCES

- Shuford, E H *A comparison of subjective probabilities for elementary and compound events* Rep No 20, The Psychometric Lab, Univer of North Carolina, 1959
- Shuford, E H Percentage estimation of proportion as a function of element type, exposure time, and task *J exp Psychol*, 1961, 61, 430-436 (a)
- Shuford, E H *Applications of Bayesian procedures based on discrete prior distributions* Rep No 31, The Psychometric Lab, Univer of North Carolina, 1961 (b)
- Shuford, E H, & Wiesen, R A *Bayes estimation of proportion The effect of stimulus distribution and exposure time* Rep No 23, The Psychology Lab, Univer of North Carolina, 1959
- Siegel, S A method for obtaining an ordered metric scale *Psychometrika*, 1956, 21, 207-16
- Siegel, S Level of aspiration and decision making *Psychol Rev*, 1957, 64, 253-262
- Siegel, S Theoretical models of choice and strategy behavior stable state behavior in the two-choice uncertain outcome situation *Psychometrika*, 1959, 24, 303-316
- Siegel, S Decision making and learning under varying conditions of reinforcement *Ann N Y Acad Sc*, 1961, 89, 766-783
- Siegel, S, & Fouraker, L E *Bargaining and group decision making* New York McGraw-Hill, 1960
- Siegel, S, & Goldstein, D A Decision making behavior in a two choice uncertain outcome situation *J exp Psychol*, 1959, 57, 37-42
- Siegel, S, & McMichael, Julia E *Individual choice and strategy behavior a general model for repeated choices* Res Bull No 11, Dept of Psychology, The Pennsylvania State Univer, 1960
- Sierpinski, W *Cardinal and Ordinal Numbers* Warsaw, Poland, 1958
- Simon, H A A behavioral model of rational choice *Quart J Econ* 1955, 69, 99-118
- Simon, H A Rational choice and the structure of the environment *Psychol Rev*, 1956, 63, 129-138
- Stevens, S S, & Galanter, E H Ratio scales and category scales for a dozen perceptual continua *J exp Psychol*, 1957, 54, 377-409
- Suppes, P The role of subjective probability and utility in decision making *Proceedings of the third Berkeley symposium on mathematical statistics and probability*, 1954-1955, Berkeley Univer of California Press, 1956, 5, 61-73 Also in R D Luce, R R Bush, & E Galanter (Eds), *Readings in mathematical psychology* Vol II New York Wiley, 1964 Pp 503-515
- Suppes, P *Introduction to logic* New York Van Nostrand, 1957
- Suppes, P Behavioristic foundations of utility *Econometrica*, 1961, 29, 186-202
- Suppes, P, & Atkinson, R C *Markov learning models for multiperson interactions* Stanford Stanford Univer Press, 1960
- Suppes, P, & Walsh, Karol A non linear model for the experimental measurement of utility *Behavioral Sci*, 1959, 4, 204-211
- Suppes, P, & Winet, Muriel An axiomatization of utility based on the notion of utility differences *Management Sci*, 1955, 1, 259-270
- Suppes, P, & Zinnes, J L Basic measurement theory In R D Luce, R R Bush, E Galanter (Eds) *Handbook of Mathematical Psychology*, I New York Wiley, 1963, Pp 1-76
- Thurstone, L L The learning function *J Gen Psychol*, 1930 3, 469-493
- Thurstone, L L The prediction of choice *Psychometrika*, 1945, 10, 237-253
- Toda, M Measurement of intuitive-probability by a method of game *Japan J Psychol*, 1951, 22, 29-40

- Toda, M. Subjective inference vs objective inference of sequential dependencies
Japan Psychol Res, 1958, 5, 1-20
- Tornqvist, L. *A model for stochastic decision making* Cowles Commission Discussion Paper, Economics 2100, 1954 (duplicated)
- Uzawa, H. Preference in rational choice in the theory of consumption. In K J Arrow, S Karlin, & P Suppes (Eds), *Mathematical Methods in the Social Sciences*, 1959 Stanford Stanford Univer Press, 1960 Pp 129-148
- Valavanis Vail, S. *A stochastic model for utilities* Univer of Michigan, 1957 (duplicated)
- von Neumann, J. Zur Theorie der Gesellschaftsspiele *Math Annalen*, 1928, 100, 295-320 English translation in A W Tucker & R D Luce (Eds), *Contributions to the theory of games, IV* Princeton Princeton Univer Press, 1959 Pp 13-42
- von Neumann, J, & Morgenstern, O. *Theory of games and economic behavior* Princeton Princeton Univer Press, 1944, 1947, 1953
- Wiener, N. A new theory of measurement A study in the logic of mathematics
Proc of the London Math Soc, 1921, 19, 181-205
- Wiesen, R. A. *Bayes estimation of proportions The effects of complete and partial feedback* Rep No 32, The Psychology Lab, Univer of North Carolina, 1962
- Wiesen, R. A., & Shuford, E. H. *Bayes strategies as adaptive behavior* Rep No 30, The Psychology Lab, Univer of North Carolina, 1961
- Wold, H., & Jureen, L. *Demand analysis, a study in econometrics* New York: Wiley, 1953

20

Stochastic Processes

J. Laurie Snell
Dartmouth College

Contents

1	Foundations	414
1 1	Definition of a stochastic process, 414	
1 2	Classification of stochastic processes, 417	
1 3	Some probability concepts, 421	
1 4	Examples and applications, 424	
2	Independent Processes	428
2 1	Limit theorems, 428	
2 2	Properties of sample sequences, 432	
2 3	Examples and special problems, 436	
3	Markov Processes	440
3 1	Classification of states for finite chains, 441	
3 2	Transient behavior for finite chains, 442	
3 3	Ergodic behavior for finite chains, 447	
3 4	Limit theorems for ergodic chains, 449	
3 5	Applications and examples of finite chains, 452	
3 6	Denumerable chains with discrete time, 458	
3 7	Recurrent denumerable chains, 460	
3 8	Applications and examples of denumerable chains, 462	
3 9	Continuous time Markov chains with a finite number of states, 464	
3 10	Continuous time Markov chains with a denumerable number of states, 467	
3 11	Applications and examples of continuous-time chains, 471	
4	Chains of Infinite Order	473
4 1	An application, 474	
5	Martingales	475
6	Stationary Processes	478
6 1	Examples and applications, 479	
7	Concluding Remarks	482
	References	484

Stochastic Processes

A stochastic process in its simplest form is a mathematical model designed to describe a sequence of experiments each of which depends in some way on chance. The subject had its origin in the analysis of simple games of chance played sequentially, but it is now used to describe basic phenomena in almost every field of knowledge. In physics, one studies, for example, the number of emissions from a radioactive source or the motion of a particle moving randomly in a liquid or a gas. In genetics, the frequency of different types of genes in a population observed through successive generations determines a stochastic process. In engineering, the study of queues or waiting lines as a stochastic process has led to important improvements in services. Recently, stochastic processes have been used in psychology to describe the actions of individuals faced with making a sequence of simple decisions. These are but a few of the diverse applications that have been made of stochastic process theory.

To apply fruitfully stochastic process theory it is necessary to have a phenomenon that takes place over an extended period of time and that is understood sufficiently well to make certain general assumptions of a probabilistic nature about the evolution of the process. This is a difficult task in the very complex phenomena studied by the behavioral scientist. In learning theory this was accomplished by restricting the type of learning studied to very simple situations.

Although probability ideas enter into many of the theories now being studied in psychology, it seems to me that only in the areas of learning theory and applications of information theory, for example, to languages, have stochastic processes been used significantly. Excellent survey articles have been written on these applications, see, for example, Luce (1960) and Chapters 9, 10, and 13 of this *Handbook*.

I shall try in this chapter simply to give some idea of the types of stochastic processes that have been studied and the problems that have been solved. It must be realized that although many of these problems arose naturally from applications, an even greater number arose because a mathematician was looking for an interesting problem to solve or from his desire to give a kind of mathematical completeness to a subject already studied. It is not possible to separate out the part which really means something in real life. No attempt will be made, therefore, to convince the reader that each result has immediate application to psychology.

At the end of the discussion of each section some indication has been given of one or more applications and special processes that have been found important in applied probability

In general, emphasis will be placed on giving the kind of results available concerning the various types of processes, and few proofs will be given. There are now excellent treatises on stochastic processes. The books of Doob (1953) and Loeve (1960) are the most complete books available on stochastic processes, but they presuppose a great deal of mathematical background. The books of Rosenblatt (1962) and Parzen (1962) require less mathematics and give a good general introduction to the theory of stochastic processes. The best introduction to stochastic processes, as well as to probability in general, is to be found in the book of Feller (1957). This book presupposes very little mathematics, includes numerous applications, and gives the reader a remarkable insight into the theory and applications of probability theory.

1 FOUNDATIONS

1.1 Definition of a Stochastic Process

Let us assume that we are observing an experiment that is the result of a sequence of trials or observations. Each observation can be described in terms of a finite set A of possible outcomes. The set A is the same for each experiment. Then a possible history for the entire experiment may be represented by a sequence $\omega = (a_1, a_2, \dots)$, where the element a_j of A represents the outcome that occurred on the j th trial. For example, in a sequence of tosses of a coin one usually takes $A = \{H, T\}$. A typical point ω would be $\omega = (H, H, T, T, T, H, T, \dots)$ indicating that heads occurred on the first two tosses, tails on the next three, etc. We denote by Ω the collection of all possible histories, that is, sequences ω . It is convenient at times to think of the entire sequence of observations as a single experiment in which we choose an element ω out of the set Ω . The sequences, ω , are often called sample sequences and the space, Ω , the sample space. By an event E we mean any subset of the sample space Ω . We say that the event E occurs if the history that evolves turns out to be an element of the set E . For example, when a coin is tossed a sequence of times, the event "heads on the first two tosses" is the set of all sequences that have H as their first two elements. The event "a head turns up on every toss" consists of the set with only the one element $\omega = (H, H, H, H, \dots)$. Our goal is to assign probabilities to events.

Of course, it would be most desirable to assign probabilities to all

events, that is, to all subsets of the sample space. This turns out not to be possible, at least not if we want our probabilities to satisfy certain quite simple and intuitive properties. This is a rather technical point, which is not discussed here. It is possible to assign probabilities to a wide class of events that seem to include all events suggested by practical problems. The probabilities for these events are all determined from a small class of basic probabilities that are now described. Let us denote by X_j the outcome of the j th experiment. Mathematically, we describe X_j as a function defined on the set of all sequences Ω . At the point $\omega = (a_1, a_2, \dots, a_j, \dots)$, X_j has the value a_j , that is, $X_j(\omega) = a_j$. Thus X_j simply picks out the j th element in the sequence. The functions X_1, X_2, \dots are sometimes called *outcome functions*.

Definition 1 A basic event is an event of the form

$$\{\omega \mid (X_1(\omega) = a_1) \wedge (X_2(\omega) = a_2) \wedge \dots \wedge (X_n(\omega) = a_n)\}$$

The notation $\{\omega \mid \dots\}$ means "the set of all ω such that \dots is true". A basic event is thus determined by specifying the first n outcomes.

It will be convenient from now on to simplify our notation slightly. We shall write the event $\{\omega \mid X_n(\omega) = a\}$ more simply as $X_n = a$. Thus our basic probabilities will be written as

$$Pr[(X_1 = a_1) \wedge (X_2 = a_2) \wedge \dots \wedge (X_n = a_n)]$$

It is sometimes more convenient to define the basic probabilities by means of conditional probabilities as follows. By

$$Pr[X_n = a_n \mid (X_{n-1} = a_{n-1}) \wedge \dots \wedge (X_1 = a_1)]$$

we denote the probability that the n th outcome is a_n when the outcomes of the first $n - 1$ experiments were a_1, a_2, \dots, a_{n-1} . Instead of giving the basic probabilities, we can equivalently specify all probabilities of the form

$$(i) \ Pr[X_1 = a_1]$$

$$(ii) \ Pr[X_n = a_n \mid (X_{n-1} = a_{n-1}) \wedge (X_{n-2} = a_{n-2}) \wedge \dots \wedge (X_1 = a_1)]$$

We shall also wish to consider situations in which the set A of possible values for each trial is an infinite set. If it is a denumerable set, that is, one that can be put in one-to-one correspondence with the integers, no change in our procedure is needed. If, however, the possible values consist of the whole real line or some interval of this line, we shall normally want the probabilities for each specific number to occur to be 0. Hence our basic probabilities, as defined earlier, are not useful. Instead we change our basic probabilities to be of the form

$$Pr[(\lambda_1 \leq X_1) \wedge (X_2 \leq \lambda_2) \wedge \dots \wedge (X_n \leq \lambda_n)]$$

for each $\lambda_1, \lambda_2, \dots, \lambda_n$ in the interval of possible outcomes

So far we have assumed that we are observing a sequence of trials. We often want to consider situations in which the outcome can be observed at any instant of time in some interval of time $0 \leq t \leq T$. For example, we might be observing the blood pressure of an animal while it is undergoing some experiment. If T is the length of time of the experiment, we would like our process to record a number for each time t in the interval $0 \leq t \leq T$. If we were actually recording this experiment, we might have an instrument that records the blood pressure in the form of a graph. Mathematically, we think of a possible outcome for the entire experiment as a function $\omega = a(t)$, where $a(t)$ is the pressure recorded at time t . Our sample space Ω in this situation is the collection of all possible functions $a(t)$ with domain the interval $[0, T]$. An event is a subset of these functions, and we are interested in the probability that the function ω that evolves is in certain subsets of possible functions. For example, the probability that the blood pressure continuously rises is the probability that the outcome function ω lies in the subset of all continuous increasing functions. The problem of assigning a measure to events in this situation is considerably more difficult than when the sample space is denumerable, however, it is still determined essentially by basic probabilities quite similar to those in the sequence case.

The basic probabilities are defined as follows: let X_t be a function with domain Ω that gives the outcome at time t . If the possible outcomes at any time t are from a denumerable set A , then the basic probabilities are of the form

$$Pr[(X_{t_1} = a_1) \wedge (X_{t_2} = a_2) \wedge \dots \wedge (X_{t_n} = a_n)],$$

where

$$[0 \leq t_1 \leq t_2 < \dots < t_n \leq T]$$

This is the probability of any possible specification of outcomes at any finite sequence of times in the interval of observation. When the outcomes are from an interval of real numbers, the basic probabilities are of the form

$$Pr[(X_{t_1} \leq \lambda_1) \wedge (X_{t_2} \leq \lambda_2) \wedge \dots \wedge (X_{t_n} \leq \lambda_n)]$$

We often wish to allow the experiment to continue indefinitely, in which case we allow time intervals $0 \leq t < \infty$.

It is convenient to consider other sequences of functions derived from the outcome functions. For example, assume that X_1, X_2, \dots are numerical outcome functions of a stochastic process. If we form the sums S_1, S_2, \dots defined by $S_n = X_1 + X_2 + \dots + X_n$, then S_1, S_2, \dots are functions defined on the space of sequences ω , and we can find the basic probabilities

$$Pr[(S_1 = a_1) \wedge (S_2 = a_2) \wedge \dots \wedge (S_n = a_n)]$$

However, the functions S_1, S_2, \dots do not represent the outcome of the n th trial in the original experiment. In general, whenever we have a sequence of real-valued functions X_1, X_2, \dots defined on a space Ω in such a way that we can find the basic probabilities

$$Pr[(X_1 = a_1) \wedge (X_2 = a_2) \wedge \dots \wedge (X_n = a_n)],$$

we shall say that X_1, X_2, \dots represent a stochastic process. We have a similar generalization when time is continuous or when the possible outcomes are an interval.

The mathematical aspects of the study of stochastic processes consist primarily of developing methods to obtain the probabilities of interesting events from the given basic probabilities. These basic probabilities are assigned by the scientist on the basis of his information about the phenomena being studied. The most general stochastic process is too general to be studied fruitfully. Therefore certain limited classes of stochastic processes have been studied, these classes being defined by certain restrictions placed on the basic probabilities.

1.2 Classification of Stochastic Processes

We have already discussed two types of classification of stochastic processes. The first is according to the nature of the trials being observed. The first case considered is called *discrete time* and the second, *continuous time*, the distinction is whether we observe an experiment at a sequence of times or observe it continuously over some interval of time. The second classification is according to the nature of the possible outcomes at any particular time. If the possible outcomes form a finite or a denumerably infinite set, we say that we have a *discrete space* experiment. If the possible outcomes form an interval, we refer to it as *continuous space process*. It is sometimes also useful to consider experiments in which the outcomes are from some more general space, and this is referred to as an *abstract space process*. For example, the outcomes at each time might be rankings of a group of individuals.

We shall consider primarily discrete time-discrete space, but we also give some consideration to the other three possibilities, discrete time-continuous space, continuous time-discrete space, and continuous time-continuous space.

For purposes of comparison, we give the definitions of the four basic types of processes at once and then comment on them in more detail. We consider discrete time-discrete space processes

Definition 2 A stochastic process X_1, X_2, \dots is an independent process if for any a_1, a_2, \dots, a_n

$$\begin{aligned} \Pr[X_n = a_n \mid (X_{n-1} = a_{n-1}) \wedge (X_{n-2} = a_{n-2}) \wedge \dots \wedge (X_1 = a_1)] \\ = \Pr[X_n = a_n]. \end{aligned}$$

Definition 3 A stochastic process X_1, X_2, \dots is a Markov process if for any a_1, a_2, \dots, a_n

$$\begin{aligned} \Pr[X_n = a_n \mid (X_{n-1} = a_{n-1}) \wedge (X_{n-2} = a_{n-2}) \wedge \dots \wedge (X_1 = a_1)] \\ = \Pr[X_n = a_n \mid X_{n-1} = a_{n-1}]. \end{aligned}$$

Definition 4 A stochastic process X_1, X_2, \dots is a stationary process if for any a_1, a_2, \dots, a_n and any h ,

$$\begin{aligned} \Pr[(X_1 = a_1) \wedge (X_2 = a_2) \wedge \dots \wedge (X_n = a_n)] \\ = \Pr[(X_{1+h} = a_1) \wedge (X_{2+h} = a_2) \wedge \dots \wedge (X_{n+h} = a_n)] \end{aligned}$$

Definition 5 A stochastic process X_1, X_2, \dots is a martingale if for any a_1, a_2, \dots, a_{n-1}

$$E[X_n \mid (X_{n-1} = a_{n-1}) \wedge (X_{n-2} = a_{n-2}) \wedge \dots \wedge (X_1 = a_1)] = a_{n-1}$$

To say that a process is an independent process is to say that the knowledge of the outcome of all of the trials up to the n th trial have no influence on the predictions for the next trial. For example, if a person makes a sequence of plays at different games in a gambling house, the appropriate model would be an independent process. The basic probabilities for such a process are determined completely by specifying the distribution of each of the random variables X_1, X_2, \dots , that is, by giving for each n the function p_n defined by

$$p_n(a_j) = \Pr[X_n = a_j]$$

An important special case of an independent process is the one in which the random variables X_1, X_2, \dots have a common distribution, that is, the functions p_n are the same for all n . To specify this process completely we need only give a single distribution function p . This model is appropriate when we repeat an experiment under conditions such that previous outcomes have no influence on future outcomes. It was the first model studied, having its origins in the attempt to answer questions about a gambler's fortune when he makes a sequence of plays of the same game, and it is the basis of classical statistics.

In defining a Markov process we weaken slightly the rather strong assumption that the past has no influence. We allow predictions to depend on the past, but when the last outcome is known we assume that the knowledge of any previous outcomes does not change our predictions. The

basic probabilities here are all determined by the initial distribution p defined by

$$p(a_i) = \Pr[X_1 = a_i]$$

and the transition probabilities

$$p_{ij}(n) = \Pr[X_n = a_j | X_{n-1} = a_i]$$

As in an independent process, an important special case is obtained when we assume that these transition probabilities do not depend on n . In other words, whenever the previous outcome is known, the predictions for the next trial are the same, independent of the number of the trial under consideration. Here we say that the Markov process has *stationary transition probabilities*.

In defining a stationary process we no longer consider the influence of the past. Instead we make the assumption that the process is such that if we arrive at any time and start making observations, our predictions about these observations are independent of the time that we happen to arrive.

To specify a stationary process we in effect need to know all the basic probabilities. The condition of stationarity simply imposes a condition on these probabilities, but unlike the condition of independence and the Markov condition it does not decrease the information necessary to prescribe the process completely.

The martingale process is again one in which we are permitted to have dependence on the complete past. However, we are now dealing with a real valued process which imposes a condition on the expected value of the next outcome given all of the previous outcomes. The expected value is given by

$$\begin{aligned} E[X_n | (X_{n-1} = a_{n-1}) \wedge (X_{n-2} = a_{n-2}) \wedge \dots \wedge (X_1 = a_1)] \\ = \sum a_j \Pr[X_n = a_j | (X_{n-1} = a_{n-1}) \wedge (X_{n-2} = a_{n-2}) \wedge \dots \wedge (X_1 = a_1)] \end{aligned}$$

The martingale condition has a simple gambling interpretation. If we think of X_1, X_2, \dots as the fortune of a gambler who is making a sequence of plays, then the resulting process is a martingale if the game is fair in the sense that his expected fortune after the next play is always just what it was on the last play. The interest in martingales lies not so much in their use as a model for a specific experimental situation as the natural role they play in the study of a large number of different processes.

In applying stochastic process theory it is important to realize that for each type of process quite severe assumptions are made. These are made in the interest of obtaining a process specific enough so that interesting mathematical results about the process can be obtained. In independent

and Markov processes the assumptions have also the advantage that they cut down very much the amount of information that the scientist has to supply in order to determine the process completely. It is often useful to use a specific type of process as an approximation to reality even though one knows that the assumptions are not in fact completely realized. An interesting example of this is Shannon's famous approximations to speech. Shannon (1948) first considered language as an independent process with a common distribution. The elements of the space are $(a_1, a_2, \dots, a_{27})$, and the outcomes are the letters of the alphabet and a space mark, the common distribution is the relative frequencies of these outcomes in the English language. He then obtained as a realization of this process the sentence,

OCRO HLI RGWR NMIELWIS EU LL NBNESEBYA TH EEI
ALHENHTTPA OOBTTVA NAH BRL

He next assumed as a model a Markov process with stationary transition probabilities p_{ij} , obtained as the relative frequency with which letter a_j follows the letter a_i in an English sentence. He obtained

ON IE ANTSOUTINYS ARE T INCTORE ST BE S DEAMY
ACHIN D ILONASIVE TUCOOWE AT TEASONARE FUSO TIZIN
ANDY TOBE SEACE CTISBE

He then assumed that the process was one in which the past through the last two outcomes was allowed to influence the future. In this case he obtained

IN NO IST LAT WHEY CRATICT FROURE BIRS CROCIO
PONDENOME OF DEMONSTURES OF THE RETAGIN IS
REGOACTIONA OF CRE

The last process is not a Markov process as it stands, but by considering the outcome of a single experiment to be a pair of letters it may be so considered. As we allow more dependence on the past, we get better approximations to English. All the processes considered are in fact also stationary processes. Much of modern information theory is based on the fact that the process producing the text is simply a stationary process, no assumption is made concerning how much of the past has an influence. In this example, we see dramatically both the improvement made by weakening the assumptions and the difficulty in determining the process when we do so.

The basic types of discrete space continuous time processes are defined in a similar way as follows. We assume that the time parameter is an interval and the outcome space is discrete.

Definition 6 A process $\{X_t, t \in I\}$ is a process of independent increments if for any $t_1 < t_2 < \dots < t_n$ the random variables $(X_{t_2} - X_{t_1}), (X_{t_3} - X_{t_2}), \dots, (X_{t_n} - X_{t_{n-1}})$ are independent.

Definition 7 A process $\{X_t, t \in I\}$ is a Markov process if for any a_1, a_2, \dots, a_n and $t_1 < t_2 < \dots < t_n$

$$\begin{aligned} \Pr[X_{t_n} = a_n \mid (X_{t_{n-1}} = a_{n-1}) \wedge (X_{t_{n-2}} = a_{n-2}) \wedge \dots \wedge (X_{t_1} = a_1)] \\ = \Pr[X_{t_n} = a_n \mid X_{t_{n-1}} = a_{n-1}] \end{aligned}$$

Definition 8 A process $\{X_t, t \in I\}$ is a stationary process if for any a_1, a_2, \dots, a_n and $t_1 < t_2 < \dots < t_n$ and any h

$$\begin{aligned} \Pr[(X_{t_1} = a_1) \wedge (X_{t_2} = a_2) \wedge \dots \wedge (X_{t_n} = a_n)] \\ = \Pr[(X_{t_1+h} = a_1) \wedge (X_{t_2+h} = a_2) \wedge \dots \wedge (X_{t_n+h} = a_n)] \end{aligned}$$

Definition 9 A process $\{X_t, t \in I\}$ is a martingale if for any a_1, a_2, \dots, a_{n-1} and $t_1 < t_2 < \dots < t_n$

$$E[X_{t_n} \mid (X_{t_{n-1}} = a_{n-1}) \wedge (X_{t_{n-2}} = a_{n-2}) \wedge \dots \wedge (X_{t_1} = a_1)] = a_{n-1}$$

Only the definition of independent increments is really new. This definition is analogous to processes that are the partial sums of independent random variables. The analog of the independent process is not interesting for continuous time. When we have an independent increment process and the distribution of $X_t - X_s$ depends only on the difference $t - s$, we say that we have an independent increment process with *stationary increments*. This type of process is analogous to sums of independent random variables with a common distribution.

1.3 Some Probability Concepts

We summarize here a few of the concepts from probability theory that we shall use in the discussion of stochastic processes. By a random variable we mean a real-valued function defined on a space Ω with a probability measure assigned to events, that is, to subsets of Ω . We assume that all events of the form $\{\omega \mid X(\omega) \leq \lambda\}$ are assigned a measure. By the cumulative distribution of X we mean the monotone function

$$F(\lambda) = \Pr[X \leq \lambda]$$

The graph of this function is an increasing graph, which is continuous except for a finite or denumerable number of jumps. If the cumulative distribution F can be written in the form

$$F(\lambda) = \int_{-\infty}^{\lambda} f(x) dx,$$

for some continuous function f , we say that f is the *density function* for X .

If the possible values of X form a finite or denumerable set $\{a_1, a_2, \dots\}$, the cumulative distribution function is determined by the probabilities

$$p_j = \text{Pr}[X = a_j]$$

In this case we often speak of $\{p_j\}$ as the distribution of X

A particularly important distribution, which has a density, is the normal distribution defined by

$$F(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\lambda} e^{-(x^2/2)} dx$$

The density function is then

$$\frac{1}{\sqrt{2\pi}} e^{-(x^2/2)}$$

An important special discrete distribution is the Poisson distribution defined by

$$p_j = \frac{\lambda^j e^{-\lambda}}{j!}$$

for $j = 0, 1, 2, \dots$

It might happen that a random variable X takes on a single value a with probability 1. In this case, the distribution of X has all of its mass concentrated at the single point a . In Fig. 1 are shown the graphs of the three distributions that we have discussed.

The mean or expected value of X , when it exists, is given in terms of its distribution function by the Stieltjes integral

$$E[X] = \int_{-\infty}^{\infty} \lambda dF(\lambda)$$

In case a density function exists, it is equivalent to

$$E[X] = \int_{-\infty}^{\infty} xf(x) dx,$$

and in the case of a discrete distribution to

$$E[X] = \sum_j a_j p_j$$

The variance of a random variable X is denoted by $\text{Var}[X]$, and it has the value

$$\text{Var}[X] = E[(X - E[X])^2]$$

If X and Y are two random variables, the covariance of X and Y is denoted by $\text{Cov}(X, Y)$ and is given by

$$\text{Cov}[X, Y] = E[(X - E[X])(Y - E[Y])]$$

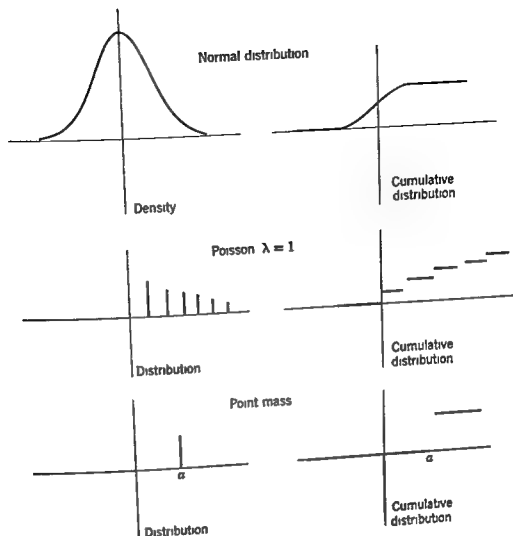


Fig 1 Examples of distributions

We shall be interested in studying the behavior of a stochastic process as we increase the number of trials. To do this it is convenient to introduce several different limiting concepts for random variables.

Definition 10 The distribution functions F_n converge to the distribution function F if for every point λ at which F is continuous

$$F_n(\lambda) \rightarrow F(\lambda)$$

The sequence of random variables X_1, X_2, \dots converge in distribution to X if the distribution of X_n converges to the distribution of X .

When we study sums of independent random variables, we shall consider the central limit theorem. This theorem states that under very general conditions sums of independent random variables, properly normalized, converge in distribution to the normal distribution. The familiar example

said to be a simple contingent schedule if these probabilities depend only on the response of the subject on the $(n+1)$ st trial. It is a double contingent schedule if it depends only on the last two responses. It is a schedule with past dependence of length m if it depends only on the $(n+1)$ st response and the previous m responses and reinforcements.

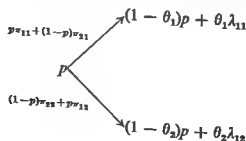
To specify the probabilities (iii) in the noncontingent case we need give only the probabilities $\pi_1, \pi_2, \dots, \pi_r$, where π_j represents the probability that the experimenter chooses the j th reinforcing event. In the simple contingent case it is necessary to specify a matrix Π with entries π_{ij} that represent the probability that the reinforcing event j is chosen when the last response of the subject was i .

An important process, which may be derived from the process just described, is the one that gives the probabilities for the various responses on the n th trial. That is, we define

$$P_n^j(a_1e_1, \dots, a_ne_n) = \Pr[A_{n+1} = j \mid e_na_n, \dots, e_1a_1]$$

Then we let $P_n = (P_n^1, P_n^2, \dots, P_n^r)$. This gives us a sequence of vector valued random variables. In the simple contingent case it is a Markov process with an infinite number of states. The reason for this is roughly the following. To show that the process P_1, P_2, \dots is a Markov process we must show that in predicting the outcome P_n in the presence of the knowledge of the $P_{n-1}, P_{n-2}, \dots, P_1$ all of the information that has any relevance is contained in the knowledge of P_{n-1} . From the basic linear axiom we see that the value of P_n depends only on the value P_{n-1} and on the experimenter's choice E_n . In the simple contingent case the choice of E_n in turn depends only on the outcome of A_n . Thus we need only convince ourselves that all of the information about the outcome of A_n is contained in the knowledge of P_{n-1} . But P_{n-1} gives us the probabilities for the occurrence of all alternatives A_n , and the knowledge of previous probabilities cannot change this given information.

An important special case of this process is the one in which there are two alternatives, one of which is reinforced on each trial. Then we need only consider the one dimensional process $\Pr[A_{n+1} = 1 \mid e_na_n, \dots, e_1a_1]$. In the simple contingent case this is a Markov process with transition probabilities



This is a Markov process which moves on the unit interval, where on each step it moves to one of two possible positions

We consider next one of the stimulus-sampling models for learning, namely, the pattern model described by Estes in Chapter 1 of Bush and Estes (1959). We assume that there are only two alternatives and that on each trial one of these is reinforced. In the pattern model it is assumed that the subject makes his decision after choosing one of a set of N stimulus patterns. We assume that these choices are equally likely so that the probability of choosing any one pattern is $1/N$. Each of the patterns is connected to one of the responses, and the subject makes the response to which the sampled pattern is connected. If the experimenter reinforces the alternative chosen by the subject, there is no change in the connection of the patterns. If he reinforces the other alternative, there is a probability c that the pattern sampled changes its connection to agree with that which was reinforced.

To illustrate the determination of the basic probabilities we assume that there are only two patterns. We indicate the responses by 1 and 2 and similarly for the connections of the patterns. A typical history for the process now is

$$\omega = (u_1 v_1 s_1 a_1 e_1 u_2 v_2 s_2 a_2 e_2 \dots)$$

where u_n is 1 or 2 according to the connection of the first pattern on the n th trial, v_n is 1 or 2 according to the connection of the second pattern, s_n is 1 or 2 according to which pattern was sampled, a_n is 1 or 2 according to which response was made, and e_n is 1 or 2 according to which response was reinforced.

We indicate the outcome functions by

$$U_1, V_1, S_1, A_1, E_1, U_2, V_2, S_2, A_2, E_2, \dots$$

where $U_n(\omega)$ is the connection of the first pattern on the n th trial, $V_n(\omega)$ the connection of the second pattern, etc.

To determine our process now we must specify the initial probabilities

$$Pr[(U_1 = u_1 \wedge V_1 = v_1)]$$

and the conditional probabilities

- (i) $Pr[U_{n+1} = u_{n+1} | e_n a_n s_n t_n u_n \dots e_1 a_1 s_1 r_1 u_1]$
- (ii) $Pr[V_{n+1} = v_{n+1} | u_{n+1} e_n a_n s_n t_n u_n \dots e_1 a_1 s_1 r_1 u_1]$
- (iii) $Pr[S_{n+1} = s_{n+1} | t_{n+1} u_{n+1} e_n a_n s_n t_n u_n \dots e_1 a_1 s_1 r_1 u_1]$
- (iv) $Pr[A_{n+1} = a_{n+1} | s_{n+1} r_{n+1} u_{n+1} e_n a_n s_n t_n u_n \dots e_1 a_1 s_1 r_1 u_1]$
- (v) $Pr[E_{n+1} = e_{n+1} | a_{n+1} s_{n+1} t_{n+1} e_n a_n s_n r_n u_n \dots e_1 a_1 s_1 r_1 u_1]$

We need only consider past histories that are possible. If ω is one of these histories

the assumptions of the model determine the conditional probabilities. Those in (i) are determined as follows. If $s_n = 1$ and $u_n \neq u_{n+1}$, the probability (i) is c . If $s_n = 1$ and $u_n = u_{n+1}$, it is $1 - c$. In other cases it is 0. Similarly, for (ii), if $s_n = 2$ and $v_{n+1} \neq v_n$, the probability (ii) is c . If $s_n = 2$ and $v_{n+1} = v_n$, it is $1 - c$. The probabilities in (iii) are always $\frac{1}{2}$ since we have assumed two patterns. The probability in (iv) is 1 if a_{n+1} agrees with the connection of the pattern last sampled and 0 otherwise. The probabilities (v) are again at the disposal of the experimenter. Since the pattern process is presumably unobservable, it would be natural to assume that these probabilities can only depend on the value of e_n and a_n . We have then the same special cases of noncontingent, simple contingent, etc., reinforcement.

Just as in the linear model, Markov theory can be used to study this process. We introduce as before the process P_1, P_2, \dots by defining

$$P_n(s_n v_n u_n, e_1 a_1 s_1 v_1 u_1) = Pr[A_n = 1 \mid s_n v_n u_n, a_1 s_1 v_1 u_1]$$

It now follows from our assumptions that the value of P_n depends only on the number of patterns conditioned to response 1, and in fact it is j/N , where j is the number connected to this response. In the noncontingent or simple contingent case the process P_1, P_2, \dots is a Markov process. We can represent this process by a finite Markov chain with states $(0, 1/N, 2/N, \dots, 1)$. Or we can take the states to be $0, 1, 2, \dots, N$. The study of this Markov chain again is easy by standard Markov chain methods. We return to this problem later.

2 INDEPENDENT PROCESSES

Let X_1, X_2, \dots be an independent process. Most of the interesting questions about such a process relate to the sums or the averages of the first n trials. Let S_1, S_2, \dots be the sequence of random variables defined by $S_n = X_1 + X_2 + \dots + X_n$. Our first interest is in theorems that relate to the distribution of S_n for large n . Such theorems are called *limit laws*. After that we consider theorems that describe the actual sequence of outcomes $S_1(\omega), S_2(\omega), \dots$.

2.1 Limit Theorems

We assume now that X_1, X_2, \dots is an independent process. Let $S_n = X_1 + X_2 + \dots + X_n$. If each X_j has a finite mean value $E[X_j] = a_j$, then the sum S_n also has a finite expected value given by

$$E[S_n] = E[X_1] + E[X_2] + \dots + E[X_n]$$

We denote $E[S_n]$ by A_n . Then $A_n = a_1 + a_2 + \dots + a_n$. If, in addition, each X_j has a finite variance $\text{Var}[X_j] = b_j^2$, then the sum S_n also has a finite variance which we denote by B_n^2 . Although it is not true in general that the variance of the sum of random variables is the sum of the individual variances, this is the case for independent random variables. Hence

$$B_n^2 = b_1^2 + b_2^2 + \dots + b_n^2$$

Definition 14 *The sequence X_1, X_2, \dots of independent random variables is said to obey the weak law of large numbers if the random variables*

$$\frac{S_n - A_n}{n}$$

converge in probability to 0

Quite general sufficient conditions exist for the weak law of large numbers to hold. One simple and useful sufficient condition is that $B_n/n \rightarrow 0$. In particular, this is true if the summands X_1, X_2, \dots are uniformly bounded. To say that a sequence of random variables is uniformly bounded is to say that all the values that any of the random variables can assume are less in absolute value than some number K .

If the random variables X_1, X_2, \dots have a common distribution, then the weak law of large numbers holds only provided that we assume the mean value exists for X_1 . If a denotes this common mean value, then the theorem states that the distribution of $(S_n/n) - a$ converges to a distribution concentrated at 0. In other words, the averages S_n/n converge in probability to a . This means that for large n it is very likely that these averages will be near a . Of course, this is the fact that enables us to use the average of a large number of identical experiments as an estimate for the mean value of each experiment when the mean is unknown. In particular, assume that we have an experiment that results in a success with probability p and a failure with probability $q = 1 - p$. We repeat this experiment a sequence of times. Then if $X_j = 1$ when the j th experiment is a success and 0 otherwise, we have a sequence of independent random variables with a common distribution and mean p . The sum $S_n = X_1 + X_2 + \dots + X_n$ represents the total number of times that the experiment was successful. The weak law of large numbers states that with high probability in a large number of experiments the proportion of successes is nearly p . This theorem justifies the frequency interpretation of probability. That is, one often defines the probability p of an event to be the fraction of times that the event occurs in a large number of repetitions of the experiment. Although we cannot in fact, be sure that we shall get a fraction p of successes, the law of large numbers does give some justification for this interpretation.

We turn now to the second important theorem about sums of independent random variables, namely, the central limit theorem

Definition 15 *The sequence X_1, X_2, \dots of independent random variables is said to obey the central limit theorem if the distribution of*

$$S_n^* = \frac{S_n - A_n}{B_n}$$

converges to the normal distribution

The effect of subtracting A_n from S_n is to produce a random variable with mean 0, and the effect of then dividing this by the standard deviation B_n is to give a random variable with variance 1. Thus, for each n , the random variable S_n^* has the same mean and variance as the unit normal distribution, and the central limit theorem states that for large n it has approximately the same distribution as this normal distribution. Necessary and sufficient conditions for this theorem to hold are known, but they are fairly complicated. A useful sufficient condition is that the individual summands X_1, X_2, \dots be uniformly bounded and that the variance B_n^2 should tend to infinity. Note that the effect of these conditions is to assure that the contribution of any one summand cannot have a large influence on the total sum for large n . Some such condition is necessary.

For identically distributed random variables a sufficient condition for the central limit theorem to hold is that the individual summands X_1, X_2, \dots have a finite variance. Assume that this is the case, and denote this common variance by b^2 . The central limit theorem then states that the

$$\frac{S_n - na}{b\sqrt{n}}$$

converges to the normal distribution

The central limit theorem is a remarkable result when one realizes its generality. The summands might arise from completely different types of observations and their distributions might be quite unsymmetrical, nonetheless, the averages of sufficiently many must have a distribution determined by a symmetrical, bell shaped density. The generality of the theorem is often offered as the explanation for the frequent occurrence of the normal distribution. For example, it is argued that the height of an individual is the cumulative effect of many small independent effects, and hence the total should be normally distributed.

Despite the generality of the central limit theorem, it is quite possible to obtain limiting distribution functions other than the normal distribution by adding up independent quantities. As an example of this, we consider next a situation that leads to the Poisson distribution.

Assume an experiment is repeated n times and that on each trial the occurrence or nonoccurrence of some event is recorded. Let p be the probability that the event occurs and $1 - p$ that it does not. Let X_1, X_2, \dots, X_n be the outcomes which are either 1 if the event occurs and 0 otherwise. Then $S_n = X_1 + X_2 + \dots + X_n$ gives the number of times the event occurs in the n trials. If we let n increase and change p in such a way that the mean of S_n , namely, np , tends to a limit λ , the distribution of S_n converges to the Poisson distribution with mean λ . One might argue that in this situation we have changed the distribution of summands as we increase n . However, this is also true in the central limit theorem since, after normalizing, the n th sum S_n is the sum of the random variables $X_{1n}, X_{2n}, X_{3n}, \dots, X_{nn}$, where $X_{jn} = (X_j - a_j)/nB_n$. Similarly, in the weak law of large number we are really talking about the distribution of the average S_n/n , and this is the sum of the n independent random variables $X_{1n}, X_{2n}, X_{3n}, \dots, X_{nn}$ where $X_{jn} = X_j/n$.

These results suggest the formulation of a very general problem. What distributions can occur as limits of sums of the form $S_n = X_{n1} + X_{n2} + X_{n3} + \dots + X_{nn}$ where each of the n summands $X_{n1}, X_{n2}, \dots, X_{nn}$ is a random variable independent of the others and the contribution of each to the sum is small for large n ? (This, of course, must be made more precise.)

This problem has been completely solved and is one of the most elegant theories in probability. The details are too complex to go into here, but suffice it to say that the class of distributions which can occur is huge, indeed infinite, in a nontrivial sense. These have been called infinitely divisible distributions, and an analytical description of the members of this class with conditions under which these distributions occur as limit distributions may be found in the book by Gnedenko and Kolmogorov (1954).

In case the reader objects to the generality of the problem posed, the following problem, more similar to the central limit theorem for identically distributed random variables, has also been completely solved. Let X_1, X_2, \dots be a sequence of identically distributed random variables. Under what conditions can one find a sequence of constants A_n and B_n so that the distributions of the random variables

$$\frac{S_n - A_n}{B_n}$$

converge to a limiting distribution, and what are the limiting distributions that can arise in this manner? The answer again is an infinite class of distributions called the *stable laws*. The only stable law with a finite variance is the normal distribution, and the corresponding limit theorem

is the central limit theorem. However, as we shall see, other limiting distributions do enter into the study of quite simple processes. As pointed out by Miller and Chomsky in Chapter 13 (pp 456-457) of this *Handbook*, distributions with infinite mean and variance occur also naturally in applications in the social sciences.

2.2 Properties of Sample Sequences

So far we have discussed only the question of the distribution of the sum of independent random variables after a large number of trials. We now wish to consider problems that relate to the entire history of the process.

Definition 16 Let X_1, X_2, \dots be a sequence of independent random variables. We say that the strong law of large numbers holds if the random variables

$$\frac{S_n - A_n}{n}$$

converge with probability 1 to 0.

A sufficient condition for the strong law of large numbers to hold is that the individual summands X_k have a finite variance b_k^2 and the sum $\sum_k b_k^2/k^2$ should be finite. This is true, in particular, when the individual summands are uniformly bounded.

For identically distributed summands the theorem states that the averages S_n/n converge to the common mean value. In this case it is necessary only to assume that this mean value $E[X_1]$ exists.

The strong law of large numbers is also helpful in understanding the frequency concept of probability. Assume that we have an experiment in which a certain event occurs with probability p . Let us repeat this experiment many times and record 1 if the event occurs and 0 if it does not occur. Then we have a sequence of identically distributed independent random variables, and the strong law of large numbers states that the fraction of times that the event occurs approaches the probability p with probability 1. For example, in a sequence of tosses of a coin it states that with probability 1 the average number of heads tends to $\frac{1}{2}$. Of course, there is nothing to stop the coin from coming up heads every time, and this is why the "probability 1" is necessary in the statement of the theorem. The point is that the theorem states that our method of assigning measures to subsets of the space of all sequences of possible outcomes is such as to assign measure 1 to the set of all sequences which converge to $\frac{1}{2}$. Note that if

our model had assigned probability $\frac{2}{3}$ to heads (that is, a biased coin), then the probability measure would assign measure 1 to the set of all sequences that converge to $\frac{2}{3}$ and so measure 0 to (among others) the set of sequences that converge to $\frac{1}{2}$.

Let us return now to our penny matching game. In Fig 2 we have shown three different histories that resulted from three games of length 100 plays each. We note that the graphs do not cross the axis very often. In each case one player is ahead the vast majority of the time. In fact, in one case one player is always ahead. This is a little surprising if we were to ask for the most likely fraction of time that one player would be ahead in such a game, we would be apt to guess $\frac{1}{2}$. In actuality this is completely wrong. The most likely thing is for one player to be ahead all the time,

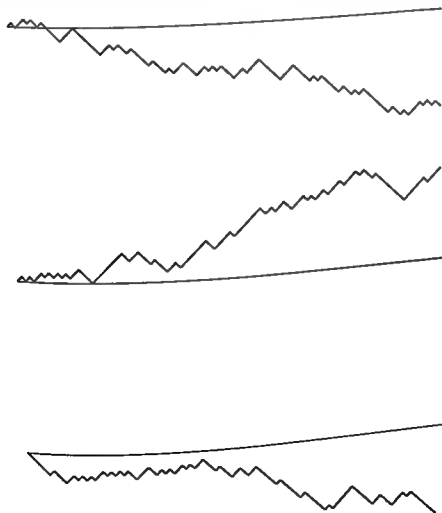


Fig 2 Three histories of 100 plays of matching pennies.

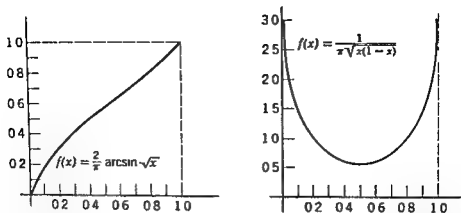


Fig 3 Limiting distribution of the fraction of time that certain random variables are positive

and the least likely thing is for a player to be ahead half the time. This is made more precise in the following limit theorem proved by E S Andersen (1953)

Theorem 1 (Arc Sine Law) Let X_1, X_2, \dots be identically distributed, independent random variables. Let $S_n = X_1 + \dots + X_n$ and let N_n be the number of times $S_j > 0$ for $j = 1, 2, 3, \dots, n$. Then if $\Pr[S_n > 0] = a_n \rightarrow a$ for $n \rightarrow \infty$ with $0 < a < 1$,

$$\Pr\left[\frac{N_n}{n} \leq \lambda\right] \rightarrow \pi^{-1} \sin \pi a \int_0^\lambda x^{a-1} (1-x)^{-a} dx$$

In particular, if the random variables X_j have a symmetric distribution, that is, if $\Pr[X_j < -\lambda] = \Pr[X_j > \lambda]$ then $\Pr[S_n > 0] \rightarrow \frac{1}{2}$. Then the distribution of the fraction of time that S_1, S_2, \dots, S_n are positive tends to a limiting distribution with density $f(x) = 1/\pi\sqrt{x(1-x)}$ on the interval $[0, 1]$. The limiting distribution in this case is $F(x) = (2/\pi) \arcsin \sqrt{x}$. A graph of this density and distribution function is given in Fig 3. This limiting distribution applies to the penny-matching game. Unlike the normal distribution, it has no maximum value, it increases as we approach the extreme values of 0 and 1, and it has a minimum at $\frac{1}{2}$.

We have seen that even in a situation of complete randomness, such as coin tossing, long trends are quite possible. This warns us against concluding a lack of randomness from the observation of such apparent trends.

Although the previous theorem warns us that it might take a long time for our penny matching friends to equalize their fortunes, it is natural to ask if it is at least true that they will eventually be equalized. The answer is yes and is a consequence of the following theorem, proved by Chung and Fuchs (1951)

Theorem 2 (Recurrence Theorem) *Let X_1, X_2, \dots be a sequence of identically distributed random variables with integer values and with mean 0. Then the partial sums S_n take on the value 0 infinitely often with probability 1.*

Returning again to our penny-matching experiment we see that the last theorem assures us that at some time the graph of the process will return to the starting point 0. In fact, since it starts afresh then, we know that with probability 1 it will return a second time. Continuing this way we can assert with probability 1 that it will return an infinite number of times. It is therefore possible to study the length of times between these returns. Let t_1 be the time required for the first return, t_2 the amount of additional time required before the second return to 0, and in general let t_n be the number of plays between the $(n-1)$ st and the n th return. Then t_1, t_2, \dots form a new sequence of integer-valued, identically distributed, independent random variables. However, in this case the common distribution has an infinite mean. Hence we cannot expect any of the standard limit theorems, such as the law of large numbers and the central limit theorem, to hold. It is possible to obtain a limit theorem for the random variables $S_n = t_1 + t_2 + \dots + t_n$, but the limiting distribution is not normal.

Results concerning the sums S_n also give information about the random variables N_n which give the number of crossings of the axis up to time n , that is, the number of equalizations of fortune in the first n plays. This is true because $Pr[N_n \geq k] = Pr[S_k \leq n]$. For a discussion of these limit laws see Feller (1957).

Let us summarize what we now know about the sample paths for our penny-matching experiment. The law of large numbers tells us that S_n/n converges to 0 with probability 1. However, we know also that S_n is 0 infinitely often. Our last theorem deals with the question of how large we can expect S_n to grow between occurrences of 0's. This is formulated as follows. Let $\{a_n\}$ be an increasing sequence of positive numbers. How fast can these increase and still permit $S_n > a_n$ infinitely often with probability 1? Let us first try $a_n = na$ for $a > 0$, that is, points on a straight line graph. The law of large numbers tells us that for $\epsilon > 0$ and almost every ω , $S_n(\omega)/n < \epsilon$ for all sufficiently large n . That is, $S_n(\omega) < \epsilon n$ for sufficiently large n . Thus the straight-line graph increases too fast. The following definition of the law of the iterated logarithm gives us a sequence $\{a_n\}$ which acts as a dividing sequence in the sense that a sequence that increases faster increases too fast and one that increases slower can be improved.

Definition 17 *Let X_1, X_2, \dots be a sequence of independent random variables. Let $S_n = X_1 + X_2 + \dots + X_n$, $A_n = E[S_n]$, $B_n^2 = \text{Var}[S_n]$. We say that the law of the iterated logarithm holds if with probability 1*

only finitely many of the sums satisfy the inequality $S_n > A_n + \lambda\sqrt{2B_n \log \log B_n}$ for every $\lambda > 1$, but for every $\lambda < 1$ this inequality holds for infinitely many S_n

As in the central limit theorem and the weak law of large numbers, very general conditions are known for which the law of the iterated logarithm holds. As in these two theorems a sufficient condition is again that the individual summands should be uniformly bounded and that the variance of the sum should tend to infinity. For a more complete discussion of these conditions and proofs see Feller (1957).

2.3 Examples and Special Problems (Examples of Independent Increments Processes)

As we have mentioned previously, the independent increments process is the continuous time analog of sums of independent random variables. We shall here discuss only two special processes of independent increments, the Brownian motion and the Poisson processes. They have played a central role in the development of stochastic process theory, and they serve as good examples of the way in which important processes arise from specific experimental situations.

The Brownian motion process was developed as a model to describe the motion of a particle of microscopic size in a fluid, say a colloidal particle. The motion is assumed to be caused by impacts on it by the other molecules.

Let X_t be the X coordinate of a particle at time t . The assumption that the medium is in microscopic equilibrium suggests that the distribution of $X_t - X_s$ for $t > s$ should be symmetric and depend only on $t - s$. Also, as a first approximation, it should be independent of previous displacements. That is, this should be an independent increments process with stationary increments. Because the displacement $X_t - X_s$ is the cumulative effect of a large number of small random impacts, the central limit theorem suggests that $X_t - X_s$ should have a normal distribution. From these assumptions the process X_t can be characterized as an independent increments process with stationary increments $X_t - X_s$ having a normal distribution with mean 0 and variance proportional to the time difference $t - s$.

As was pointed out earlier, a typical sample path for a continuous time process is a function, and we can think of the history as described by the graph of the function. The basic probability measure assigns measures to certain subsets of functions. Thus we can speak of the probability that an outcome function is one of a certain class of functions. For example, we can ask for the probability that the outcome function is continuous. For

the case of Brownian motion the probability assigned to the subset of continuous functions is 1. We thus say that with probability 1 the outcome function, or sample path, of a Brownian motion will be continuous. Somewhat less satisfactory is the fact that with probability 1 the sample paths turn out to have infinite length in any finite time interval. Doob (1960) has pointed out that this undesirable outcome cannot be willed away by adding a hypothesis that the sample paths should have finite length. The fact is that the hypotheses already made determine the process and determine the fact that the paths have infinite length. Doob suggested that examples like this show that some care must be used in setting down hypotheses about a process so that one does not end up with inconsistent hypotheses. He also suggested that often when the model leads to conclusions repugnant to the scientist he is simply taking the model too seriously. The fact is that the Brownian motion model describes the behavior of the particle in the small and here it does very well.

The Poisson process is again one with stationary independent increments. The increments are integral valued, and the distribution of $X_t - X_s$ is a Poisson distribution with mean proportional to the time difference $t - s$, that is, $c|t - s|$. The sample functions of this process are monotone increasing, but increasing only in jumps of unit magnitude. The expected number of jumps in an interval of length t is ct . The Poisson process is often used as a model for events occurring randomly in time. An event occurs when the sample function increases. The rate at which the events occur is the constant c . Then X_t is the number of events that have occurred prior to time t .

Assume now that we want a model for the random occurrence of events in time. For example, a radioactive source is emitting particles in such a way that the time of emission of a particle is random. Let us denote by X_t the number of events that have occurred by time t . We take $X_0 = 0$. Then, if we assume that X_t is a process of stationary independent increments and in addition that only a finite number of events occur in any finite time interval, we are led to the Poisson process. Again very simple hypotheses lead to a quite specific process.

The Poisson process has been used, for example, in the study of queues or waiting lines. The assumption is often made that customers arrive according to a Poisson process. A consequence of the Poisson model is that the times between the occurrence of the j th and $j + 1$ event are independent random variables with common distribution. If Y_1, Y_2, \dots are these times, then the distribution is given by

$$Pr[Y_1 \leq t] = 1 - e^{-ct}, \quad t \geq 0,$$

that is, an exponential distribution. A consequence of this is that if we

are using this model for arrival of customers in a line then, given that no customer has arrived by time s , the probability that a customer has arrived by time $s + t$ is independent of s . Again, if this consequence seems unreasonable, this means that the Poisson process is not appropriate or that we are being too fussy about the applicability of our model.

RENEWAL THEORY A renewal process is a special case of a sequence of independent random variables. We assume that X_1, X_2, \dots are non-negative, integral valued independent random variables. We interpret X_i as the lifetime of the i th article which at the end of its lifetime is replaced. A simple example is the replacement of light bulbs. We can also consider a situation in which a subject repeats a task a number of times and let X_i be the length of time required to complete the task on the i th opportunity.

We write as usual, $S_n = X_1 + X_2 + \dots + X_n$. Then S_n is the time of the n th renewal. We write N_t for the largest value of n for which $S_n \leq t$. That is, it is the number of renewals that have occurred by time t . These two quantities are connected by the fact that

$$\Pr[S_n \leq t] = \Pr[N_t \geq n]$$

It is intuitively clear that if $m = E[X_1]$, then in time t we should expect about t/m renewals. The following theorem summarizes this fact.

Theorem 3 *If N_t is the number of renewals in time t , then*

$$\frac{E[N_t]}{t} \rightarrow \frac{1}{m},$$

where $m = E[X_1]$. If m is infinite, the limit is 0. If X_1 has finite variance σ^2 , then

$$E[N_t] - tm \rightarrow \frac{\sigma^2}{2m^2} - \frac{3}{2}$$

$$\text{Var}[N_t] \sim \frac{\sigma^2}{m^3} t$$

and

$$\Pr\left[N_t \geq \frac{t}{m} - \frac{\lambda\sigma}{m}\sqrt{\frac{t}{m}}\right] \rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\lambda} e^{-y^2/2} dy$$

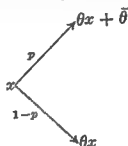
The limit theorem is obtained by applying the central limit theorem to S_n . If the variance is infinite, limit theorems for N_t can also be obtained, but these will involve nonnormal distributions. For a more complete discussion of this theorem, see Smith (1958).

The results may be applied to recurrent Markov chains as follows. Start a recurrent chain in state 0 and let X_i be the length of time between

the $(i - 1)$ st and i th return to 0. Then X_1, X_2, \dots is a renewal process. In a finite chain the means and variances are finite. They may be infinite in a denumerable chain, as in returns to the origin in our penny-matching game.

AN EXAMPLE FROM PSYCHOLOGY In our discussion of limit laws for sums of independent random variables we stressed that limit theorems are obtained when it is assumed that the contribution of any one summand is small compared with the total sum. It is interesting that in the Bush-Mosteller learning model one meets a situation in which this is not the case, and as a result we have a quite different behavior.

Specifically, consider the Bush-Mosteller model for the noncontingent case. Then we are led to a Markov process with state space the unit interval and transition probabilities given by



Here $\bar{\theta} = 1 - \theta$. It was shown in Kemeny and Snell (1957) that if the process is started at 0, the distribution of the position after n steps is the same as the distribution of the sum

$$S_n = \sum_{j=1}^n \epsilon_j \theta \bar{\theta}^{j-1},$$

where the ϵ_j are a sequence of independent random variables which take on the value 1 with probability p and the value 0 with probability $1 - p$. That is, if X_1, X_2, \dots represent the Markov process started in state 0,

$$Pr[X_n \leq \lambda] = Pr[S_n \leq \lambda]$$

The sums S_n converge to a finite total sum

$$S = \sum_{j=1}^{\infty} \epsilon_j \theta \bar{\theta}^{j-1}$$

Hence $\lim_n Pr[S_n \leq \lambda] = Pr[S \leq \lambda]$. Therefore

$$\lim_n Pr[X_n \leq \lambda] = Pr[S \leq \lambda]$$

In the case $\theta = \frac{1}{2}$ and $p = \frac{1}{2}$, S has the uniform distribution. That is, $Pr[a < S < b] = b - a$ for $0 \leq a < b \leq 1$. However, in the case $\theta = \frac{1}{3}$

the distribution is a famous pathological distribution in mathematics called the *Cantor function*. It has all its mass on a set of points that has length 0 according to the usual measure of length (Lebesgue measure). It is still an interesting outstanding problem in mathematics to describe the type of limiting distribution that is obtained for other choices of θ and p .

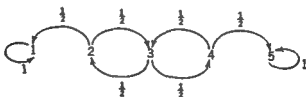
The fact that distributions which mathematicians consider as pathological occur in such a simple model in psychology indicates the danger of trying to assume that everything will be simple in applications.

3 MARKOV PROCESSES

We begin our treatment of Markov processes by considering a discrete-time, finite-state Markov chain with stationary transition probabilities. We denote the states by integers $1, 2, \dots, r$. Recall that such a process is completely determined by specifying how it starts, that is, by the initial probabilities $Pr[X_1 = i]$, and by the transition probabilities p_{ij} which represent

$$Pr[X_{n+1} = j | X_n = i]$$

It is customary to exhibit the transition probabilities in the form of a matrix $P = \{p_{ij}\}$. These probabilities can also be exhibited schematically. For example, consider the simple random walk that moves through the integers $1, 2, 3, 4, 5$. We assume that when it is in state $2, 3, 4$, it moves with equal probability one step to the right or left. When in 1 or 5 it remains in this state. Then these probabilities may be indicated by a diagram as follows



or as a matrix

$$P = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{matrix}.$$

Matrix theory plays an important role in the study of Markov chains. For example, the n th power of the transition matrix P has the following interpretation. The ij th entry, $p_{ij}^{(n)}$, of the matrix P^n is the probability that a chain which started in state i will be in state j after n steps. Similarly, if the initial probabilities are given by the vector $p = (p_1, p_2, \dots, p_r)$, then the vector pP^n gives the probabilities of being in each of the states after n steps.

3.1 Classification of States for Finite Chains

The study of finite chains is greatly simplified by classifying them into several different types and studying each type separately. Knowledge about the behavior of these special chains answers questions about the most general chain.

Definition 18 *A state i is called transient if when the process is started in i , the probability of ever returning to this state is less than 1. It is recurrent if the probability of returning is 1.*

We denote the set of all states by S , the set of all recurrent states by R , and the set of all transient states by T . Then $S = R \cup T$. In a finite chain it is possible to have all states recurrent, but they cannot be all transient. The set R of recurrent states is further subdivided as follows. We say that states i and j communicate if it is possible to go both from i to j and from j to i . The relation of communication is an equivalence relation and as such partitions the set R into equivalence classes which we denote by R_1, R_2, \dots, R_d . If the process starts in equivalence class R_j , it remains in this class for all time. Choose a state i in R_j , and consider the times n such that $p_{ii}^{(n)} > 0$. These are the times that it is possible to return to state i . It can be shown that if d is the least common multiple of these times, then all sufficiently large multiples of d are in the set of possible return times. The number d is called the *period* of the state. All states in the same equivalence class R_j have the same period. Thus we speak of the class R_j as having period d . It is possible to subdivide R_j further into d mutually disjoint subsets C_1, C_2, \dots, C_d called *cyclic subsets*. If the process is started in C_1 it can move only to a state in C_2 , from there only to a state in C_3 , etc., until it reaches C_d . From C_d it can go only to C_1 . That is, it moves cyclically through the d subclasses.

If we start the process in a transient state it eventually reaches one of the recurrent classes, and from that time it remains in this class, moving cyclically through the subclasses. We analyze the transient behavior—that is, the behavior up to the time it reaches a recurrent set—first and then consider the behavior when it begins in (or gets into) a recurrent set.

The two theories can then be put together to give the most general behavior. To carry out this program, it is convenient to define a special type of chain called an absorbing chain.

Definition 19 *A state is absorbing if it is impossible to leave the state. That is, i is absorbing if $p_{ii} = 1$. A chain is absorbing if it is possible to reach an absorbing state from every state.*

3.2 Transient Behavior for Finite Chains

We want here to study the behavior of a finite chain as it moves among the transient states. Once it reaches a recurrent state it can never go back to a transient state. Thus we are interested in the behavior of the chain up to the first time that it hits a recurrent state. To study this it is convenient to form a new chain by making all the recurrent states into absorbing states, thereby obtaining an absorbing chain. We assume that we have made this modification in our chain, and we renumber the states so that the absorbing states come first. We can then put our transition matrix in the form

$$P = \begin{array}{c} \text{absorbing states} \\ \text{transient states} \end{array} \begin{bmatrix} I & 0 \\ R & Q \end{bmatrix}$$

The entries of the matrix R give the probabilities of moving from the transient states into the absorbing states in one step. The entries of Q give the probabilities of moving in one step from a transient state to the other transient states. The matrix I is the identity matrix of dimension equal to the number of absorbing states, and 0 is the matrix of all 0 's with the number of rows equal to the number of absorbing states and the number of columns equal to the number of transient states.

When the transition matrix is put in the preceding canonical form, its n th power has the form

$$P^n = \begin{bmatrix} I & 0 \\ B^{(n)} & Q^n \end{bmatrix}$$

The ij th entry of $B^{(n)}$ represents the probability that starting in the transient state i the process is absorbed in state j during the first n steps. These probabilities are nonincreasing and hence tend to a limiting value which we denote by b_{ij} . Then b_{ij} is the probability that starting in state i the process is eventually absorbed in state j .

The ij th entry of Q^n gives the probability that starting in the transient state i the chain is in the transient state j after n steps. Since the process is eventually absorbed, these probabilities must tend to 0 . Thus, the matrix

P^n approaches a limiting matrix A of the form

$$A = \begin{bmatrix} I & 0 \\ B & 0 \end{bmatrix}$$

This means that the components of P^n approach the corresponding components of A

Since the matrix Q^n is simply the n th power of the matrix Q , it is often possible to give an explicit expression for this quantity. This is particularly simple when it is possible to diagonalize Q in the form

$$Q = U \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & & \lambda_3 \end{bmatrix} U^{-1},$$

where the middle matrix has only nonzero entries on the diagonal. This involves finding the eigenvalues $\lambda_1, \lambda_2, \lambda_3$. These are real numbers λ such that, for some vector u , $uQ = \lambda u$. In most cases this is quite difficult. Once done, however, we can write

$$Q^n = U \begin{bmatrix} \lambda_1^n & 0 & 0 \\ 0 & \lambda_2^n & 0 \\ 0 & & \lambda_3^n \end{bmatrix} U^{-1}$$

From this a simple expression for $B^{(n)}$ can be obtained since

$$B^{(n)} = (I + Q + Q^2 + \dots + Q^{n-1})R$$

$$= U \begin{bmatrix} \frac{1 - \lambda_1^n}{1 - \lambda_1} & 0 & 0 \\ 0 & \frac{1 - \lambda_2^n}{1 - \lambda_2} & 0 \\ 0 & & \frac{1 - \lambda_3^n}{1 - \lambda_3} \end{bmatrix} U^{-1}R$$

When we have a Markov chain model we are interested in studying a variety of descriptive quantities which relate to the process under observation. Many of the descriptive quantities for absorbing chains can be obtained by simple matrix operations on one matrix called the *fundamental matrix* in Kemeny and Snell (1960). This is the matrix

$$N = I + Q + Q^2 + \dots$$

This matrix is also the inverse of $I - Q$, that is,

$$N(I - Q) = (I - Q)N = I$$

The entries of this matrix have a simple probabilistic interpretation. The entry n_{ij} is the mean number of times that the process is ever in state j (counting the initial state) when it is started in state i . The matrix B may be expressed simply in terms of N by

$$B = NR$$

Consider next the time t to absorption. Clearly t is a random variable whose distribution and moments depend upon the starting state. Let us denote by $g_i^{(k)} = E_i[t^k]$ the expected value of t^k when the process is started in state i . In particular, $g^{(1)}$ is a vector whose components give the mean number of steps to absorption for the various starting states. Then $g^{(k)}$ give the k th moment for these times. We first observe that

$$g^{(1)} = N1,$$

where 1 is a column vector with all entries 1 . To see this, we observe that n_{ij} is the mean number of times in state j starting in state i and hence

$$\sum_{j \text{ transient}} n_{ij}$$

is the mean number of times in a transient state starting in i . That is, $g_i^{(1)}$ is the sum of the components in the i th row of N . We shall give the method of computing $g^{(k)}$ in some detail since it is typical of an important technique in Markov chain theory. We first observe that

$$g_i^{(k)} = E_i[t^k] = \sum_{j \text{ transient}} p_{ij} E_j[(t+1)^k] + \sum_{j \text{ absorbing}} p_{ij}$$

This expresses the mean value of t^k in terms of the possible outcomes of the first step and the conditional expectations given these possible outcomes. Using the binomial theorem, we can write this in vector form as

$$g^{(k)} = Qg^{(k-1)} + \sum_{m=1}^{k-1} \binom{k}{m} Qg^{(m)} + 1$$

or

$$(I - Q)g^{(k)} = \sum_{m=0}^{k-1} \binom{k}{m} Qg^{(m)} + 1$$

Thus, since N is the inverse of $I - Q$,

$$g^{(k)} = N \left[\sum_{m=0}^{k-1} \binom{k}{m} Qg^{(m)} + 1 \right]$$

This is a recursion relation for $g^{(k)}$ in terms of the values for smaller k . This relation has the following solution. Let

$$A(r, n) = \sum_{m=0}^n (-1)^{m+n} \binom{n}{m} m^r,$$

with $A(r, 0) = \delta_{r,0}$ and $A(r, 1) = 1 - \delta_{r,0}$. Then

$$g^{(k)} = \sum_{m=0}^k (-1)^{k+m} A(k, m) N^m 1$$

We thus see that all the moments of t can be found simply by arithmetic operations on N . The matrix N itself is simply the inverse of $I - Q$. Thus it is a simple matter, using machine computations, to find these from a transition matrix for an absorbing chain. It is instructive to use the quantity t to compare the matrix method of analysis used previously with the method used when the eigenvalues are available.

The probability that $t = n$, as a function of the various starting states, is given by the components of the vector

$$Q^{n-1} R 1$$

That is, for t to equal n it must be the case that after $n - 1$ steps the process was in some transient state and that on the next step it moved into some absorbing state. If we can diagonalize Q , then the probabilities are

$$U \begin{bmatrix} \lambda_1^{n-1} & & \\ & \ddots & \\ & & \lambda_n^{n-1} \end{bmatrix} U^{-1} R 1$$

That is, given the diagonalization of Q , we can find the exact distribution and, of course, from this any of its moments. The point is that eigenvalues are often very hard to find, whereas N is easy to obtain as $(I - Q)^{-1}$.

A particularly simple quantity to analyze is the length of time that the chain remains in a particular state. The probability of remaining in state j exactly k times and then leaving is $p_{jj}^{k-1}(1 - p_{jj})$. This type of distribution is called a *geometric distribution*. It has the following property. Given that

the process has been in state j for the last m steps, the probability that it remains k more steps is independent of m . The mean length of time that the process remains in state i is $1/(1 - p_{ii})$, and the variance is $p_{ii}/(1 - p_{ii})^2$.

It is also a simple matter to compute the probability h_{ij} that starting in the transient state i the transient state j is ever entered. Then the sum $\sum_j h_{ij}$ gives the mean number of different transient states that are ever reached starting in the state i . Formulas for these quantities in terms of the fundamental matrix N may be found in Kemeny and Snell (1960, Chapter 3).

It is often convenient to modify the given chain and to study the resulting related chain. For example, suppose we are interested in the number of different changes of state that take place in our process. Then we simply form a new chain that has a transition matrix P' obtained from P by replacing p_{ii} for the transient states by 0 and renormalizing the rows so they sum to 1. This new chain represents the old chain watched only at times that it changes state. The time to absorption in this chain is the same as the total number of changes of state in the original chain. Thus its distribution and moments give information about the original chain.

As a second example of forming a new chain, assume that we are interested in observing the original chain only when it is in a subset E of the transient states. This gives us a new absorbing chain, and its transition matrix P^E is easily obtained from the fundamental matrix N . In fact, if N_E is simply the matrix N restricted to rows and columns corresponding to states in E , then $N_E^{-1} = I - Q^E$, where Q^E is the Q part of the new transition matrix P^E . Then

$$P^E = \begin{bmatrix} I & 0 \\ R_E & Q^E \end{bmatrix}$$

There is an important connection with classical potential theory and Markov chains. Of course, classical potential theory seems a long way from psychology, but this connection has led to new results for general Markov chains, so we mention it briefly here.

A potential is defined by a function g over the transient states that has the form $g = Nf$, where f is a nonnegative function on the transient states. (Here we are representing functions as column vectors so that f_i is the value of f at the state i .) The support of the potential is the set on which the charge f has a value different from 0. We can give a game interpretation for a potential in which a player receives an amount f_j every time the process is in state j . Then if the process is started in the transient state i , the value of the potential g at the state i , that is, g_i , is the expected total winnings.

If E is a subset of transient states, then the equilibrium potential is defined to be the one with support in E that has the constant value 1 on the set E . In our game interpretation, we are asked to adjust the payments so that the player is paid only when in E and such that his expected winnings are a constant no matter where in E he starts. Restricting everything to E , we are asked to solve the equations

$$1 = N_E f_E$$

for f . But this is $f = (I - Q^E)1$. The entries of f may be interpreted as follows. f_i is the probability that from i the chain steps out of E on the first step and never returns. We call these escape probabilities. The values of the corresponding potential $g = Nf$ is 1 on E . For states not in E the value of g represents the probability that starting at i E is ever reached.

In classical physics the capacity of a set is the total equilibrium charge. In this case, we define capacity as follows. Let $\alpha = \{\alpha_1, \alpha_2, \dots, \alpha_r\}$ be any row vector such that $\alpha \geq \alpha P$. For example, a row of N has this property. Then the capacity relative to α is defined as

$$c(E) = \sum_i \alpha_i e_i,$$

where $\{e_i\}$ is the charge of the equilibrium potential. It is then possible to prove, as in the classical case, that as the set increases, the capacity increases. This gives information about the escape probabilities. For example, as we increase the size of the set, it is harder to escape and hence the escape probabilities decrease. Since the capacity increases, they cannot decrease too fast.

3.3 Ergodic Behavior for Finite Chains

We assume now that we have a recurrent chain and that it has a single recurrent class. Such a chain is called an ergodic chain. Let us assume that it has period d , so that there are d cyclic subclasses C_1, C_2, \dots, C_d . For an absorbing chain, we observed that the successive powers P^n of the transition matrix converged to a limiting matrix A . This fact is replaced here by the following theorem.

Theorem 4 $\frac{I + P + P^2 + \dots + P^{n-1}}{n} \rightarrow A$, where A is a matrix each row of which is the same vector $\alpha = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$. The vector α is the unique probability vector such that $\alpha P = \alpha$.

A proof of this theorem may be found in Kemeny and Snell (1960, Chapter 5)

The j th entry of $I + P + P^2 + \dots + P^{n-1}$ is the mean number of times the chain is in state j in the first n steps. Dividing by n , we obtain the mean value for the fraction of time spent in state j . This theorem states that the mean value of the fraction of time spent in state j approaches a limiting value a_j , which is independent of the state in which the chain started. We shall see later that a stronger result holds, namely, that the fraction of the time in state j approaches a_j with probability 1.

The limiting matrix A is easily found since α may be determined simply by writing down the linear equations necessary for α to be a probability vector and $\alpha = \alpha P$. This vector also has the interpretation that if it is the initial probability vector, then the probability of finding the chain in the states at any later time is also α . In fact, with this choice and only with this choice, the resulting process is a stationary process.

In the case $d = 1$, the basic theorem can be strengthened to state that $P^n \rightarrow A$. That is, the probabilities $p_{ij}^{(n)}$ of finding the chain in state j after n steps approaches a limiting value a_j , independent of the starting state i .

In a recurrent class the process simply moves around through the states returning to each state infinitely often with probability 1. Hence we are interested in quantities which have to do with the behavior in going from one state to another as well as in quantities having to do with the average time spent in the states.

The fundamental matrix for transient chains is $N = (I - Q)^{-1} = I + Q + Q^2 + \dots$. In the recurrent case Kemeny and Snell (1960, Chapter 5) showed that many of the descriptive quantities can be obtained in terms of the matrix $Z = (I - P + A)^{-1}$. In the regular case this matrix has the power series expression

$$Z = I + (P - A) + (P^2 - A) + \dots$$

It does not have as simple an interpretation as the matrix N .

The mean first passage time m_{ij} is defined to be the mean time starting in state i to reach state j for the first time. If $i = j$, m_{ii} is the mean time to return to i starting in i . The matrix $M = \{m_{ij}\}$ is called the mean first passage matrix. The entries m_{ii} are particularly simple, namely, $1/a_i$. This is intuitively quite plausible since we expect to be in the state i a fraction a_i of the time, so it should take, on the average, $1/a_i$ steps to return. The remaining entries are obtained from Z by $m_{ij} = 1/a_j(z_{ji} - z_{ij})$ for $i \neq j$.

The variance of the time to go from i to j can also be obtained from the fundamental matrix A . In particular, if the process is started in state s_i ,

the variance of the time required to return to state s_i is given by $(2z_{ii} - a_i - 1)/a_i$. Let us start the process in state s_i and let T_1 be the time required to return for the first time, T_2 the additional time required to return for the second time, and in general T_n the time between the $(n-1)$ st and n th return. Then T_1, T_2, \dots are identically distributed random variables, and we have just shown how to obtain their mean and variance. Hence the central limit theorem can be applied to the sum $S_n = T_1 + T_2 + \dots + T_n$. If we denote by N_n the number of visits to state s_i in the first n steps, then interpreting T_1, T_2, \dots as a renewal process we can obtain results about N_n as indicated in Sec. 13. We shall mention more general limit theorems in the next section.

Another quantity of interest is the covariance of the times in two different states. Let us start the process in state i and let Y^j be the number of visits to state j before returning to state i . We let

$$b_i^{jk} = \text{Cov}[Y^j, Y^k]$$

Let S_n^j be the number of visits to state j in the first n steps. Then it can be shown that

$$\lim_{n \rightarrow \infty} \left\{ \left(\frac{1}{n} \right) \text{Cov}[S_n^j, S_n^k] \right\} = c_{ij}$$

exists independent of the way in which the chain is started. The quantities c_{ij} can be computed from the fundamental matrix Z by

$$c_{jk} = a_j z_{ij} + a_j z_{ji} - a_i \delta_{ij} - a_i a_j$$

The quantities c_{jk} and b_i^{jk} are related, and it is possible to go from one set of quantities to the others. The formulas which enable one to do this are

$$b_i^{jk} = \frac{1}{a_i} \left(c_{ij} - \frac{a_i}{a_k} c_{kj} - \frac{a_j}{a_k} c_{ik} + \frac{a_i a_j}{a_k^2} c_{kk} \right)$$

and

$$c_{ij} = a_i \left[b_k^{ij} - a_j \sum_l b_k^{il} - a_l \sum_l b_k^{lj} + a_i a_j \sum_m b_l^{im} \right]$$

This connection was observed by Kolmogorov (1962).

3.4 Limit Theorems for Ergodic Chains

Let X_1, X_2, \dots be the process for which X_m is the state occupied in an ergodic chain. Let f be any function defined on the state space. We

consider the new process $f(X_1), f(X_2), \dots$. That is, this new process records $f(j)$ whenever the original process was in state j . We are interested in studying the sums $S_n = f(X_1) + f(X_2) + \dots + f(X_n)$. If f is the function that has the value 1 on the state j and 0 on all other states, then $f(X_n)$ is 1 if the chain is in state j on the n th step and 0 otherwise. Thus S_n represents the total number of times that the chain is in state j in the first n steps. It is helpful in thinking of the general case to interpret our process as a game in which the player receives $f(j)$ every time the chain is in state j . The expected winning then on each play when the process is in equilibrium is

$$m = \sum_j a_j f(j)$$

The first theorem, the law of large numbers, states that the player's average winning in the long run is equal to this mean value.

Theorem 5 (Law of Large Numbers for Markov Chains) *Let f be any function on the state space of an ergodic Markov chain $\{X_n\}$. Let $S_n = f(X_1) + f(X_2) + \dots + f(X_n)$. Then the averages S_n/n converge with probability 1 to the constant value $m = \sum a_j f(j)$ independent of the starting distribution.*

For the law of large numbers, the constant m plays the role of the common mean when the random variables are independent. For the central limit theorem we need a quantity b^2 that will play the role of the common variance per experiment in independent processes. This is provided by the following result.

Theorem 6 *Under the assumption of the previous theorem, $(1/n) \text{Var}[S_n]$ converges as n tends to infinity to a constant $b^2 = \sum_{j=1}^r f_j c_{jj} f_j$.*

Theorem 7 (Central Limit Theorem) *If $b^2 > 0$, the distribution of*

$$\frac{S_n - nm}{b\sqrt{n}}$$

converges to a normal distribution.

Theorem 8 (Law of the Iterated Logarithm) *Suppose $b^2 > 0$ if $\lambda > 1$, then with probability 1*

$$S_n < nm + \lambda b \sqrt{2n \log \log n}$$

only a finite number of times, and if $\lambda \leq 1$, then with probability 1

$$S_n > nm + \lambda b \sqrt{2n \log \log n}$$

infinitely often.

The proofs of these theorems involve the following device. Interpreting $f(X_n)$ again as the winnings on the n th play let us define Z_n to be the total winning between the n th and the $(n+1)$ st return to the state i . Then Z_1, Z_2, \dots is a sequence of identically distributed and independent random variables. Let t_n be the length of time between the n th and the $(n+1)$ st return to state i . The mean value of t_n is $1/a_i$, and the mean value of Z_n is $\sum_j f_j a_j / a_i$. Thus the random variables are independent and have mean 0. By applying the standard theorems for independent random variables to these random variables, the preceding theorems are obtained. The basic quantity B^2 that enters into these theorems is also the variance of the random variables $Z_n - t_n m$. Details of these proofs may be found in Chung (1960).

There is also a slightly different type of central limit theorem available for ergodic chains. Suppose that we want to keep track of the number of times the process is in each of the states up to a certain time n . Recall that S_n^j is the number of times in state j in the first n steps. Then we can form the vector $S_n = (S_n^1, S_n^2, \dots, S_n^s)$ if there are s states. Then it is possible to prove a central limit theorem for the random vector

$$\frac{S_n - n\alpha}{n}$$

The limiting distribution is an $s-1$ dimensional normal distribution determined by the matrix C of covariances c_{ij} . For details of this result see Kolmogorov (1962).

As in transient behavior, potential theory has also suggested a number of new probabilistic results for Markov chains. We mention some typical results.

Let E be any set of states, let us start the chain in state i , and let u_i be the mean number of times that the process is in state i before returning to i provided that it has entered the set E at least once. Let $u = (u_1, u_2, \dots, u_s)$. Then u is proportional to the limiting vector α . In fact, $u = [1/c(E)]\alpha$, where $c(E)$ is obtained as follows. We make the chain into a transient chain by stopping it when it returns to i . Then $c(E)$ is the capacity of E relative to α for this transient chain.

A very peculiar result suggested by potential theory is the following. Let M be the first passage matrix, then the vector $M\alpha^T$ has constant elements. (Here α^T means the row vector α made into a column vector.) This result is so simple that there should be some probabilistic interpretation of it, but none has yet been found.

If a Markov chain is observed only at times when it is in a subset of states, a new Markov chain is obtained. By means of recurrent potential

theory it is possible to obtain simple matrix expressions for the transition matrix of this new process

Details of this application of potential theory may be found in Kemeny and Snell (1961, pp 114-115)

3.5 Applications and Examples of Finite Chains

As an example of the application of finite Markov chains, we discuss briefly the pattern stimulus model defined in Sec 1.4. In the simple contingent case the basic outcome process is a Markov chain with stationary transition matrices. When there are two patterns, a state is a vector of the form $(u \ v \ s \ a \ e)$, where u is the connection of the first pattern, v the connection of the second pattern, s the stimulus pattern sampled, a the response made by the subject, and e the response reinforced by the experimenter.

The type of chain that results depends on the reward schedule of the experimenter. Recall that this is determined by a matrix

$$\pi = \begin{matrix} & \begin{matrix} 1 & 2 \end{matrix} \\ \begin{matrix} 1 \\ 2 \end{matrix} & \begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{bmatrix} \end{matrix},$$

where π_{ij} is the probability that response j is rewarded when response i is made. Assume first that the reward matrix is of the form

$$\pi = \begin{matrix} & \begin{matrix} 1 & 2 \end{matrix} \\ \begin{matrix} 1 \\ 2 \end{matrix} & \begin{bmatrix} 1 & 0 \\ \pi_{21} & \pi_{22} \end{bmatrix} \end{matrix}$$

That is, the experimenter is sure to reward response 1 when it is made. When the patterns become connected to 1, their connections can never change and the subject will continue to make response 1. Then the states $(1 \ 1 \ 1 \ 1)$ and $(1 \ 1 \ 2 \ 1)$ are recurrent states. All other states are transient. The process eventually reaches one of these two states, and then moves between these two states. To study the process in detail, we would treat these two states as a single absorbing state, and so have an absorbing chain with a single absorbing state. The case

$$\pi = \begin{bmatrix} \pi_{11} & \pi_{12} \\ 0 & 1 \end{bmatrix}$$

is similar

If the matrix is of the form

$$\pi = \begin{matrix} & \begin{matrix} 1 & 2 \end{matrix} \\ \begin{matrix} 1 \\ 2 \end{matrix} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{matrix},$$

the experimenter always rewards the response made by the subject. In this case we have four recurrent states (1 1 1 1), (1 1 2 1), (2 2 1 2), and (2 2 2 2). The first two constitute a single recurrent class and the last two, a second recurrent class. All other states are transient.

In all other cases the set of all states forms a single recurrent class. The period is one except when $c = 1$ and

$$\pi = \begin{matrix} & \begin{matrix} 1 & 2 \end{matrix} \\ \begin{matrix} 1 \\ 2 \end{matrix} & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{matrix},$$

in which case the period is 2. For example, from state (1 1 1 2) the process is sure to move to either (2 1 1 2) or (2 1 2 1). From (2 1 2 1) it is possible to move to (1 1 1 2).

To study the cases where transient states exist, we make the recurrent states absorbing and use transient theory. The length of time to absorption can be interpreted as the time to complete learning. From the time he is absorbed the subject makes the same response and is always reinforced. If the fundamental matrix is known, the moments of this time can be obtained. If eigenvalue expansions are available, the actual distribution of this time may be obtained.

Other interesting quantities are the number of times that response 1 is made and the number of times that response 1 is reinforced. These quantities may be represented as follows. Define a function f by $f(u, v, s, a, e) = 1$ if the state is transient and $a = 1$, and $f(u, v, s, a, e) = 0$ otherwise. That is, f is the characteristic function of the set of transient states in which response 1 is made. Similarly, let g be the characteristic function of the set of transient states in which response 1 is reinforced. Then we are interested in the random variables $S = f(X_1) + f(X_2) + \dots$ and $T = g(X_1) + g(X_2) + \dots$, where X_1, X_2, \dots are the outcome functions for the chain. It is possible to obtain expressions in terms of the fundamental matrix for the means and variances of these quantities and the covariances using the methods for functions of a recurrent chain in Kemeny and Snell (1960, pp. 84-88).

In the recurrent cases we are, of course, interested in the limiting probabilities. Let f be the characteristic function of the states in which the response is 1 and g the characteristic function of the states in which

the response 1 is reinforced. Then $f(X_1) + f(X_2) + \dots + f(X_n)$ gives the number of responses of type 1 and $g(X_1) + g(X_2) + \dots + g(X_n)$ the number of reinforcements of response 1 in the first n trials. We know from the law of large numbers that

$$\frac{f(X_1) + f(X_2) + \dots + f(X_n)}{n}$$

converges with probability 1 to a limiting value a . This is the average number of responses of type one in the long run. Similarly,

$$\frac{g(X_1) + g(X_2) + \dots + g(X_n)}{n}$$

converges with probability 1 to a limiting value b . It is a peculiarity of these learning models that $a = b$. That is, the subject and the experimenter in the long run match each other in the frequencies with which a response is made and reinforced. We know that these quantities have limiting variances and covariances.

We know also that we can apply a central limit theorem to the number of type one responses and type one reinforcements. Similarly, we have the law of the iterated logarithm available.

Estes has proceeded somewhat differently in studying this model. He has focused on the connection process which, as we observed, is a Markov chain. It is somewhat simpler than the chain that we have been considering, and he was able to compute the basic quantities for this chain. The transition matrix for the connection process is obtained as follows. Recall that a state is now the number of stimulus patterns connected to response 1. If there are N stimulus patterns, there are $N + 1$ states $0, 1, 2, \dots, N$. Assume that the chain is in state i . Then if $i = 0$, it either remains in this state or moves to 1. If it is a state with $0 < i < N$, then it can remain in this state or move to $i + 1$ or $i - 1$. If it is in state N , it either remains there or moves to $N - 1$. This is because on a single trial only one pattern can be affected. Assume that $0 < i < N$. We shall illustrate the computation of $p_{i, i-1}$. To go from state i to state $i - 1$ the subject must sample one of the patterns connected to state response 1, he must make response 1, then the experimenter must reinforce response 2, and, finally, the pattern sampled must change its connection. The probability that a pattern connected to response 1 is sampled is i/N . The subject then makes response 1. The experimenter reinforces response 2 with probability π_{12} . The connection then changes with probability c . Hence

$$p_{i, i-1} = \frac{i}{Nc\pi_{12}}.$$

In a similar manner we find the other transition probabilities. They are

$$p_{00} = 1 - c\pi_{21},$$

$$p_{i, i-1} = \frac{i}{Nc\pi_{12}}, \quad i = 1, 2, \dots, N,$$

$$p_{i, i} = 1 - \left(\frac{1-i}{N} \right) c\pi_{21} - \frac{i}{Nc\pi_{12}}, \quad i = 1, 2, \dots, N-1,$$

$$p_{i, i+1} = \left(\frac{1-i}{N} \right) c\pi_{21}, \quad i = 0, 1, 2, \dots, N-1,$$

$$p_{N, N} = 1 - c\pi_{12}$$

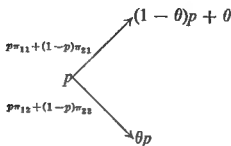
Note that if we think of this as a chain moving on the nonnegative integers, it can go only to neighboring states. Such a chain is called a random walk by Karlin and McGregor (1959), and they have developed elaborate methods for studying them. We shall discuss these processes later.

However, for the chain considered here, Estes has given a detailed study and computed a large number of interesting descriptive quantities. See Bush and Estes (1959).

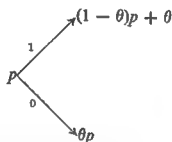
Another stimulus model which has been studied a great deal by Markov chain methods is the *stimulus component* model. In this model it is assumed that there are a finite number of stimulus elements. Each element is connected to one response, but the subject samples a subset of the elements. He makes response 1 with a probability equal to the fraction of elements connected to response 1 in the sample chosen. If no stimulus elements are sampled, some convention is necessary. All elements in the set sampled have their connections changed, if necessary, to agree with the choice of the experimenter. It is assumed that each stimulus element is sampled with probability θ and that the samplings are independent. The connection process, that is, the number connected to state 1, is again a Markov chain in the noncontingent or simple contingent case. In this model the connection process is an absorbing chain with state N absorbing if 1 is always rewarded. It is absorbing with state 0 absorbing if 2 is always rewarded. It is absorbing with both states absorbing if the response made by the subject is always rewarded. Otherwise it is always a single recurrent class. Discussions of this chain can be found in Kemeny and Snell (1957, 1960) and in Bush and Estes (1959).

In the linear model of Bush and Mosteller there is also an underlying Markov process which we can think of as generating the responses and in turn the reinforcements. However, this is a Markov process with state space the unit interval $[0, 1]$. No general theory for Markov processes

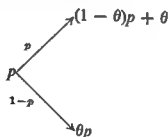
with continuous state space exists which seems satisfactory for these applications. The result is that these processes have had to be studied individually. The most complete treatment on the asymptotic behavior of these processes that has been given is by Karlin (1953). The behavior is quite similar to the connection process in the finite stimulus model. For example, consider the special case



If response 1 is always rewarded, then we have the case



The process then goes with probability 1 to 1 as a limiting value. Unlike the absorbing chain, it never actually reaches 1. If the response made by the subject is always rewarded then the transition probabilities are



In this case as the process gets near to 0 it is extremely likely to move to the right, that is, to get even closer to 0. Similarly, if it gets near 1 it is very likely to move even closer. As this suggests, the sample paths of the process can be shown to converge with probability 1 to either 0 or 1. Thus, the remaining problem is the probability that the process ends up in each of these possible positions.

When either response may be rewarded independent of what the subject

does, we are led to a process which acts like a recurrent Markov chain. For any interval $[a, b]$, if the process is started in this interval it returns to the interval infinitely often. The probabilities

$$Pr[X_n \leq \lambda]$$

converge to a limiting distribution as n tends to infinity. The distribution of the position X_n converges to a limiting distribution that is independent of the manner in which the chain is started. Not much is known about these distributions. Even in the contingent case, as we remarked in Sec. 2, they can be quite complicated.

Analysis of the response processes and the resulting experimenter's processes must also be done by special methods. A detailed account of the types of questions that can be answered for this model exists in the book by Bush and Mosteller (1955).

In the treatment given by Estes and Suppes it is again necessary to use special techniques to obtain the descriptive quantities. This has been done, for example, in the article by Estes and Suppes in Bush and Estes (1959). See also Chapters 8, 10, 11, 12, 14 of Bush and Estes (1959).

Another application of finite Markov chains is found in the model developed by B. Cohen (1963) for experiments relating to conformity. In these experiments a subject is run with a group of pretrained confederates of the experimenter. The subject is led to believe that he is participating in a group experiment on visual perception. On each of a sequence of trials, each member of the group is required to choose aloud that line one from among three comparison lines that has the same length as a standard line.

The subject's turn to choose on each trial comes only after he has heard the unanimous, although incorrect, responses of the confederates. Presumably, he is motivated then to answer according to his perceptions, on the one hand, and to conform to the unanimous choice of the group, on the other hand.

The data consist of a sequence of responses, each response being one of two alternatives, a or b , where a indicates a correct, that is, nonconforming, result and b an incorrect, that is, conforming, response.

Cohen constructed the following four-state Markov chain model to describe the data. He assumed that on any one trial the subject is in any one of four mental states

State 1	Nonconforming
State 2	Temporary nonconforming
State 3	Temporary conforming
State 4	Conforming

The model assumes that the subject starts in state 2. The transition matrix is assumed to be of the form

$$P = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ \alpha & 1 - \alpha - \beta & \beta & 0 \\ 0 & \gamma & 1 - \gamma - \epsilon & \epsilon \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{matrix}.$$

This is a four-state absorbing Markov chain. The entries α , β , γ , and ϵ are parameters to be estimated from the observed data. It is assumed that when the subject is in state 1 or 2, he gives a nonconforming response, and when he is in state 3 or 4, he gives a conforming response b . When the experiment is performed, the experimenter is not able to observe the states of this chain, but only when the subject conforms or does not conform.

If we denote by X_1, X_2, \dots the outcomes of the Markov chain, then the response can be described as the process $f(X_1), f(X_2), \dots$, where f is a function on the state space that has the value a on states 1 and 2 and the value b on states 3 and 4. This process is not a Markov process. For example, let us assume that the underlying process is started in states 2 and 3 with equal probability. Let us compare the two probabilities.

$$(i) \quad \Pr[f(X_2) = b \mid f(X_2) = b \wedge f(X_1) = a]$$

and

$$(ii) \quad \Pr[f(X_2) = b \mid f(X_2) = b \wedge f(X_1) = b].$$

The information that $f(X_1) = a$ tells us that $X_1 = 2$, and then if $f(X_2) = b$, we know that $X_2 = 3$. Hence the first probability is $1 - \gamma$. In the second case it is possible that $X_2 = 3$ or $X_2 = 4$. Since in the latter alternative it is certain that $f(X_2) = b$ we see that (ii) will be a larger number.

Since the observed process is not a Markov chain, we cannot apply Markov chain methods directly to it but must instead apply them to the underlying chain and from this obtain information about the observed process.

3.6 Denumerable Chains with Discrete Time

We consider now a chain with a denumerable set of states that we label $0, 1, 2, \dots$. The basic decomposition into transient and recurrent states is the same as in the finite case, but, of course, we may have a denumerable

number of different recurrent classes. However, each class still has a finite number d of cyclic subclasses. In the finite case it is necessary to have at least one ergodic class because the process leaves the transient states eventually when there are only a finite number of possibilities and it must have somewhere to go. However, with an infinite number of states this is not so. For example, the trivial chain in which the particle moves with probability 1 to the next higher integer on each trial has only transient states and no recurrent states.

In studying transient behavior, it is again convenient to make the recurrent states absorbing thereby obtaining the canonical form

$$P = \begin{bmatrix} I & 0 \\ R & Q \end{bmatrix}$$

There is no general method for diagonalizing infinite matrices, however, many of the methods developed using the fundamental matrix $N = I + Q + Q^2 + \dots$ still apply. In particular, this infinite series converges and the entries n_{ij} again give the mean number of times the process enters state j given it started in state i .

It is still true that

$$N(I - Q) = (I - Q)N = I,$$

so that N is an inverse of $I - Q$. However, in general, there are other matrices which are also inverses of $I - Q$, and so N is not uniquely determined by the equations of an inverse. The reason for this is that the equation

$$(I - Q)g = 0$$

may have a nontrivial solution, this could not happen in the finite case. If it does, adding such a solution to any row of N gives another inverse for $I - Q$. The matrix N is characterized as the minimal nonnegative inverse of $I - Q$ in the sense that any other nonnegative matrix such that $N'(I - Q) = (I - Q)N' = I$ has the property that $N' \geq N$, that is, $N'_{ij} \geq N_{ij}$, for all i and j .

If there are absorbing states, the matrix NR still gives the probabilities of absorption in the various absorbing states for a given starting state. It is no longer true that these probabilities necessarily add up to 1. That is, there may be a positive probability that the process is never absorbed. An example of this is the chain in which 0 is absorbing and on the positive integers a step is made to the right with probability $\frac{1}{2}$ and to the left with probability $\frac{1}{2}$.

If the probability of absorption is one, it is still possible to have an infinite mean time to absorption. However, under the assumption that the relevant moments of time to absorption are finite, the formulas given

in the finite case apply. Recall that, in the derivation of these quantities, we showed that the moments of the absorption time represented by the vector g satisfied an equation of the form

$$(I - Q)g = f,$$

where f is a known nonnegative function. We then concluded that $g = Nf$. In the infinite case, as we have pointed out previously, the equation $(I - Q)g = f$ may have solutions other than $g = Nf$. Some additional argument is needed to show that $g = Nf$ is the solution we are looking for. Recall that a function of the form $g = Nf$ was called a potential, in analogy with classical potential theory. Thus we see that we must show that the moments g are a potential. In problems such as this the study of potential theory for Markov chains has greatly increased our knowledge of solutions of very simple probability problems for denumerable chains.

We have observed that a basic difference between the transient theory for the finite and for the infinite case is that in the former the chain has a final position—namely, one of the absorbing states. When the Markov chain is infinite, this need not be so. We gave a single example, but a less trivial one is a three-dimensional random walk. This process moves on the lattice points in three dimensions, that is, on points of the form (n_1, n_2, n_3) in Euclidean three space with integer coordinates. It moves with probability $\frac{1}{6}$ to each of the six neighboring points

$$(n_1 + 1, n_2, n_3), (n_1 - 1, n_2, n_3), (n_1, n_2 + 1, n_3), \\ (n_1, n_2 - 1, n_3), (n_1, n_2, n_3 + 1), (n_1, n_2, n_3 - 1)$$

This is a transient process. As a result the process eventually leaves any finite set. It goes off to infinity. In order to give the transient chain a final position, a boundary called the Martin boundary has been added. This consists of adding ideal states to the state space and then proving that in a well defined sense the process ends up at an absorbing state or at one of these new ideal points.

This boundary has also been useful in finding solutions of the equation $(I - Q)g = f$ when f is known. As we remarked, such equations arise in a natural way when we study descriptive quantities of a transient chain. A discussion of the Martin boundary is given by Doob (1959).

3.7 Recurrent Denumerable Chains

We consider now a single recurrent class with period d . Starting at state i , the process eventually returns to i with probability 1, however,

the mean time required for this return may be finite or it may be infinite. If it is finite for one state, the same is true for any other state and, in addition, the mean first passage times are also finite. This leads to a basic classification of recurrent chains.

Definition 20 A recurrent class is called recurrent positive if the mean time to return to a state is finite for any state in the class. It is called recurrent null if this mean time is infinite.

An example of a recurrent null chain is the simple random walk that moves on the integers, making a step to each of the neighboring points with probability $\frac{1}{2}$. The same is true for the corresponding random walk in two dimensions. Recall that in three dimensions the random walk process becomes transient. Of course, any finite chain is recurrent positive. An example of an infinite chain that is recurrent positive is the one that moves on the nonnegative integers making a step to the left with probability $\frac{2}{3}$ and to the right with probability $\frac{1}{3}$ at all states except 0. When at 0, it returns to state 1.

The basic ergodic theorem for recurrent chains is the following.

Theorem 9 Let P be the transition matrix of a recurrent chain with a single recurrent class. Then

$$\frac{I + P + P^2 + \dots + P^n}{n+1}, \quad n = 1, 2, \dots,$$

converges to a limiting matrix A . In the ergodic positive case, each row of A is the unique probability vector α such that $\alpha P = \alpha$. In the ergodic null case, all entries of A are 0. If the period is one, $P^n \rightarrow A$.

A proof of this theorem may be found in Chung (1960).

Thus, recurrent positive chains behave just as finite ones. Recurrent null chains, however, are quite different. For example, the probability of finding the process in state i after a large number of steps tends to 0. This is like the transient chain. However, whereas the transient chain wanders off to infinity, the recurrent chain keeps coming back. It comes back so rarely, however, that at any given large time it is, in fact, most likely to be far away from any prescribed finite set.

In the null chain there is still a nonnegative solution to the equation $\alpha = \alpha P$. It is unique up to a multiplicative constant, but the sum of the components is infinite so it cannot be made into a probability vector. Its components still have interpretations. For example, let $N_{ij}^{(n)}$ be the mean number of entries into state j in the first n steps starting at state i .

Theorem 10 For any recurrent chain with a single recurrent class

$$\frac{N_{ij}^{(n)}}{N_{ii}^{(n)}} \rightarrow \frac{a_j}{a_i}$$

as n tends to infinity, where $\alpha = (\alpha_1, \alpha_2, \dots)$ is any nonnegative solution of $\alpha P = \alpha$

Thus, although the mean time spent in a state tends to 0, in a null chain, the ratio of the mean time for two different states approaches a limit different from zero

It is also true that for any recurrent chain the mean number of entries into state j between occurrences of state i is finite and has the form a_j/a_i , as in the finite case. A proof of these facts as well as a proof of Theorem 10 may be found in Chung (1960)

3.8 Applications and Examples of Denumerable Chains

Denumerable chains seem not yet to have found much application in psychology. Nonetheless, it seems worthwhile to mention some rather general classes of processes that have found quite general application and about which a great deal is known. The first is a class of random walk processes studied by Karlin and McGregor (1959)

A random walk is a denumerable Markov chain with state space the integers 0, 1, 2, ... It has the property that on any one step it can move at most one unit to the right or left. To specify the process completely it is necessary to give three probabilities for each state n : q_n is the probability of moving one step to the left from state n , r_n is the probability of remaining in state n , and p_n is the probability of moving one step to the right. Thus $q_n + r_n + p_n = 1$. We assume that $q_0 = 0$. We have already remarked that the connection process for the stimulus pattern model is of this type, although with only a finite number of states.

Karlin and McGregor have introduced a powerful analytic method for studying these processes. They define a system of polynomials $Q_i(x)$, $i = 0, 1, 2, \dots$ by setting $Q_0(x) = 1$ and solving recursively the equations

$$xQ_n(x) = p_n Q_{n+1}(x) + r_n Q_n(x) + q_n Q_{n-1}(x)$$

They show that associated with these polynomials is a unique measure ψ defined on the interval $[-1, 1]$ such that

$$\int_{-1}^1 Q_n(x) Q_m(x) d\psi(x) = 0 \quad n \neq m$$

$$\pi_n \int_{-1}^1 Q_n^2(x) d\psi(x) = 1$$

where $\pi_n = (p_0 p_1 \dots p_{n-1}) / (q_1 q_2 \dots q_n)$. In terms of this measure, called the *spectral measure*, they show that

$$P_{ii}^{(n)} = \pi_i \int_{-1}^1 x^n Q_i(x) Q_i(x) d\psi(x)$$

They are able, using this representation theory, to obtain criteria for the states to be transient, recurrent ergodic, and recurrent null. In addition, they give methods for computing descriptive quantities such as the first passage distribution and any of the other quantities that we have discussed. To find the explicit distributions it is necessary to know a great deal about the polynomials that enter into the process. Fortunately, many of the random walks that occur in practice are associated with classical orthogonal polynomials about which a great deal is known. For example, a useful class of processes is obtained by assuming that

$$p_n = \frac{1}{2} \left(1 + \frac{a}{n+a} \right), \quad n > 0,$$

$$q_n = \frac{1}{2} \left(1 - \frac{a}{n+a} \right),$$

$$p_0 = 1,$$

where a is a constant $> -\frac{1}{2}$. The polynomials associated with these random walks are the classical ultraspherical polynomials, the spectral measure is

$$d\psi = (1 - x^2)^{a-1/2} dx$$

This class of chains provides examples of all three basic types of denumerable chains. For $a < \frac{1}{2}$ the chain is recurrent positive, for $a = \frac{1}{2}$ it is recurrent null, and for $a > \frac{1}{2}$ it is transient.

The second class of denumerable chains that we mention are those called branching processes. As in Markov chains, the first application of branching processes was to a problem in the social sciences, namely, the survival of family names. We assume that in the n th generation, that is, at time n , there are a number of particles, or persons, or names, and that each of these reproduces j of its kind for the next generation and dies. We adopt the neutral terminology "particles." We assume that a single particle produces j offspring with probability p_j , that this is the same for all particles, and that they act independently. We then take as a state for our Markov chain the total number of particles at any given time. If at time n the process is in state i , the next state is determined as follows. It is the sum of i independent experiments, each of which has the common distribution $p = \{p_0, p_1, \dots\}$. The state 0 is an absorbing state. We interpret the process as dying out when 0 is reached. We normally start in state 1, that is, with one particle.

The probability of absorption is, of course, an interesting property of the process. Let the mean number of particles produced by a single particle be m . Then if $m \leq 1$, it turns out that the population dies out with probability 1. If $m > 1$, there is a positive probability that it will continue indefinitely, and if so it tends to ∞ .

In the case $m \leq 1$, we would like to know the time to reach 0, that is, to be absorbed. The mean time is ∞ if $m = 1$ and finite if $m < 1$.

It is also of interest to determine the distribution of the total number of particles ever born. When $m > 1$, we can look for limit theorems for the total size of the population at time n , and there is a very simple result. It has been proved that the distribution of X_n/m^n converges to a limiting distribution F which has a density function f . Furthermore, the process X_n/m^n converges almost everywhere to a limiting random variable z_∞ .

It is illuminating to compare this limit theorem with the law of large numbers, which also leads to convergence with probability 1 but the limit is a constant value. The branching process is one of the few examples of a limit theorem in which the paths themselves converge with probability 1, but to different values in general.

The more general model in which a number of different types of particles that reproduce differently has also been studied in great detail. All these results are discussed in a review paper by Harris (1951).

3.9 Continuous-Time Markov Chains with a Finite Number of States

We turn now to a process that we observe continuously as it moves through a finite number of states. The intuitive picture is this: the process starts in some state, remains there a random length of time, and then

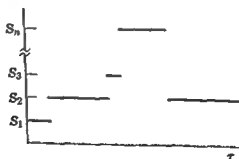


Fig. 4. Typical history of continuous-time Markov chain with a finite number of states.

jumps to another state where again it remains for a random length of time, etc. A typical history for such a process is a graph such as that shown in Fig. 4.

For every t with $0 \leq t < \infty$, we now have a matrix of probabilities $P(t) = \{p_{ij}(t)\}$, where

$$p_{ij}(t) = \Pr[X_t = j \mid X_0 = i]$$

These matrices satisfy the equation

$$P(s+t) = P(s)P(t)$$

It is customary to assume also that

$$\lim_{t \downarrow 0} P(t) = I$$

That is, in a very short time there is small probability of the process moving

The classification procedure is quite similar to that of discrete-time chains. We say that state i communicates with state j if for some s , $p_{ij}(s) > 0$ and if for some t , $p_{ji}(t) > 0$. We define equivalence classes as in discrete-time process. A state i is recurrent if when the process is started in i it leaves i and returns with probability 1. A class is recurrent if its states are recurrent. Equivalently a state is recurrent if and only if $\int_0^\infty \beta_{ii}(t) dt$ is infinite. That is, if and only if the mean time spent in the state is infinite. A state is transient if it is not recurrent.

An essential simplification occurs in continuous time chains, namely, the cyclic behavior that can occur in discrete-time chains cannot occur in continuous time chains. In fact, in a recurrent class, $p_{ii}(t) > 0$ for all sufficiently large t .

Associated with any continuous time process are discrete-time chains called *skeleton chains*. These are defined as the discrete time chains with transition matrices $P(h)$ for $h > 0$. Such a chain corresponds to observing the original chain only at times that are multiples of h . The recurrent classes of the chain $P(h)$ are, for any h , the same as those of the continuous chain. Also the recurrent classes of these chains always have period one. It is often useful to approximate a continuous chain by skeleton chains by taking h smaller and smaller.

In the discrete-time process the entire process was determined by a single transition matrix P . In the continuous time case this matrix is replaced by the matrix $Q = \{q_{ij}\}$, whose entries are the derivatives of the $p_{ij}(t)$ at 0. It follows from our assumptions that as a function of t $p_{ij}(t)$ has a derivative. Then,

$$q_{ij} = \lim_{t \downarrow 0} \frac{p_{ij}(t) - \delta_{ij}}{t}$$

The matrix Q has the property that the diagonal elements are nonpositive and the off diagonal elements are nonnegative. The row sums are 0. The transition matrices then satisfy two basic systems of differential equations. The first, called the backward equation, is

$$P'(t) = QP(t).$$

and the second, called the forward equation, is

$$P'(t) = P(t)Q$$

The analogy with linear equations suggests that the solution should be

$$P(t) = e^{Qt}$$

This is indeed so, but we need to say what e^A means when A is a matrix. We define this by the power series

$$e^A = I + A + \frac{A^2}{2} + \frac{A^3}{3} + \dots$$

Theorem 11 *Let $P(t)$ be a continuous-time Markov chain with a finite number of states. Then*

$$\lim_{t \rightarrow \infty} P(t) = A$$

exists. If $P(t)$ has a single recurrent class, each row of A is the probability vector such that $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_r)$. This vector is the unique probability vector such that $\alpha Q = 0$.

The stronger theorem is possible here because we do not have the cyclic case, hence, we do not ever have to take averages.

The quantities q_{ij} have simple probabilistic interpretations. Let $q_i = -q_{ii}$. Then if the process starts in state i , the length of time that it remains in this state has an exponential distribution with mean $1/q_i$. That is, if T is the length of time that the process remains in state i , then

$$\Pr\{T \geq t\} = e^{-q_i t}$$

Next define a new transition matrix $R = \{r_{ij}\}$ by

$$r_{ij} = \begin{cases} \frac{q_{ij}}{q_i}, & i \neq j \\ 0, & i = j \end{cases}$$

Then r_{ij} represents the probability of the process jumping to state j given that it is in state i . That is, if we were to record simply the states to which the chain moves, we would obtain a history that could be described as the outcome of a discrete time Markov chain with transition matrix R . Quantities that have only to do with the number of times different states are entered can be computed from discrete time theory applied to R . In fact, any result obtained by applying discrete time theory to R has an interpretation in the continuous time process.

A discussion of the computation of basic descriptive quantities in this situation is contained in Kemeny and Snell (1960).

3 10 Continuous-Time Markov Chains with a Denumerable Number of States

We start, as in the finite case, with the transition matrices $P(t)$ satisfying

$$P(s+t) = P(s)P(t)$$

and

$$\lim_{t \rightarrow 0} P(t) = I$$

In this case considerably more complex things can happen. Although the limit

$$q_{ii} = \lim_{t \rightarrow 0} \frac{p_{ii}(t)}{t} \geq 0$$

still exists, it is possible to have

$$q_{ii} = \lim_{t \rightarrow 0} \frac{p_{ii}(t) - 1}{t} = -\infty$$

In the finite case $-1/q_{ii}$ was the mean length of time the chain remains in i when it is started in i . The limit q_{ii} being $-\infty$ suggests that the mean time the process remains in i is 0, that is, that the process instantaneously moves out of this state into another state. Although this can happen, it is reasonable for most applications to assume that this does not happen. In other words, we assume that $q_{ii} = -q_{ii} < \infty$ for all states i in the chain. Such states are called *stable*.

In finite chains the sample paths were simple step functions. We now have to allow for more complicated paths. For example, suppose that a chain can only move to the next higher integer. If we make the mean length of time that it stays in a state smaller and smaller as it moves higher up the ladder, it is conceivable that we could go through all states in a finite time. Since $1/q_i$ is the mean length of time that the chain stays in state i , if we can find a process such that $\sum_i 1/q_i < \infty$, then the mean time to go through all the states would be finite. This would mean that a sample path would start out as the one shown in Fig. 5. Here we observe that the process has moved through all of the states by time t_0 . We say that the path reaches infinity at time t_0 .

If our process goes to infinity, it has to reappear at later times since $\sum_j p_{ij}(t) = 1$. This can happen in the two different ways as indicated in Figs. 6 and 7. In Fig. 6 the process simply climbs down from infinity, and in Fig. 7 it jumps from infinity to a specific state. Just as a process

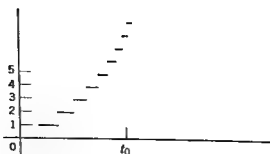


Fig 5 Sample path when chain moves only to next higher integer, the mean length of time in state i is $1/q_i$ and $\sum 1/q_i < \infty$

can jump from infinity, it is also possible to have it jump to infinity, as indicated in Fig 8

The definition of recurrent and transient states is the same as for finite, continuous-time chains. As in discrete-time chains, we now have a further classification of recurrent states into recurrent null and recurrent positive. A state is recurrent positive if the mean time to return to it after leaving it is finite. It is recurrent null otherwise. The basic convergence theorem is the following

Theorem 12 Let $P(t)$ be the transition matrix of a continuous-time denumerable-state chain. Then if $\lim_{t \rightarrow 0} P(t) = I$,

$$\lim_{t \rightarrow \infty} P(t) = A$$

exists. If the chain has a single recurrent class, the limit has all rows equal to a probability vector $\alpha = (\alpha_1, \alpha_2, \dots)$. Also $\alpha P(t) = \alpha$ for all t . If the class is either recurrent null or transient, $A = 0$.

It is no longer true that α is the unique solution to $\alpha Q = 0$. A proof of Theorem 12 may be found in Chung (1960). In finite chains, Q determines uniquely the transition probabilities $P(t)$. This is not true in infinite chains and the reason is roughly the following. The matrix Q , being the derivative at 0 of $P(t)$, involves only the initial behavior of the process. If from a given Q we were able to construct a process such that it goes to

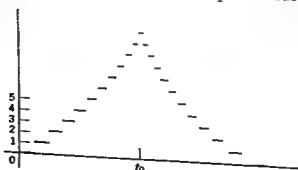


Fig 6 One type of return when the sample path goes to infinity

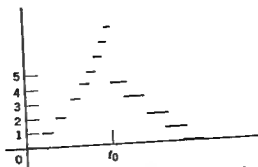


Fig 7 A second type of return when the sample path goes to infinity,

infinity in a finite time with positive probability, then we should be able to bring it back in different ways. For example, we might assume that when at infinity it comes immediately back to state 0. We might also try to make it come back to state 1. Examples can be constructed to show that this is possible, each leading to different processes with the same initial behavior, that is, that behave in exactly the same way until infinity is reached for the first time. The result is that $P(t)$ is different for the two processes, but Q is the same.

The probabilistic interpretations of q_{ij} are still valid. That is, if the process starts in state i , the length of time that it remains there has an exponential distribution with mean $1/q_i$. Also $r_{ij} = q_{ij}/q_i$ for $i \neq j$ and $r_{ii} = 0$ are the probabilities that when the process is started in state i the first jump is to state j . It is no longer true that $R = \{r_{ij}\}$ has row sums equal to one. All that can be said is that it has row sums less than or equal to one. The row sums are less than one, for example, if there is a positive probability that the process has a sample function such as that of Fig 8. When the row sums of R are equal to 1, the system is called *conservative*, that is, when $\sum_j q_{ij} = 0$ for all i .

We recall that the transition probabilities for finite chains satisfy two differential equations

$$P'(t) = P(t)Q.$$

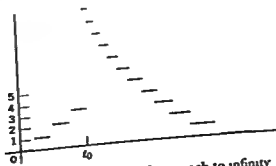


Fig 8 A second type of approach to infinity

which is called the *forward equation*, and

$$P'(t) = QP(t)$$

which is called the *backward equation*. In the denumerable case neither system of equations need hold, however, it is always true that

$$P'(t) \geq QP(t)$$

$$P'(t) \geq P(t)Q$$

The question of the validity of the differential equations can also be described in terms of the behavior of sample paths. The i th row of the backward equation holds if and only if whenever the process is started in state i it has as its first discontinuity a jump to ∞ with positive probability. Recall that r_{ij} is the probability that, when it jumped, it jumped to state j . Hence we could expect a jump to infinity to take place with positive probability if and only if $\sum_j r_{ij} < 1$, that is, if and only if $\sum_j q_{ij} \neq 0$,

which is in turn equivalent to the system not being conservative.

Whereas the backward equation has to do with the manner in which the process goes to infinity, the forward equation has to do with the manner in which the process returns from infinity (thus showing that the equations have been named backwards). Specifically, the j th column of the forward equation holds if and only if whenever the process is conditioned to be at state j at time t there is a positive probability that the process arrives at j by means of a jump from infinity.

Unfortunately, even when both equations are valid, a unique process still is not determined. That is, it is possible to have different processes with the same Q and satisfying both basic differential equations. It is true, however, that given a matrix with the properties required for a Q matrix of a continuous-time chain, there is always at least one process $P(t)$ whose transition probabilities satisfy both the backward equation $P'(t) = QP(t)$ and the forward equation $P'(t) = P(t)Q$. The process is called the *minimal process* because if \bar{P} is any other process with the same Q , then $\bar{P}(t) \geq P(t)$ for every t . The minimal process has the property that in any finite time interval there are at most a finite number of jumps. It is not necessarily an honest process in the sense that the row sums of $P(t)$ equal one, they may be less than one. That is, in this process there may be a positive probability that the process will disappear. The behavior of any process with the same Q as the minimal process is the same as the minimal process up to the first time that it reaches infinity. For this reason, if the process has the property that with probability 1 it does not reach infinity in finite time, we can be sure that it is, in fact, the minimal process.

A detailed discussion of these points may be found in Chung (1960)

3.11 Applications and Examples of Continuous-Time Chains

BIRTH AND DEATH PROCESSES Continuous time chains do not seem to have found application in psychology yet, however, an important class of continuous time chains, about which much is known, have been widely applied in other fields. These are the birth and death processes, which are distinguished by the feature that they move by a jump of at most one unit.

A birth and death process is a denumerable state continuous time Markov chain with transition functions $P(t)$ that satisfy the following conditions (as $t \rightarrow 0$)

$$p_{i,i+1}(t) = \lambda_i t + o(t),$$

$$p_{i,i-1}(t) = \mu_i t + o(t),$$

$$p_{i,i}(t) = 1 - (\lambda_i + \mu_i)t + o(t),$$

where $\lambda_i > 0$ for $i \geq 0$, $\mu_i > 0$ for $i \geq 1$, and $\mu_0 = 0$. The notation $o(h)$ means a quantity so small that $o(h)/h$ approaches 0 as h approaches 0.

From our assumptions it follows immediately that the Q matrix must be of the form

$$q_{i,i+1} = \lambda_i, \quad i \geq 0$$

$$q_{i,i} = -(\lambda_i + \mu_i), \quad i \geq 0$$

$$q_{i,i-1} = \mu_i, \quad i > 0$$

The quantities λ_i are often called birth rates and μ_i , death rates. This terminology arises from applications in which the state is the population at time t in some sort of growth process. Since the Q matrix is conservative, we know that the backward differential equations are satisfied, that is,

$$P'(t) = QP(t)$$

The forward equations,

$$P'(t) = P(t)Q,$$

need not be satisfied. Examples for which this equation fails can be constructed by making the birth rates increase so fast that there is a positive probability that the process will go to infinity in finite time, and the process is then brought back to a regular state by some probability distribution.

The question of when the two equations are satisfied has been completely answered by Feller (1959) who gave simple conditions on the birth and death rates for all possible situations. He showed that the possibilities for uniqueness of solutions of these equations have to do with the nature

of infinity as a boundary point. Roughly speaking, he showed the following. Infinity acts as an exit-type boundary point if starting from any state there is a positive probability that in finite time the process will drift off to infinity before hitting 0. It acts as an entrance point if starting from infinity it can reach 0 in a finite time with a positive probability. More precisely, there is a number $c > 0$ such that the probability of reaching 0 before time t is greater than c for any starting state, no matter how far away from 0 it is. Infinity is called a natural boundary point if it can act neither as an exit nor as an entrance point. In this case the forward and backward equations have a unique solution which is, of course, the minimal solution. If infinity can act as an exit boundary but not an entrance boundary, there is a unique solution to the forward equation and an infinite number of different solutions to the backward equations. If infinity can act as an entrance boundary point but not an exit, the forward equation has an infinite number of different solutions and the backward one a unique solution. Finally, if infinity can act both as an entrance and an exit boundary point, both equations have an infinite number of different solutions.

In the first three cases, a unique process, namely, the minimal process, satisfies both systems of differential equations. Recall that this process agrees with any other process up to the time that infinity is reached for the first time. Karlin and McGregor (1957) studied in detail this minimal process when it is the unique solution. They found simple criteria in terms of the birth and death rates for the process to be recurrent or transient. In a recurrent process, they gave criteria to distinguish the recurrent positive and recurrent null chains. They gave methods for computing a large number of descriptive quantities for both transient and recurrent chains. These include the actual distribution of first passage times in the recurrent case. Their method is similar to that used to study random walks in that they obtain an expression for $p_{ij}(t)$ for the minimal process which is of the form

$$p_{ij}(t) = \pi_j \int_0^\infty e^{-xt} Q_i(x) Q_j(x) d\psi(x),$$

where $\pi_0 = 1$, $\pi_n = \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n}$, and Q_0, Q_1, \dots is a sequence of orthogonal polynomials with respect to the measure ψ . The polynomials satisfy the recursive equations

$$\begin{aligned} Q_0(x) &\equiv 1, & Q_{-1}(x) &\equiv 0, \\ -xQ_n(x) &= -(\lambda_n + \mu_n)Q_n(x) + \lambda_n Q_{n+1}(x) + \mu_n Q_{n-1}(x). \end{aligned}$$

Many important special processes correspond to well-known systems of classical polynomials, and when this happens a method by Karlin and McGregor (1957) for computing the minimal process can be used for a great variety of basic quantities, such as distributions of first passage times for recurrent chains and time to absorption for absorbing chains. They have also studied in great detail a specific type of birth and death process called the linear birth and death processes. These are processes in which

$$\lambda_n = \lambda n + a, \quad \lambda > 0$$

$$\mu_n = \mu n + b, \quad \mu > 0$$

For details see Karlin and McGregor (1958)

4 CHAINS OF INFINITE ORDER

Consider now a stochastic process in which the outcomes are from a finite set A . In defining a chain of infinite order, we allow the complete past to have an influence on the next prediction, but we try to capture the idea that the influence of the distant past is small. The results discussed in this section are discussed more completely by Lamperti and Suppes (1959)

We shall make two assumptions, labeled *A* and *B*, about the process $\{X_n\}$. When these assumptions hold we say that $\{X_n\}$ is a chain of infinite order.

Assumption A *There is some state a_0 and some integer n_0 , and a number $\delta > 0$ such that*

$$Pr[X_{m+n_0} = a_0 \mid a_m a_{m-1} \dots a_1] > \delta$$

for every choice of $a_1 a_2 \dots a_m$.

In other words, no matter what the past history, the probability that we will be at state a_0 after n_0 steps is greater than δ .

For the second assumption, we wish to make precise the idea that the distant past does not have much influence on the prediction for the next experiment. Let $a_1 \dots a_{r+s}$ and $b_1 \dots b_{r+s}$ be two histories of length $r+s$ in which the final s outcomes are the same. Assume further that the outcome a_0 of assumption *A* occurs at least m times in these final s outcomes. Given these two different past histories, consider the difference between the probabilities of outcome a , that is,

$$\begin{aligned} &Pr[X_{r+s+1} = a \mid a_1 \dots a_{r+s}] \\ &- Pr[X_{r+s+1} = a \mid b_1 \dots b_{r+s}] \end{aligned}$$

Let ϵ_m be the smallest number such that these differences are less in absolute value than ϵ_m no matter what choice we make for a and for the

sequences $a_1 \dots a_r c_{r+1} \dots c_{r+s}$ and $b_1 \dots b_r c_{r+1} \dots c_{r+s}$ provided that the c 's contain at least $m a_0$'s

Assumption B The numbers ϵ_m have the property that $\sum_m \epsilon_m < \infty$.

The principal result for chains of infinite order is the following ergodic-type theorem

Theorem 13 If X_1, X_2, \dots is a chain of infinite order, then as n tends to infinity

$$Pr[X_{m+n} = a \mid a_m a_{m-1} \dots a_1]$$

converges to a limiting value π_a which is independent of $a_1 a_2 \dots a_m$; moreover, $\sum \pi_a = 1$

Note that as functions of $a_1 a_2 \dots a_m$ the probabilities

$$Pr[X_{m+n} = a \mid a_m a_{m-1} \dots a_1]$$

are random variables. In some applications the moments of these random variables are important. Lamperti and Suppes (1959) proved that these moments have limiting values as n tends to infinity. This paper also includes a proof of Theorem 13

4.1 An Application

Lamperti and Suppes applied the preceding results to the linear model described by Estes and Suppes, Chapter 8 of Bush and Estes (1959). See Sec. 1 for a definition of this model. Recall that a typical history ω is of the form

$$\omega = (a_1 e_1 a_2 e_2 \dots),$$

where a_n is the response of the subject on the n th trial and e_n is the reinforcement event of the experimenter. Denoting the outcome functions by $A_1, E_1, A_2, E_2, \dots$, the basic linear axiom is

$$\begin{aligned} Pr[A_{n+1} = j \mid k a_n e_{n-1} a_{n-2} \dots e_1 a_1] \\ = (1 - \theta_k) Pr[A_n = j \mid e_{n-1} a_{n-1} \dots e_1 a_1] + \theta_k \lambda_{jk}, \end{aligned}$$

where $0 \leq \theta_k \leq 1$ and $\sum_k \lambda_{jk} = 1$.

We consider then the probabilities

$$Pr[A_n = j \mid e_{n-1} a_{n-1} \dots e_1 a_1],$$

which for j fixed is a random variable. Let $\alpha'_{j,n}$ denote the n th moment of this random variable, that is,

$$\alpha'_{j,n} = \sum \{Pr[A_n = j \mid e_{n-1} a_{n-1} \dots e_1 a_1]\}^n Pr[e_{n-1} a_{n-1} \dots e_1 a_1],$$

where the sum is taken over all possible histories $a_1 e_1 \dots a_n e_n$. In particular, the first moment gives $Pr[A_n = j]$. Using the results of chains of infinite order, Lamperti and Suppes proved the following theorem

Theorem 14 *Assume that a linear model satisfies the following conditions*

- (i) *the reinforcement schedule has past dependence of length m and*
- (ii) *for some k , $0_k \neq 0$, and for this k there is an a and n_0 such that for all sequences $a_1, e_1, \dots, a_n e_n$*

$$Pr[E_{n+n_0} = k \mid e_n a_n \dots e_1 a_1] \geq \delta > 0$$

Then $\lim_{n \rightarrow \infty} \alpha_{j,n}^ = \alpha_j^*$ exists and is independent of the initial distribution of responses*

Note, in particular, that when the hypotheses are satisfied, $Pr[A_n = j]$ has a limiting value as n tends to infinity, and this limit is independent of the initial distribution. More generally, the probability of any finite sequence of outcomes occurring has a limiting value as n tends to infinity.

The model most often used in applications assumes that $\theta_k = 0$ for $k \neq 0$, $\theta_0 = 0$, $\lambda_{jj} = 1$, $\lambda_{jk} = 0$ for $j \neq k$. If we have a reinforcement schedule with lag v for this case, for the hypotheses of Theorem 14 to be satisfied we need only assume that some response k has a positive probability of being reinforced on the n th trial no matter what the outcome of A_{n-v} was, similarly, for the double contingent case.

These results were extended to cover the two person learning situations in which the reinforcements of each subject depend on the responses of both subjects. This again leads to a situation in which Markov theory is not applicable, and it is necessary to resort to the more general theory of chains of infinite order to prove asymptotic results.

5 MARTINGALES

Recall that a stochastic process with discrete outcomes is a martingale if

$$E[X_{n+1} \mid (X_n = a_n) \wedge X_{n-1} = a_{n-1}) \wedge (X_1 = a_1)] = a_n,$$

which is to say that at each trial the expected value of the next outcome is just what the last outcome was. Interpreting X_n as a gambler's fortune in a game the martingale property means that the game is fair in the sense that his expected fortune after each play is always just what it was before the play. It follows from the martingale property that $E[X_1] = E[X_2] = \dots = E[X_n]$.

There are two basic types of theorems for martingales, one is known as a system theorem and the other as a convergence theorem. System theorems

are suggested by the following game interpretation. If a game is fair, that is, has the martingale property, then it should remain fair if the gambler is allowed to use certain systems of play. Of course, there must be some restrictions on the systems allowed. He should not be allowed, for example, to look into the future. We illustrate this in terms of one kind of system called *optional stopping*. This means that the player is allowed to watch how his fortune proceeds and to stop playing at some time which seems advantageous to him. We describe this mathematically by introducing a random variable t such that the decision to stop at time n is represented by $t(\omega) = n$. Here ω is a sample sequence for the process X_1, X_2, \dots . The condition that the future is not available to the player means that we must be able to tell if $t(\omega) = n$ by knowing only the first n components of $\omega = (a_1, a_2, \dots, a_n, a_{n+1}, \dots)$. Sometimes it is convenient to allow $t(\omega) = \infty$, which means that the player does not stop, but we shall assume that the probability is zero that this happens. Given a stopping time t , we can associate a final random variable X_t which in the gambling interpretation represents the player's final fortune. If $\omega = (a_1, a_2, \dots, a_n, \dots)$ and $t(\omega) = n$ then $X_t(\omega) = a_n$. Now if the game is to remain fair under an optional stopping system of play, it must be the case that $E[X_t] = E[X_1]$. That is, in terms of expected value the system provides no advantage. The first theorem that we mention states that under a suitable hypothesis this is true.

Theorem 15 *Let X_1, X_2, \dots be a martingale observed until the random stopping time t . Assume that during the period of observation all the outcomes are in absolute value less than some constant K . Then if X_t is the final observation, $E[X_t] = E[X_1]$.*

A proof of Theorem 15 can be found in Doob (1953, Chapter 7).

A sufficient condition for the theorem to hold for any stopping time is for the process X_1, X_2, \dots to be uniformly bounded. That some such condition is necessary can be seen from the penny-matching game, which clearly is a fair game and hence a martingale. If each player has an infinite amount of money and if play can continue as long as he pleases, then one player can be sure to win by adopting the following stopping rule. Play until he is one penny ahead. Then his initial winning is 0 and his final winning with probability 1 will be 1. That is, $E[X_1] = 0$ and $E[X_t] = 1$. If we add the hypothesis that each player has only a finite amount of money, then the theorem applies. It can be used, in fact, to find the probability that the game ends in favor of one player. Assume, for example, that Peter has A pennies and Paul has B pennies and they play until either Peter has won all of Paul's money or Paul has won all of Peter's. The process that represents Paul's winnings is simply an unlimited coin-tossing process stopped the first time it reaches $+A$ or $-B$. All the

time that play continues the outcomes are bounded by the maximum of A and B , hence the theorem applies. Let p be the probability that Paul wins an amount A . Then he loses B with probability $1 - p$. Hence his expected final fortune is

$$pA - (1 - p)B$$

However, his expected initial winning was 0 so that by the theorem

$$0 = pA - (1 - p)B$$

or $p = B/(A + B)$. This is typical of how a gambling type interpretation enables us to prove a pure probability result.

Most applications of the convergence properties of a martingale come from the following theorem.

Theorem 16 *Let X_1, X_2, \dots be a martingale with nonnegative random variables. Then with probability 1 $X_n(\omega)$ converges to a limiting value $X_\infty(\omega)$. Furthermore $E[X_\infty] \leq E[X_1]$. If in addition the X_n are uniformly bounded in n , then $E[X_\infty] = E[X_1]$.*

This theorem is proved in Doob (1953).

As an illustration of the use of this convergence theorem, recall that a branching process is a denumerable state Markov chain with states $0, 1, 2, \dots$, such that if the chain is in state j , then j independent experiments with integer valued outcomes are performed. The next state is the sum of the outcomes. Assume we start in state 1 that is with one particle and let X_1, X_2, \dots be the resulting process. Let m be the expected number of particles produced by a single particle. We define $Z_n = X_n/m^n$. Then Z_1, Z_2, \dots is a nonnegative martingale, and by the theorem it converges to a limiting random variable Z_∞ . If $m < 1$, then X_n/m^n increases to infinity unless at some time X_n becomes 0. Since we know that the limit is finite it must be that X_n does sometime equal 0. That is the process dies out with probability 1.

Consider next the case in which the mean number of offspring per generation is one. Then $\{X_n\}$ is a martingale and so it converges. Since the sample paths are integers convergence can occur only if the population remains fixed from some time on which in turn can happen with positive probability only by dying out. Hence, again the process eventually dies out with probability 1. Here is an example of a martingale for which the expected value of the limit is less than the expected value of the initial outcome.

If $m > 1$, then with positive probability the process continues indefinitely. If so it increases to infinity more slowly than m^n . The probability of the process dying out can also be found by martingale theory as follows. If $m > 1$ there is a positive solution $a < 1$ to the equation

$$x = p_0 + p_1x + p_2x^2 + \dots$$

where $\{p_0, p_1, p_2, \dots\}$ is the distribution of the number of offspring produced by a single particle. Then the process $Y_n = a^{X_n}$ is a martingale. The outcomes of this process are bounded, so the expected value of the final outcome is the same as Y_1 , which in turn is a . However, the limit of this martingale is 0 on a path where X_n does not die out and 1 on a path where it does. Hence the expected value of the limit of Y_n is the probability that the process dies out. In other words, a is the probability that the process dies out. For a complete discussion of martingales see Doob (1953, Chapter 7).

6 STATIONARY PROCESSES

Recall that a discrete stochastic process $\{X_n, n = 1, 2, 3, \dots\}$ is stationary if

$$Pr[(X_h = a_1) \wedge (X_{h+1} = a_2) \wedge \dots \wedge (X_{h+n} = a_n)]$$

is the same for every h , which in words means that in predicting a particular sequence of outcomes the time at which we start our predictions is not relevant.

We have already seen two examples of stationary processes. The first was a sequence of independent random variables with a common distribution. The second example arose as follows. Let P be the transition matrix of a recurrent Markov chain, and let α be the stationary measure, that is, $\alpha P = \alpha$. Then if α is a probability vector, we can use α as a starting distribution and the resulting process is a stationary process.

For each of these examples we have seen that the averages

$$\frac{X_1 + X_2 + \dots + X_n}{n}$$

converge with probability 1 as n tends to infinity. When $\{X_n\}$ is an independent process, this average converges almost everywhere to a constant value, namely, to $E[X_1]$. For Markov chains, the limit is the same for all paths that start in the same recurrent class but differs among classes. Thus, for both the independent process and the ergodic Markov chain, we have a limiting random variable, but it may not be a constant with probability 1. These results are all consequences of a very general theorem known as the ergodic theorem for stationary processes.

Theorem 17 *Let $\{X_n\}$ be a stationary process and let f be a function such that $E[f(X_1)]$ is finite. Then*

$$\frac{f(X_1) + f(X_2) + \dots + f(X_n)}{n}$$

converges with probability 1.

A proof of Theorem 17 may be found in Doob (1953, Chapter 10)

In addition to knowing that the averages converge, it is important to know under what conditions the averages converge to the mean of the individual trials. For this we need a new concept. Let T be the transformation of the points ω such that if $\omega = (a_1, a_2, \dots)$, then $T\omega = (a_2, a_3, \dots)$. Thus T , which is called the *shift transformation*, simply replaces the history ω by the history ω' obtained from ω by dropping the first outcome and shifting the others to the left by one.

If A is a subset of Ω , the set TA is the set of all points ω' of the form $T\omega$ for ω in A . A set A is said to be an *invariant set* if A and TA differ by at most a set of ω 's of probability 0. The space Ω is clearly invariant under the shift operation since $T\Omega = \Omega$, and the same is true for the empty set.

Definition 21 A stationary process $\{X_n\}$ is called *metrically transitive* if the shift operation has only sets of measure 0 or 1 as invariant sets.

An independent process is metrically transitive, although this requires a proof. The second example above, that of a Markov chain started in equilibrium, need not be metrically transitive. Indeed, let us assume that there are two recurrent classes A and B . Then the set S of all sequences that begin with a state in A is an invariant set. Its measure is $\sum_{s_i \in S} a_i$,

which is not 0 because the components of α are positive, and not 1 because the sum of all the a_i adds up to 1. If there is only a single class, the resulting process is metrically transitive. In the two examples that we have given which are metrically transitive, namely, independent processes and recurrent Markov chains with a single class, the law of large numbers states that the averages converge to a constant. These are special cases of the following theorem for stationary processes.

Theorem 18 Let X_n be a stationary process that is metrically transitive. If f is a function such that $E[f(X_1)]$ is finite, then

$$\frac{f(X_1) + f(X_2) + \dots + f(X_n)}{n} \rightarrow E[f(X_1)]$$

with probability 1.

See Doob (1953, Chapter 10) for a proof of this theorem.

6.1 Examples and Applications

A third example of a stationary process arises when we average data obtained from an independent process. For example, assume that Y_1, Y_2, \dots is an independent process with common distribution. Instead of recording this process, however, suppose we record at each stage the average of the last three outcomes. That is, we start at time three and

record

$$X_1 = \frac{Y_1 + Y_2 + Y_3}{3},$$

$$X_2 = \frac{Y_2 + Y_3 + Y_4}{3},$$

$$X_n = \frac{Y_n + Y_{n+1} + Y_{n+2}}{3}$$

Then the process X_1, X_2, \dots is a stationary process. More generally, if Y_1, Y_2, \dots is an independent process with common distribution, and if

$$X_n = c_0 Y_n + c_1 Y_{n+1} + \dots + c_r Y_{n+r}$$

for any fixed c_0, c_1, \dots, c_r , then X_1, X_2, \dots is a stationary process which is called a process of *moving averages*.

Another way in which stationary processes have arisen is the following. Suppose X_1, X_2, \dots is a process that is

1 stationary, and

$$2 \quad X_n = c_0 X_{n-1} + c_1 X_{n-2} + \dots + c_r X_{n-r} + Y_n,$$

where Y_1, Y_2, \dots is a sequence of identically distributed random variables. The first question is whether such a process can exist at all, and, if so, whether it is in some sense unique. This is quite similar to the procedure used when observing a deterministic phenomenon $X(1), X(2), \dots$ when we postulate that the $X(n)$ satisfy a difference equation of the following form

$$X(n) = c_0 X(n-1) + c_1 X(n-2) + \dots + c_r X(n-r) + Y(n)$$

where $Y(n)$ is a known function of n .

Procedures for solving this stochastic difference equation have been developed and conditions under which a stationary solution exists are known. A process that is stationary and satisfies a difference equation of the preceding type is known as one of *autoregression*. For a discussion of these processes see Rosenblatt (1962).

A problem that has received a great deal of attention is that of predicting future outcomes in a stationary process when the outcomes up to a certain time are known. To describe this problem more precisely, it is convenient to assume that we are observing a process that has been going on for an infinite length of time and that will continue indefinitely. More precisely, the process is described by random variables of the form

$$, X_{-2}, X_{-1}, X_0, X_1, X_2,$$

Stationarity is generalized to mean that

$$Pr[(X_n = a_1) \wedge (X_{n+1} = a_2) \wedge \dots \wedge (X_{n+r} = a_r)]$$

is independent of n for both positive and negative n

The problem, then, is Given the outcomes up to time n , how do we predict the outcome X_{n+r} ? To solve this problem we must have a way of telling what is the best prediction We therefore introduce a measure of the error that we make If our prediction is \bar{X}_{n+r} , then our error is the random variable $|X_{n+r} - \bar{X}_{n+r}|$, and we agree to take

$$E[(|X_{n+r} - \bar{X}_{n+r}|)^2]$$

as a measure of the error made Of course, this measure of error is rather arbitrary, but it has been chosen for mathematical convenience The problem is then to find a method of determining the estimate \bar{X}_{n+r} that minimizes the expected value of the sequence of the errors, that is, that minimizes $E[(|X_{n+r} - \bar{X}_{n+r}|)^2]$ This problem has been solved provided that the estimate is assumed to be a linear combination of previous outcomes,

$$\bar{X}_{n+r} = c_n X_n + c_{n-1} X_{n-1} +$$

for some $c_n, c_{n-1},$
(1962)

Details of this solution are given in Rosenblatt

As a final application, we mention the notion of entropy of a stationary process, which is an important concept in the study of information theory as introduced by Shannon If an experiment results in outcomes a_1, a_2, \dots, a_r with probabilities $p(a_1), p(a_2), \dots, p(a_r)$, then the entropy is defined to be

$$-\sum p(a_i) \log p(a_i)$$

If X is the outcome function for this experiment, then we can write

$$H = E[-\log p(X)]$$

We would, however, like to think of the experiment as the entire sequence of text $\omega = (a_1, a_2, \dots)$ Of course, in general, $p(\omega) = 0$ so we cannot use the preceding definition to define entropy of an arbitrary stationary source However, using our basic probabilities, we can assign an entropy to the partial source X_1, X_2, \dots, X_n This is

$$\sum_{(a_1, a_2, \dots, a_r)} p(a_1 a_2 \dots a_r) \log p(a_1 a_2 \dots a_r),$$

where $p(a_1 a_2 \dots a_r) = Pr[(X_1 = a_1) \wedge (X_2 = a_2) \wedge \dots \wedge (X_r = a_r)]$ and the sum is taken over all possible first r outcomes We can also write this as

$$-E[\log p(X_1, X_2, \dots, X_r)]$$

record

$$X_1 = \frac{Y_1 + Y_2 + Y_3}{3},$$

$$X_2 = \frac{Y_2 + Y_3 + Y_4}{3},$$

$$X_n = \frac{Y_n + Y_{n+1} + Y_{n+2}}{3}$$

Then the process X_1, X_2, \dots is a stationary process. More generally, if Y_1, Y_2, \dots is an independent process with common distribution, and if

$$X_n = c_0 Y_n + c_1 Y_{n+1} + \dots + c_r Y_{n+r}$$

for any fixed c_0, c_1, \dots, c_r , then X_1, X_2, \dots is a stationary process which is called a process of *moving averages*.

Another way in which stationary processes have arisen is the following. Suppose X_1, X_2, \dots is a process that is

1 stationary, and

$$2 \quad X_n = c_0 X_{n-1} + c_1 X_{n-2} + \dots + c_r X_{n-r} + Y_n,$$

where Y_1, Y_2, \dots is a sequence of identically distributed random variables. The first question is whether such a process can exist at all, and, if so, whether it is in some sense unique. This is quite similar to the procedure used when observing a deterministic phenomenon $X(1), X(2), \dots$ when we postulate that the $X(n)$ satisfy a difference equation of the following form

$$X(n) = c_0 X(n-1) + c_1 X(n-2) + \dots + c_r X(n-r) + Y(n)$$

where $Y(n)$ is a known function of n .

Procedures for solving this stochastic difference equation have been developed and conditions under which a stationary solution exists are known. A process that is stationary and satisfies a difference equation of the preceding type is known as one of *autoregression*. For a discussion of these processes see Rosenblatt (1962).

A problem that has received a great deal of attention is that of predicting future outcomes in a stationary process when the outcomes up to a certain time are known. To describe this problem more precisely, it is convenient to assume that we are observing a process that has been going on for an infinite length of time and that will continue indefinitely. More precisely, the process is described by random variables of the form

$$\dots, X_{-2}, X_{-1}, X_0, X_1, X_2,$$

Stationarity is generalized to mean that

$$Pr[(X_n = a_1) \wedge (X_{n+1} = a_2) \wedge \dots \wedge (X_{n+r} = a_r)]$$

is independent of n for both positive and negative n

The problem, then, is: Given the outcomes up to time n , how do we predict the outcome X_{n+r} ? To solve this problem we must have a way of telling what is the best prediction. We therefore introduce a measure of the error that we make. If our prediction is \bar{X}_{n+r} , then our error is the random variable $|X_{n+r} - \bar{X}_{n+r}|$, and we agree to take

$$E[(\bar{X}_{n+r} - X_n)^2]$$

as a measure of the error made. Of course this measure of error is rather arbitrary, but it has been chosen for mathematical convenience. The problem is then to find a method of determining the estimate \bar{X}_{n+r} that minimizes the expected value of the sequence of the errors that is that minimizes $E[(\bar{X}_{n+r} - X_n)^2]$. This problem has been solved provided that the estimate is assumed to be a linear combination of previous outcomes,

$$\bar{X}_{n+r} = c_n X_n + c_{n-1} X_{n-1} +$$

for some $c_n, c_{n-1},$
(1962)

Details of this solution are given in Rosenblatt

As a final application, we mention the notion of entropy of a stationary process, which is an important concept in the study of information theory as introduced by Shannon. If an experiment results in outcomes a_1, a_2, \dots, a_r with probabilities $p(a_1), p(a_2), \dots, p(a_r)$ then the entropy is defined to be

$$-\sum p(a_i) \log p(a_i)$$

If X is the outcome function for this experiment, then we can write

$$H = E[-\log p(X)]$$

We would, however, like to think of the experiment as the entire sequence of text $\omega = (a_1, a_2, \dots)$. Of course, in general, $p(\omega) = 0$ so we cannot use the preceding definition to define entropy of an arbitrary stationary source. However, using our basic probabilities, we can assign an entropy to the partial source X_1, X_2, \dots, X_n . This is

$$\sum_{(a_1, a_2, \dots, a_r)} p(a_1 a_2 \dots a_r) \log p(a_1 a_2 \dots a_r),$$

where $p(a_1 a_2 \dots a_r) = Pr[(X_1 = a_1) \wedge (X_2 = a_2) \wedge \dots \wedge (X_r = a_r)]$ and the sum is taken over all possible first r outcomes. We can also write this as

$$-E[\log p(X_1, X_2, \dots, X_r)]$$

Let

$$H_n = -\frac{1}{n} E[\log p(X_1, X_2, \dots, X_n)]$$

Then if $H_n \rightarrow H$, it is reasonable to call H the entropy per signal. Actually more than this has been proved. The sequence

$$Y_n = -\frac{1}{n} \log p(X_1, X_2, \dots, X_n)$$

has been shown to converge with probability 1 to a constant H and $H_n = E[Y_n] \rightarrow H$ provided only that the source is a stationary, metrically transitive process. The proof of this assertion involves using both the ergodic theorem and the basic martingale theorem. An interesting discussion of the idea of this proof may be found in Chung (1962).

7 CONCLUDING REMARKS

We have considered four basic types of stochastic processes, namely, independent processes, Markov processes, martingales, and stationary processes. This is a rather small number of types, considering the wide variety of applications that have been found for stochastic processes. Although it is easy to define new types of processes, it is not always easy to work with them mathematically. A good example of the fact that one cannot neglect the mathematical difficulties is to be found in the observation that every process is, in fact, a Markov process if one just looks at it correctly. To see this, assume that X_1, X_2, \dots is a discrete process taking on values in a set A . We form a Markov chain in which a state is a pair $(n, a_1, a_2, \dots, a_n)$, where n is any integer and a_1, a_2, \dots, a_n are n possible values from A . If the first n outcomes of the process X_1, X_2, \dots, X_n are a_1, a_2, \dots, a_n respectively, then we say that our chain is in the state $(n, a_1, a_2, \dots, a_n)$. It moves to a state of the form

$$(n+1, a_1, a_2, \dots, a_n, a_{n+1})$$

with probability given by

$$P_{(n, a_1, a_2, \dots, a_n) \rightarrow (n+1, a_1, a_2, \dots, a_n, a_{n+1})} = \\ Pr[X_{n+1} = a_{n+1} | (X_n = a_n) \wedge (X_{n-1} = a_{n-1}) \wedge \dots \wedge (X_1 = a_1)]$$

Since the study of this Markov chain is equivalent to the study of the original process, we can reduce the study of the most general discrete process to the study of denumerable Markov chains. The difficulty with

this approach is that the state space is quite complicated and unnatural from the point of view of mathematical analysis

It is clear that some reasonable compromise must be made between the assumptions that one wants to make about the process being observed and assumptions that lead to a process that can be studied. A good example of just such a compromise arises in the theory of queues, which is devoted to processes that arise in the following manner. Persons arrive at random at a service station where they are served, the service time also being a random quantity, and they leave. One must, of course, make specific the nature of the random arrival times and the random service times. The study started with quite simple assumptions and, for many situations, unrealistic ones. It was discovered that even with simplifying assumptions, quite useful qualitative information was gained about practical queueing problems. The subject has by now been studied intensively, the voluminous literature includes a great variety of assumptions to fit a wide range of different situations. For an introductory treatment of this subject, see Cox and Smith (1961).

Many users of present-day stochastic process theory object to the emphasis on stationarity. The two processes studied in greatest detail, independent trials and Markov chains, each have a stationarity assumption. With independent trials, each trial looks the same. In a Markov chain, the transition probabilities are assumed to be the same for each time. It is natural in applications to suggest that these assumptions be weakened. Results on Markov chains with nonstationary transition probabilities are known, but this class of processes is so general that little of a specific nature can be said. Even the two state chain is extremely complicated to analyze, and widely different types of behavior are possible depending on the nature of the transition probabilities. Some ergodic theorems and some limit theorems for these processes have been discovered. See, for example, Dobrusin (1956). It seems necessary to find some specific manner in which the transition probabilities change before a detailed study becomes possible. Presumably, if nonstationary situations are important in applications in the social sciences, some way to formulate these assumptions will be suggested by the substantive assumptions.

We have given only the barest introduction to the theory of stochastic processes. Doob, in his book written in 1953, was able to survey the main results then known about stochastic processes. Today, almost any chapter in Doob's book would have to be expanded to a book at least the size of his in order to provide any degree of completeness. Despite the growth of the subject, it is still possible to describe very simple processes about which little or nothing is known. For example, recent studies on the growth of cancer have suggested the following simple problem. From a

large number of unit cubes, choose one and attach another randomly to one of its faces. Add a third randomly to one of the available faces of the pair. Continue in this way, each time adding a block at random to whatever faces happen to be available. Practically nothing is known about this very simple process. For example, it is not known whether the growth approximates a sphere after a large number of additions or whether, once a distortion occurs by chance, distortions tend to become exaggerated.

It seems to the author that, just as psychology suggested the study of interesting processes in learning theory, other areas of psychology should become the source of new and interesting processes for the probabilists.

References

- Andersen, E. S. On the fluctuations of sums of random variables. *Mathematica Scand.*, 1953, 1, 263-285.
- Anderson, T. W. The use of factor analysis in the statistical analysis of multiple time series. *Psychometrika*, 1963, 28, 1-25.
- Bush, R. R., & Estes, W. K. *Studies in mathematical learning theory*. Stanford, California: Stanford Univ. Press, 1959.
- Bush, R. R., & Mosteller, F. *Stochastic models for learning*. New York: Wiley, 1955.
- Chung, K. L. *Markov chains with stationary transition probabilities*. Berlin: Springer, 1960.
- Chung, K. L. The Ergodic theorem of information theory, *Recent developments in information and decision processes*. New York: Macmillan, 1962.
- Chung, K. L., & Fuchs, W. H. T. On the distribution of values of sums of random variables. *Mem. Amer. Math. Soc.*, 1951, 6, 1-12.
- Ciucu, G., & Theodorescu, R. *Probleme cu Legături Complete* (Rumanian). Bucharest, 1960.
- Cohen, B. P. *Conflict and conformity*. Cambridge, Mass.: MIT Press, 1963.
- Cox, D. R., & Smith, W. L. *Queues*. New York: Wiley, 1961.
- Dobrusin, R. L. Central limit theorem for non stationary Markov chains. *Teor. Veroyatnost. i Primenen.*, 1956, 1, 72-89.
- Doob, J. L. *Stochastic processes*. New York: Wiley, 1953.
- Doob, J. L. Discrete potential theory and boundaries. *J. Math. Mech.*, 1959, 8, 433-458.
- Doob, J. L. Some problems concerning the consistency of mathematical models. In R. E. Machol (Ed.) *Information and decision processes*. New York: McGraw-Hill, 1960. Pp. 27-33.
- Feller, W. Fluctuation theory of recurrent events. *Trans. Amer. Math. Soc.*, 1949, 67, 98-119.
- Feller, W. *An introduction to probability and its applications*, Vol. 1 (2nd Ed.). New York: Wiley, 1957.
- Feller, W. The birth and death processes as diffusion processes. *J. Math. Pures Appl.*, 1959, 38, 301-345.
- Gnedenko, B. V., & Kolmogorov, A. N. *Limit distributions for sums of independent random variables*. Cambridge, Mass.: Addison-Wesley, 1954.

REFERENCES

- Harris, T E Some mathematical models for branching processes, *Second Berkeley Symposium on Mathematical Statistics and Probabilities* Berkeley, California Univer of California Press, 1951
- Karlin, S Some random walks arising in learning models I *Pacific J Math*, 1953, 3, 725-756 Reprinted in R D Luce, R R Bush, & E Galanter (Eds) *Readings in mathematical psychology* New York Wiley, 1963 Pp 381-403
- Karlin, S, & McGregor, J The classification of birth and death processes, *Trans Amer Math Soc*, 1957, 86, 366-400
- Karlin, S, & McGregor, J Linear growth birth and death processes *J Math Mech*, 1958, 7, 643-662
- Karlin, S, & McGregor, J Random walks *Illinois J Math*, 1959, 3, 66-81
- Kemeny, J G, & Snell, J L Markov processes in learning theory *Psychometrika*, 1957, 22, 221-230
- Kemeny, J G, & Snell, J L *Finite Markov chains* Princeton, New Jersey Van Nostrand, 1960
- Kemeny, J G, & Snell, J L Finite continuous time Markov chains *Teor Veroyatnost i ee Primeneni*, 1961, 6, 111-115
- Kolmogorov, A N A local limit theorem for Markov chains In *Selected translations in Mathematical statistics and probability*, Vol 2, American Mathematical Society, 1962
- Lamperti, J, & Suppes, P Chains of infinite order and their applications to learning theory *Pacific J Math*, 1959, 9, 739-754 Reprinted in R D Luce R R Bush, & E Galanter (Eds) *Readings in mathematical psychology* New York Wiley, 1963 Pp 413-428
- Loeve, M *Probability theory* (2nd Ed) Princeton, New Jersey Van Nostrand, 1960
- Luce, R D (Ed) *Developments in mathematical psychology* Glencoe, Illinois Free Press, 1960
- Parzen E *Stochastic processes* San Francisco Holden Day, 1962
- Rosenblatt, M *Random processes* New York Oxford Univer Press, 1962
- Shannon, C E A mathematical theory of communication *Bell System Tech J*, 1948, 27, 319-423, 623-656
- Smith, W L Renewal theory and its ramifications *J Roy Statist, Soc Ser B*, 1958, 20, 243-302

21

Functional Equations

Richard Bellman

The RAND Corporation

Contents

1	Difference Equations and Iteration	491
1 1	Difference equations, 491	
1 2	Iteration, 492	
1 3	The Abel-Schroder equation, 494	
1 4	Heuristic derivation of the linearizing function, 495	
1 5	Stochastic iteration, 496	
2	Differential Equations and Functional Equations	496
2 1	Introduction, 496	
2 2	Causality and functional equations, 497	
2 3	Functional equation for exponential, 498	
2 4	Cauchy's problem, 499	
2 5	The linear function, 499	
2 6	Associated functional equations, 500	
2 7	General functional equations, 500	
2 8	Approximate functional equations, 501	
2 9	Erdos' functional equation, 501	
2 10	A limit theorem, 502	
3	Probability Theory	502
3 1	Preliminary remarks, 502	
3 2	Markov processes, 504	
3 3	Random walks, 505	
3 4	Learning processes, 505	
4	Dynamic Programming and Markovian Decision Processes	506
4 1	Dynamic programming, 506	
4 2	Allocation processes and the maximum transform, 508	
4 3	Stochastic multistage decision processes, 509	
4 4	Adaptive processes, 509	
	References	510

Functional Equations

In studying mathematics and its manifold applications to science, the first equations that one meets are algebraic equations. We learn, progressively, algorithms for the solution of general linear equations of the form $ax + b = c$, for the solution of linear systems

$$\sum_{j=1}^N a_{ij} x_j = c_i, \quad i = 1, 2, \dots, N,$$

for quadratic equations such as

$$ax^2 + 2bx + c = 0,$$

and then for the numerical solution of more complicated equations such as $x^5 + ax + b = 0$ or $x + e \sin x = c$.

In all these cases, the problem is to determine particular values of a known function, or functions if we are dealing with simultaneous equations. In elementary calculus, we encounter, for the first time, the question of determining an unknown function. Indeed, the basic problem of integral calculus is that of obtaining the function $f(x)$ whose derivative is the given function $g(x)$. This is the simplest type of differential equation,

$$\frac{df}{dx} = g(x)$$

In courses in mechanics, and physics in general, more complicated classes of differential equations arise. For example, an investigation of Hooke's law for the behavior of a spring under small deflections from equilibrium leads to the equation

$$\frac{d^2f}{dx^2} + mf = 0,$$

the motion of the pendulum introduces the equation

$$\frac{d^2f}{dx^2} + k \sin f = 0,$$

and so on. These are all examples of equations involving unknown functions.

In several of the following introductory sections we shall sketch briefly how it is that equations involving derivatives and differences dominate the

scientific domain. Nonetheless, other types of equations expressing properties of unknown functions play extremely important roles. To illustrate this, let us discuss one way in which an equation of a quite different nature arises in psychological research.

Suppose that the reaction to a stimulus is being studied by means of a combination of experimental and theoretical techniques. If the stimulus is of physical nature, say a light source, we can attach various parameters to it, intensity, duration, etc. For simplicity, suppose that there is only one parameter x and one reaction, measured by $f(x)$.

Sometimes we can ascertain all the desired values of $f(x)$ experimentally, sometimes a priori theoretical arguments suffice. Most frequently an amalgam of previous theoretical results, experiments, plausibility arguments, and bold extrapolation yield certain relations for $f(x)$ which, we trust, determine $f(x)$ completely. The values obtained from this analysis are then compared with previous results of like nature and experimental observation, leading to further experiments, revisions of the assumptions, etc., see Bellman and Brock (1960).

For example, in the situation described, we may surmise that the response to an increase in the stimulus parameter is additive, namely,

$$f(x + y) = f(x) + f(y)$$

This is an equation for the unknown function $f(x)$, a simple, but extremely important, example of a *functional equation*.

The function $f(x) = kx$, for any constant k , satisfies this equation. Is it, however, the only solution? As we shall see in the following sections, sometimes simple equations of this nature uniquely characterize a function, and sometimes they do not.

The term "functional equation" is such a broad one that we shall make no effort to define it. Operationally, we can conceive of all equations that involve unknown functions and that escape the conventional areas of differential, difference, and integral equations as constituting the domain of functional equations. From this it is obvious that there is no routine way to provide a survey of what is known about functional equations and their applications nor any time-worn introduction to their study. Were we even to restrict ourselves to the new and rapidly expanding area of mathematical psychology, the task of classification and analysis would still be overwhelming.

What we shall attempt to do in the following pages is to provide the reader with a background of classical analysis that motivates the investigation of many important classes of functional equations and that explains some of their origins. From time to time we shall indicate a few of the

connections with contemporary research in psychology and provide some references to current literature

The most important single reference is Aczel (1961) This work presents a lucid and detailed exposition and contains an almost complete bibliography

1 DIFFERENCE EQUATIONS AND ITERATION

1.1 Difference Equations

A standard mathematical approach to the study of physical processes, where the adjective physical is here used in the broad sense, involves the introduction of a number of state variables, x_1, x_2, \dots, x_n , each a measure of a different property of the system These may be position and velocity, voltage and current, behavior patterns, generalized coordinates, probability distributions, etc Usually, these properties change over time Hence, we write $x_1(t), x_2(t), \dots, x_n(t)$

In order to ascertain the future behavior of the system, we must make various assumptions concerning the local and global behavior of the functions $x_i(t)$ One of the simplest, and yet perhaps most important, mathematical models of the time history of a system (or a process—we shall use the terms interchangeably) is obtained by assuming that the immediate future depends only on the present, with complete independence of the past To derive an analytic consequence of this hypothesis, let us proceed as follows

Suppose that time varies in a discrete fashion, so that t assumes only the values $0, 1, 2, \dots$ The supposition that the values $x_1(t+1), x_2(t+1), \dots, x_n(t+1)$ (the state of the system in the immediate future) depends only on the present values $x_1(t), x_2(t), \dots, x_n(t)$ is equivalent to the set of relations

$$x_i(t+1) = g_i[x_1(t), x_2(t), \dots, x_n(t)] \quad i = 1, 2, \dots, N, \quad (1)$$

for $t = 0, 1, 2, \dots$, where the g_i are functions which we are free to prescribe Various assumptions concerning these functions lead to different predicted behaviors of the system Comparing these predictions with observations and other theoretical analyses, we can test the validity of our conceptual framework

A particularly interesting case is that for which the g_i are linear functions of their arguments,

$$x_i(t+1) = \sum_{j=1}^n a_{ij}x_j(t) \quad i = 1, 2, \dots, N \quad (2)$$

It is also important in a number of investigations to consider processes in which the dependence changes in the course of time. This is most simply done by letting the coefficients a_{ij} depend on time, that is, $a_{ij} = a_{ij}(t)$.

The introduction of elementary vector notation greatly simplifies the typography. Letting

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_N(t) \end{bmatrix}, \quad g(x) = \begin{bmatrix} g_1(x) \\ g_2(x) \\ \vdots \\ g_N(x) \end{bmatrix}, \quad (3)$$

we can write Eq. 1 in the compact form

$$x(t+1) = g[x(t)] \quad (4)$$

If A denotes the $N \times N$ matrix (a_{ij}) , $i, j = 1, 2, \dots, N$, the linear relations of Eq. 2 may be written

$$x(t+1) = Ax(t) \quad (5)$$

Equations of the preceding type, Eqs. 4 and 5, are called difference or recurrence equations. An excellent introductory account may be found in Goldberg (1958), for a discussion of the connections between matrix theory and equations of the type appearing in Eq. 5, see Bellman (1960).

1.2 Iteration

From the previous discussion, in which the analytic aspects are merely translations into mathematical terminology of rather immediate ideas, we are led rather quickly to some interesting and significant problems. Suppose that we start with the difference equation

$$u(t+1) = g[u(t)], \quad u(0) = c, \quad t = 0, 1, 2, \dots, \quad (6)$$

and write, for $n = 0, 1, 2, \dots$,

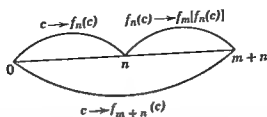
$$u(n) = f_n(c),$$

indicating in an explicit fashion the dependence on the initial value c as well as the dependence on the time n .

It is easy to establish inductively that the function $f_n(c)$ satisfies the basic functional equation

$$f_{m+n}(c) = f_m[f_n(c)], \quad (7)$$

for $m, n = 0, 1, 2, \dots$, and all c . This may be regarded as an expression of the uniqueness of solution of Eq 6, as an expression of the law of causality or as the obvious multiplicative characteristic of iteration, see the discussion in Bellman (1961). Schematically,



Verbally, this states that a system in state c left initially to run for a time $m + n$ ends up in the same state as the same system allowed to proceed for a time n , stopped, and then, starting in the new state $f_n(c)$, allowed to proceed for a time m .

It is remarkable that a number of the basic functional equations of analysis, the equations governing e^t , $\log t$, $\sin t$, etc., stem from this observation. We shall discuss these matters subsequently in connection with differential equations.

Observe that Eq 7 is valid only for discrete values of m and n , $m, n = 0, 1, 2, \dots$. It turns out to be quite significant to ask whether or not we can find a function of two variables, $g(c, t)$, defined for all c and all $t \geq 0$, with the property that

$$g(c, n) = f_n(c), \quad (8)$$

for $n = 0, 1, 2, \dots$, and that

$$g(c, t + s) = g[g(c, t), s] \quad (9)$$

for all nonnegative t and s .

This is an *interpolation* problem. We are attempting to imbed a discrete process into a continuous process satisfying the same conditions of causality. It is reasonable to suspect that sometimes this can be done, and sometimes not. Since the general problem is one of great difficulty, we shall discuss only some simple, but particularly important cases.

Parenthetically, let us point out that Euler solved a particular problem of this general type when he observed that

$$\Gamma(x + 1) = \int_0^\infty t^x e^{-t} dt$$

(the gamma function), defined in this way for $\text{Re}(x) > 0$, and by analytic continuation as a meromorphic function of x over the entire complex

plane, satisfies the functional equation

$$\Gamma(x+1) = x\Gamma(x)$$

Hence $\Gamma(x+1) = x!$ if x is a positive integer. Thus, the function defined previously by the integral represents a continuous interpolation of the factorials

Iteration and the theories that flow from this fundamental concept occupy a central role in classical and modern analysis (Hadamard, 1944, Montel, 1957). In a natural way, iteration generalized to produce the theory of branching processes (Harris, 1963) and the theory of semigroups (Hille & Phillips, 1948). In connection with multistage decision processes and modern control theory, it leads to dynamic programming (Bellman, 1957, 1961, Bellman & Dreyfus, 1962), in mathematical physics, it leads to invariant imbedding (Bellman & Kalaba, 1956, Bellman, Kalaba, & Prestrud, 1963).

1.3 The Abel-Schroder Equation

A second problem, apparently unrelated to the interpolation problem posed in the foregoing section, is the following. Given the function $f(c)$, can we find a function $g(c)$ and a constant k such that

$$g[f(c)] = kg(c)? \quad (10)$$

In effect, g linearizes f . Alternatively, we can regard g as a *relative invariant* of the transformation $f(c)$. Equation 10 is a very famous equation in analysis, the equation of Abel-Schroder.

To show the intimate connection between the two problems, let us proceed as follows. Suppose that a function $g(c)$ exists that satisfies Eq 10. Then

$$g\{f[f(c)]\} = kg[f(c)] = k^2g(c),$$

and an easy induction shows that

$$g[f^{(n)}(c)] = h^n g(c),$$

for n a positive integer. Hence

$$f^{(n)}(c) = g^{-1}[k^n g(c)]$$

Introduce the function

$$G(c, t) = g^{-1}[k^t g(c)],$$

for $t \geq 0$. It follows immediately that

$$G(c, t+s) = G[G(c, t), s]$$

This is an answer to the interpolation problem posed in Eq 9, Sec 1.2

A psychological problem that gives rise to the Abel-Schroder equation is the classical one of Fechner to find those transformations G of the physical continuum for which the "just noticeable difference" is constant. If the physically measured jnd at c is $h(c)$, then the requirement on G is simply

$$G[c + h(c)] - G(c) = K,$$

where K is a positive constant. If we let $f(c) = c + h(c)$, $g = e^G$ and $k = e^K$, then it is clear that this condition is equivalent to Eq 10. This problem is discussed in detail in Luce and Edwards (1958) and in Luce and Galanter (1963, pp 206-213).

1.4 Heuristic Derivation of the Linearizing Function

The function $g(c)$, in many important situations, is connected with the asymptotic or steady-state behavior of the n th iterate of $f(c)$.

Consider the special case where $f(c)$ is an analytic function of c in a circle $|c| \leq c_1$,

$$f(c) = b_1 c + b_2 c^2 + \dots,$$

with $b_1 \neq 0$. Introduce the function

$$F_n(c) = \frac{f^{(n)}(c)}{b_1^n},$$

and suppose that the sequence $\{F_n(c)\}$ has a limit as $n \rightarrow \infty$. Let

$$g(c) = \lim_{n \rightarrow \infty} \frac{f^{(n)}(c)}{b_1^n} \quad (11)$$

Then

$$\begin{aligned} g[f(c)] &= \lim_{n \rightarrow \infty} \frac{f^{(n)}[f(c)]}{b_1^n} \\ &= \lim_{n \rightarrow \infty} \frac{f^{(n+1)}(c)}{b_1^n} \\ &= b_1 \lim_{n \rightarrow \infty} \frac{f^{(n+1)}(c)}{b_1^{n+1}} \\ &= b_1 g(c) \end{aligned}$$

Thus we obtain the function required in Eq 10.

In the case where $0 < |b_1| < 1$, it is not difficult to establish the existence of an analytic function $g(c) = c + a_2 c^2 + \dots$ in a sufficiently small neighborhood of the origin. The result was first given by Schroder (1881) and Koenigs (1885). For a discussion of the two-dimensional case and references to other and previous work on the multidimensional case, see Bellman (1952b). For detailed accounts, see the books by Harris and Montel referred to previously. The area is one which requires very careful analysis and often some deep and delicate considerations.

1.5 Stochastic Iteration

In the study of some situations, we encounter processes for which there is a possibility of one of several transformations being applied at any stage. Considering the simple case in which there are two possible transformations, $f_1(c)$ and $f_2(c)$, with respective probabilities p_1 and p_2 , we are led to the functional equation

$$p_1 g[f_1(c)] + p_2 g[f_2(c)] = b f(c)$$

See Bellman (1964) for a further discussion.

2 DIFFERENTIAL EQUATIONS AND FUNCTIONAL EQUATIONS

2.1 Introduction

We are led to systems of differential equations of the form

$$\frac{dx_i}{dt} = g_i(x_1, x_2, \dots, x_N), \quad x_i(0) = c_i, \quad i = 1, 2, \dots, N, \quad (12)$$

either as limiting forms of the class of difference equations already considered or as direct consequences of postulating that the rates of change of each state variable depend only on the current values of the state vector $x(t)$.

In the following sections, we wish to indicate connections between the solutions of equations of this nature, the preceding results, and functional equations in general.

2.2 Causality and Functional Equations

Uniqueness of solution, which is to say causality, enables us to derive a fundamental functional equation. Write

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

and

$$x(t) = f(c, t),$$

indicating explicitly the dependence on the initial value c .

Then, as before, assuming uniqueness, we obtain the fundamental relation

$$f(c, t+s) = f[f(c, t), s], \quad (13)$$

for $s, t \geq 0$, and the values of c for which a solution of Eq. 12 can be continued over time.

The connection with iteration is now readily established. The value of the solution of Eq. 12 at $t = 1$ is a vector function of c ,

$$u(1) = \phi(c)$$

It follows that

$$u(2) = \phi[\phi(c)] = \phi^{(2)}(c),$$

and, generally,

$$u(n) = \phi^{(n)}(c)$$

If

$$g(u) = a_1 u + a_2 u^2 + \dots,$$

for $|u| \leq c_1$, it is clear that $\phi(c)$ is analytic around the origin,

$$\phi(c) = k_1 c + k_2 c^2 + \dots$$

Conversely, can every such one-dimensional analytic transformation be obtained from a differential equation of the type

$$\frac{du}{dt} = g(u) \quad u(0) = c?$$

For the case where $|k_1| < 1$, we can proceed as follows. Let

$$I(c) = \lim_{n \rightarrow \infty} \frac{\phi^{(n)}(c)}{k_1^n}. \quad (14)$$

and write

$$f(c, t) = h^{-1}[k_1^{-1}h(c)] \quad (15)$$

Using the fact that f satisfies Eq 13, we have

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial s} [f(c, t), s] \Big|_{s=0} = g[f(c, t)],$$

where $g(u)$ is defined by

$$g(u) = \frac{\partial f(u, t)}{\partial t} \Big|_{t=0}$$

This yields $g(u)$ explicitly in terms of $h(c)$, Eq 14, if we use the representation of Eq 15. The multidimensional analog is readily obtained

2.3 Functional Equation for Exponential

The case where $g(u) = au$ leads to an interesting result. Write the solution of the linear differential equation

$$\frac{du}{dt} = au, \quad u(0) = c$$

in the form $u = E(t)c$. The basic functional equation of Eq 13 yields the relation

$$E(t+s)c = E(s)[E(t)c]$$

for all c , and therefore

$$E(t+s) = E(s)E(t), \quad (16)$$

which is the fundamental property of the exponential function $E(t) = e^{at}$.

Similarly, defining the matrix exponential e^{At} as the solution of the matrix differential equation

$$\frac{dX}{dt} = AX, \quad X(0) = I$$

(where I is the identity matrix), we obtain the functional equation

$$e^{A(t+s)} = e^{At}e^{As},$$

for all t and s . However, unlike the scalar case, $e^{At}e^{Bt} = e^{Bt}e^{At}$ if and only if $AB = BA$, see Bellman (1960) for further discussion of the matrix exponential. For general operators A , a study of the exponential leads to the domain of semigroups (Hille & Phillips, 1948).

2.4 Cauchy's Problem

Cauchy was the first to ask whether or not Eq. 16 uniquely characterizes the exponential function. We are thus led to study the functional equation

$$u(t+s) = u(t)u(s)$$

as an equation for an unknown function $u(t)$. As our subsequent discussion will show, the problem is more sophisticated than we might imagine.

2.5 The Linear Function

Taking logarithms in Eq. 16, we are led to the equation

$$F(t+s) = F(t) + F(s), \quad (17)$$

assumed to hold for $-\infty < s, t < \infty$, for $F(t) = \log E(t)$. Hence it is sufficient at the moment to examine when the solution of Eq. 17 is given by $F(t) = kt$ for some constant k .

To make the problem precise, we must define the class of functions under consideration. Suppose, for example, that we ask for all solutions which possess derivatives for all finite t . Then, differentiating with respect to t and then with respect to s , we obtain the two equations

$$F'(t+s) = F'(t),$$

$$F'(t+s) = F'(s)$$

A standard device in the study of functional equations is to employ differentiation to reduce them to differential equations. Thus, and this is also a standard step in all such procedures,

$$F'(t) = F'(s) = k,$$

a constant. It follows that $F(t) = kt + b$. Since $F(0) = 0$, from Eq. 17, we have $b = 0$.

Despite this very simple and satisfying solution, many difficult questions remain. Suppose that we require only that $F(t)$ be continuous, or Lebesgue integrable, or merely defined for each t . In the first two cases, the solution remains the linear function. If, however, we admit nonmeasurable functions and the axiom of choice, then solutions of Eq. 17 can be constructed which are not of the form $F(t) = kt$.

Many ingenious techniques have been devised to treat Eq. 17 in the scalar case, in the vector case, and in the matrix case. See Aczél (1961).

for a very detailed and lucid discussion of these matters and for many further references, for a brief discussion of the matrix case, see Bellman (1960)

The solutions of Eq 17 which are nonlinear are quite weird and cannot be characterized simply Their existence, as noted previously, depends upon an acceptance of the axiom of choice

2 6 Associated Functional Equations

Starting with Eq 17, we are led to the equations

$$\begin{aligned}F_1(t+s) &= F_1(t)F_1(s), & (F_1 &= e^{it}), \\F_2(ts) &= F_2(t) + F_2(s), & (F_2 &= \log t), \\F_3(ts) &= F_3(t)F_3(s), & (F_3 &= t^k),\end{aligned}$$

all equivalent, under various conditions of regularity, to Eq 17 upon changes of dependent and independent variable These equations arise frequently in mathematical analysis and applications Once again, the reader is referred to Aczel (1961)

The matrix equation

$$F(AB) = F(A)F(B)$$

is quite interesting since under various assumptions it characterizes the determinant of A , see Aczel (1961)

2 7 General Functional Equations

It is clear that we have discovered a new game Start with any elementary functions, say $\sin t$ and $\cos t$, and write down their addition theorems

$$\begin{aligned}S(t+s) &= S(t)C(s) + C(t)S(s), \\C(t+s) &= C(t)C(s) - S(t)S(s)\end{aligned}$$

Under what conditions do these relations uniquely determine the original functions?

Continuing in this vein, we may ask for the determination of a function $f(t)$ satisfying a given polynomial relation

$$P[f(t), f(s), f(t+s)] = 0$$

The general problem is quite difficult since we enter the domain of elliptic and Abelian functions, following the lead of Weierstrass, see Rausenberger

(1884) and Weierstrass and Schwarz (1893) A number of particular equations of this type are discussed in Aczel (1961)

2.8 Approximate Functional Equations

A fundamental class of problems in analysis and mathematical physics is that associated with the concept of "stability" In verbal terms, the question may be stated in the following form "If the equation is altered slightly, are the solutions altered slightly?"

For example, if we have a polynomial equation

$$r^n + a_1 r^{n-1} + \dots + a_n = 0,$$

it is not difficult to show that small changes in the coefficients produce correspondingly small changes in the roots In other words, the roots are continuous functions of the coefficients

If we have a differential equation, such as

$$\frac{d^2 u}{dt^2} + a(t)u = 0,$$

it is a much more difficult problem to determine when small changes in the properties of $a(t)$ produce correspondingly small changes in the properties of u , see Bellman (1957)

In general, in scientific investigations, it is the study of inequalities that is far more important than the study of equalities Given an equation $F(u) = 0$ with a solution u , we want to know when $|F(t)| \leq \epsilon$, where ϵ is a small quantity, implies $t \cong u$ These problems are often quite difficult

In connection with the functional equations we have been discussing in the previous section, we can ask whether the inequality

$$|f(x+y) - f(x) - f(y)| \leq \epsilon$$

for $-\infty < x, y < \infty$ implies that $|f(x) - kx| \leq \delta$ for some constant k For results of this nature, see Ulam and Hyers (1952) In this case the answer is affirmative

2.9 Erdos' Functional Equation

Alternatively, we can preserve the equality in the functional equation and restrict the set of x and y for which the equation holds For example, suppose that we assume that $f(n)$ is defined only for $n = 1, 2, \dots$ and that

$$f(mn) = f(m)f(n)$$

if m and n are relatively prime. It is not true, in general, that $f(n) = n^k$, as in Sec 2.6

Number-theoretic functions satisfying the preceding equation are called *multiplicative*. For example, $d(n)$, the number of divisors of an integer n , is a multiplicative function. If we add the condition that $f(n+1) > f(n)$, then it is true that $f(n) = n^k$ for some constant k , see Erdos (1946).

The problem of determining when a polynomial equation of the form

$$P[f_1(n), f_2(n), \dots, f_n(n)] = 0$$

can hold, with each $f_i(n)$ a multiplicative number-theoretic function, has also been studied (Bellman & Shapiro, 1948).

2.10 A Limit Theorem

If $u(x+y) = u(x) + u(y)$ for $x, y \geq 0$, and $u(x)$ is continuous in x , we know that $u(x) = kx$ for some constant k . If this equation holds only for $x, y = 0, 1, 2, \dots$, we can draw the same conclusion.

In many investigations, we can only deduce that

$$u(m+n) \leq u(m) + u(n)$$

for $m, n = 0, 1, 2, \dots$. Nevertheless, we can assert that $u(n)$ acts like kn for large n in the sense that

$$\lim_{n \rightarrow \infty} \frac{u(n)}{n} = k$$

exists, see Polya and Szego (1945).

3 PROBABILITY THEORY

3.1 Preliminary Remarks

The theory of probability is a breeding ground for functional equations. In very natural ways, investigations of utilities, means, laws of probability theory, etc., pursued in axiomatic fashion, lead to the question of determining all acceptable solutions of various classes of functional equations (Aczél, 1961, Luce, 1958, 1959a, b, Marcus, 1962, Richter, 1952a, b, c, 1954, Shannon & Weaver, 1949). Many additional references may be found in Aczél (1961).

Many areas in modern probability theory and its applications can be considered to pertain to the iteration of stochastic transformations, see Bharucha-Reid (1960) and Harris (1963). In particular, we most frequently encounter the problem of describing the result of a large number of random effects, which is to say, of determining the limiting distribution of

$$\begin{aligned} s_n &= x_1 + x_2 + \dots + x_n \\ &= s_{n-1} + x_n, \end{aligned}$$

where the x_i are independent random variables with given distribution functions. More general stochastic iteration processes are treated in Harris (1963), Dvoretzky (1956), and Bellman (1964).

Interesting functional equations arise in the following fashion. Let x be a random variable with the density function $g(x)$ and y a random variable with the density function $h(y)$. Then $z = x + y$ has the density function $k(z)$, where

$$\begin{aligned} k(z) &= \int_{-\infty}^{\infty} g(z-y)h(y) dy \\ &= \int_{-\infty}^{\infty} h(z-y)g(y) dy \end{aligned}$$

Suppose that g and h are both members of the same family of functions, that is, there exists a function f of two variables such that $g(x) = f(x, a)$, $h(y) = f(y, b)$. When is $k(z)$ also a member of this family, say $k(z) = f(z, r(a, b))$?

For example, if x and y have Gaussian density functions

$$g(x) = \frac{e^{-(x-a_1)^2/2k_1}}{\sqrt{2\pi k_1}}$$

and

$$h(y) = \frac{e^{-(y-a_2)^2/2k_2}}{\sqrt{2\pi k_2}},$$

then the random variable $z = x + y$ also has a Gaussian density function. Under suitable assumptions, the converse holds. For example, Titchmarsh (1937) showed that the equation

$$\frac{1}{c} f\left(\frac{x}{c}\right) = \frac{1}{ab} \int_{-\infty}^{\infty} f\left(\frac{y}{a}\right) f\left(\frac{x-b}{b}\right) dy$$

under the assumptions

$$f(x) \geq 0, \quad \int_{-\infty}^{\infty} f(x) dx < \infty, \quad \int_{-\infty}^{\infty} x^2 f(x) dx < \infty,$$

leads to the conclusion that $f(x) = e^{-x^2/2k}/\sqrt{2\pi k}$, with $c^2 = a^2 + b^2$. A deeper study of this question leads to the theory of infinitely divisible distributions (Gnedenko, 1962)

3.2 Markov Processes

Both as a preparation for the "learning" and adaptive processes we shall discuss subsequently, and as a further illustration of the way in which probability theory leads to functional equations, let us discuss some simple Markov processes

Consider a process in which a system S may be in one of N different states, which we designate by the numbers $1, 2, \dots, N$, at any of the discrete times $t = 0, 1, 2, \dots$. At time t , there is a probability a_{ij} that a system in state i will be transformed into a system in state j . The matrix $A(t) = (a_{ij})$, $i, j = 1, 2, \dots, N$, is called the *transition matrix*. For simplicity, let us consider the stationary case, for which the transition matrix is constant

It is easily seen that a Markov process is an elegant device for handling on a phenomenological level systems that we do not understand well enough to treat by means of differential or difference equations. For a systematic and very readable account of the application of these techniques to the domains of mathematical physics and biology, see Bharucha-Reid (1960)

Let $u_i(t)$ be the probability that a system initially in state j is in state i at time t , $j = 1, 2, \dots, N$. Letting one time interval go by, we readily see that

$$u_i(t) = \sum_{k=1}^N a_{ik} u_k(t-1), \quad t = 1, 2, \dots,$$

with $u_i(0) = \delta_{ij}$

Observe that the evolution of this Markov process is equivalent to the iteration of the matrix A . Once again, the study of the time-history of a system introduces iteration

If we consider the case where we iterate random matrices, say

$$X_1 = Z_1 Z_{1-1} \quad Z_1 Z_0, \quad Z_0 = X,$$

where the Z_i are independent random matrices, we obtain other types of interesting functional equations. For example, if

$$Z = \begin{cases} A & \text{with probability } p, \\ B & \text{with probability } 1 - p, \end{cases}$$

setting

$$f_N(X) = e^{q(X_N)},$$

we have

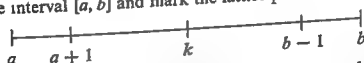
$$f_N(X) = pf_{N-1}(AX) + (1-p)f_{N-1}(BX), \quad f_0(X) = X$$

The study of the limiting distribution of the values of the elements of X_N is quite difficult, see Bellman (1954), and Kesten and Furstenberg (1960), for a generalization of Markovian processes using positive definite matrices as probabilities, see Bellman (1953a)

For an introduction to the theory of Markov chains, see Kemeny and Snell (1960)

3 3 Random Walks

A most interesting type of Markov process is the following Suppose we consider the interval $[a, b]$ and mark the lattice points



and assume that a particle at an arbitrary point k has a probability p_k of moving one step to the left and a probability $1 - p_k$ of moving one step to the right in a unit time

Starting at k , we may want to know the expected time to get to a boundary, the probability of arriving at a before arriving at b , etc. Processes of this nature have been extensively studied, see Bharucha Reid (1960) and Feller (1957)

To treat the second question, we let

u_k = the probability of arriving at a before arriving at b ,

for $a < k < b$. Then, it is easy to see that

$$u_k = p_k u_{k-1} + (1 - p_k) u_{k+1}, \quad (18)$$

with $u_a = 1$, $u_b = 0$

An alternate approach to problems of this nature, based on different types of functional equations, may be obtained from the theory of invariant imbedding (Bellman & Kalaba, 1956, Bellman, Kalaba, & Wing, 1960)

3 4 Learning Processes

The last two sections set the stage for a sketch of one of the most important applications of mathematics to psychology, the theory of learning

processes Functional equations are basic in this area, see, for example, Bellman, Harris and Shapiro (1953), Bush and Estes (1959), Bush and Mosteller (1955), Kanai (1962), Karlin (1953), Luce (1959a, 1964)

To see how learning processes lead to functional equations under suitable hypotheses, consider the random walk model in Sec 3 3 and suppose that experience modifies behavior in such a way that a move to the right or to the left changes the subsequent probability of moving to the right or left

Taking only the case in which the p_k are independent of k and all equal to p , let

$T_1(p)$ = the probability of moving to the left if the preceding move has been to the left,

$T_2(p)$ = the probability of moving to the left if the preceding move has been to the right

In place of the function u_k of Sec 3 3, we introduce the function

$u_k(p)$ = the probability of arriving at a before arriving at b , starting at k with a probability p of moving to the left

The same reasoning as before yields the recurrence functional equation

$$u_k(p) = pu_{k-1}[T_1(p)] + (1 - p)u_{k-1}[T_2(p)] \quad (19)$$

Equations of this type, as well as far more complicated ones arising from somewhat more complicated psychological models, are discussed in the papers and books cited previously The analytic and computational aspects of equations of this nature are surprisingly complex

4 DYNAMIC PROGRAMMING AND MARKOVIAN DECISION PROCESSES

4 1 Dynamic Programming

A further extension of classical Markovian processes is the class of Markovian decision processes, a particular and important type of dynamic programming process In place of the recurrence relation of Eqs 18 or 19, we have equations of the form

$$u_k(p) = \max_q \{g(p, q) + p(q)u_{k-1}[T_1(p, q)] + [1 - p(q)]u_{k-1}[T_2(p, q)]\} \quad (20)$$

These equations arise in a very natural way in equipment and replacement theory, in inventory, in control theory, sequential analysis, throughout

operations research, etc. See Bellman (1957, 1961) and Bellman and Dreyfus (1962)

Since dynamic programming will be required for a discussion of how the functional equations associated with adaptive processes arise, let us briefly review some of the fundamental concepts of the theory

Dynamic programming is a mathematical theory of multistage decision processes. As opposed to classical analysis, the emphasis is on policies, a generalization and extension of the fundamental engineering concept of feedback control. A *policy*, quite simply, is a rule for determining what you do in terms of where you are and what you know. In these terms, it is clear that dynamic programming can be used in the study of a number of psychological processes.

The determination of optimal policies, those yielding maximum returns or most desirable objectives, is facilitated by the following intuitive principle of optimality.

PRINCIPLE OF OPTIMALITY *An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.*

To obtain functional equations which enable us to study the analytic structure of optimal policies and to derive computational algorithms, we proceed in the following fashion. Let S be a system which at time $t = 0$ is specified by a point p in generalized phase space, the state vector. A decision is equivalent to a transformation $T(p, q)$, where q is the decision variable. As a result of specifying the value of q , a return of $g(p, q)$ is obtained. The problem is to make N consecutive decisions, q_1, q_2, \dots, q_N , which maximize the over-all return.

Let

$f_N(p)$ = the return from an N -stage process using an optimal policy, starting in state p

Then the principle of optimality yields the following functional equation

$$f_N(p) = \max_q \{g(p, q) + f_{N-1}[T(p, q)]\}, \quad N \geq 2,$$

with

$$f_1(p) = \max_q g(p, q)$$

For extensive discussions of equations of this nature, together with applications across the scientific board and with numerical illustrations, see the books by Bellman (1957, 1961) and Bellman and Dreyfus (1962).

In particular, let us note that the theory of dynamic programming has particular relevance to the study of feedback control processes. With its

aid, we can formulate the calculus of variations as a semigroup process involving decisions at each stage, and we can study stochastic and adaptive variational processes (Bellman, 1961) and Bellman and Dreyfus (1962)

4.2 Allocation Processes and the Maximum Transform

As an example of the use of these methods, let us consider the problem of maximizing the function

$$R_N = g_1(x_1) + g_2(x_2) + \dots + g_N(x_N)$$

under the constraints

$$x_1 + x_2 + \dots + x_N = x, \quad x_i > 0, \quad i = 1, 2, \dots, N$$

Considering this as a multistage process, write

$$f_N(x) = \max_{x_i} R_N \quad (21)$$

Then

$$\begin{aligned} f_N(x) &= \max_{0 < x_N < x} [g_N(x_N) + f_{N-1}(x - x_N)], \quad N \geq 2, \\ f_1(x) &= g_1(x) \end{aligned} \quad (22)$$

Remarkably, this equation can be solved by means of transform techniques. Write

$$F(y) = \max_{x \geq 0} [f(x) - xy] = M(f),$$

the maximum transform. Then, under mild constraints on $f(x)$,

$$f(x) = \min_{y \geq 0} [F(y) + xy]$$

Furthermore,

$$M\left\{\max_{0 \leq x_1 \leq x} [g(x_1) + h(x - x_1)]\right\} = M(g) + M(h)$$

We thus have an analog to the Laplace transform, particularly with reference to its ability to disentangle convolutions.

Applying the maximum transform to Eq. 22, we have

$$M(f_N) = M(g_N) + M(f_{N-1})$$

Hence

$$\begin{aligned} M(f_N) &= \sum_{k=1}^N M(g_k), \\ f_N &= M^{-1} \left[\sum_{k=1}^N M(g_k) \right] \end{aligned}$$

For detailed discussions, see Bellman and Karush (1961, 1962a, b, 1963), where the connection with duality theory and convexity is discussed. Further results are found there.

4.3 Stochastic Multistage Decision Processes

Returning to the abstract multistage decision process discussed in Sec. 4.1, let us suppose that both the return, $g(p, q, r)$, and the transformation, $T(p, q, r)$, are stochastic, depending on the random variable r with known distribution function. In this case, if we let

$$f_v(p) = \max_q \exp R_v,$$

we obtain the functional equation

$$f_{\lambda}(p) = \max_q \left(\int \{g(p, q, r) + f_{\lambda-1}[T(p, q, r)]\} dG(r) \right),$$

$$f_1(p) = \max_q \int g(p, q, r) dG(r)$$

In the same fashion, we can treat multistage games (Bellman, 1957)

4.4 Adaptive Processes

Let us now introduce a further stage of complexity by assuming that the process contains a number of unknown features. For example, the state may not be completely known, the effect of a decision may not be known, the return may not be known, and even the overall goal may not be known initially.

In a situation of this type, it is clear that learning and adaptation play basic roles. Starting with certain initial estimates for the unknown quantities and qualities, we proceed as if these estimates were correct and then revise the estimates on the basis of experience. The theory of dynamic programming is ideally designed to treat processes of this nature. The generalized state variable now contains both the information concerning the physical state and the best estimates to date. Applications of this approach have been made to mathematical economics, operations research, statistics, control theory, biology, and psychiatry. See Bellman, Friend, and Kurland (1961), Bellman and Kalaba (1959a, b, c), Flood (1954a, b), and Tou (1963).

To illustrate the way problems of this nature are approached, let us consider the following simple adaptive process. Suppose we are shown

two slot machines, one with a known fixed probability p of yielding a return of 1 unit and the other with a fixed, but unknown, probability of yielding a return of 1 unit. Given N opportunities to try the machines, how should we apportion our trials so as to maximize the expected gain?

Let r represent the fixed, but unknown, probability of success on the second machine, and $dG(r)$ represent an a priori probability distribution function for r . After m successes and n failures on the second machine, we agree to transform $dG(r)$ into an a posteriori probability distribution function

$$dG_{m,n}(r) = \frac{r^m(1-r)^n dG(r)}{\int_0^1 r^m(1-r)^n dG(r)}$$

The expected probability of success after m successes and n failures is then

$$p_{m,n} = \frac{\int_0^1 r^{m+1}(1-r)^n dG(r)}{\int_0^1 r^m(1-r)^n dG(r)}$$

Let

$f_{m,n}(N)$ = expected return using an optimal policy obtained from N remaining trials, having previously observed m successes and n failures

Then the principle of optimality yields the functional equation

$$f_{m,n}(N) = \max \left[\begin{array}{l} p_{m,n}[1 + f_{m+1,n}(N-1)] \\ + (1 - p_{m,n})f_{m,n+1}(N-1), pN \end{array} \right]$$

This process has been extensively discussed, see for example, Bellman (1956) and Karlin, Bradt, and Johnson (1956)

References

- Aczél, J. Some general methods in the theory of functional equations *Uspek Math Nauk*, 1951, 6, 3-68
 Aczél, J. A solution of some problems of K. Borsuk and L. Jánossy *Acta Phys Acad Sci Hung*, 1955, 4, 351-362
 Aczél, J. *Vorlesungen über functional Gleichungen und ihre Anwendungen* Basel Birkhäuser Verlag, 1961
 Aczél, J., & Daroczy, Z. Charakterisierung der Entropien positiver Ordnung und der Shannonschen Entropie *Acta Math Acad Sci Hung*, 1963, 14, 95-121

REFERENCES

- Bellman, R. A note on scalar functions of matrices *Amer Math Mon*, 1952, 59, 391 (a)
- Bellman, R. The iteration of power series in two variables *Duke Math J*, 1952, 38, 716-719 (b)
- Bellman, R. A generalization of classical probability theory—I Markov chains *Proc Nat Acad Sci USA*, 1953, 39, 1075-1077 (a)
- Bellman, R. *Stability theory of differential equations* New York McGraw-Hill, 1953 (b)
- Bellman, R. Limit theorems for noncommutative operations—I *Duke Math J*, 1954, 21, 491-500
- Bellman, R. A problem in the sequential design of experiments *Sankhya*, 1956, 16, 221-229
- Bellman, R. *Dynamic programming* Princeton Princeton Univer Press, 1957
- Bellman, R. *Introduction to matrix analysis* New York McGraw-Hill, 1960
- Bellman, R. *Adaptive control processes A guided tour* Princeton Princeton Univer Press, 1961
- Bellman, R. Stochastic transformations and functional equations In R. Bellman (Ed.), *Proceedings of the Symposia in Applied Mathematics*, 17 Providence Rhode Island Am Math Soc, 1964
- Bellman, R., & Brock, P. On the concepts of a problem and problem solving *Amer Math Mon*, 1960, 67, 119-134
- Bellman, R., & Dreyfus, S. *Applied dynamic programming* Princeton Princeton Univer Press, 1962
- Bellman, R., Friend, M. B., & Kurland, L. *Psychiatric interviewing and multistage decision processes of adaptive type* The RAND Corporation, RM-3732 NIH, 1961
- Bellman, R., Harris, T. E., & Shapiro, H. N. Some functional equations occurring in the theory of decision processes The RAND Corporation, P 382, 1953
- Bellman, R., & Kalaba, R. On the principle of invariant imbedding and propagation through inhomogeneous media *Proc Nat Acad Sci USA*, 1956, 42, 629-632
- Bellman, R., & Kalaba, R. Random walk, scattering and invariant imbedding—I One-dimensional case *Proc Nat Acad Sci USA*, 1957, 43, 930-933
- Bellman, R., & Kalaba, R. A mathematical theory of adaptive control processes *Proc Nat Acad Sci USA*, 1959, 45, 1288-1290 (a)
- Bellman, R., & Kalaba, R. Functional equations in adaptive processes and random transmission *Trans International Symposium on Circuit and Information Theory*, 1959, 271-282 (b)
- Bellman, R., & Kalaba, R. On adaptive control processes *IRE National Convention Record*, 1959, part 4, 3-11 (c)
- Bellman, R., & Kalaba, R. Interrupted stochastic control processes *Information and Control*, 1961, 4, 346-349
- Bellman, R., & Kalaba, R. *Dynamic programming and modern control theory* New York McGraw Hill, 1964
- Bellman, R., Kalaba, R., & Prestrud, M. *Invariant imbedding and radiative transfer in slabs of finite thickness* New York American Elsevier, 1963
- Bellman, R., Kalaba, R., & Wing, G. M. Invariant imbedding and mathematical physics—I Particle processes *J Math Phys*, 1960, 1, 280-308
- Bellman, R., & Karush, W. On a new functional transform in analysis the maximum transform *Bull Amer Math Soc*, 1961, 67, 501-503
- Bellman, R., & Karush, W. Mathematical programming and the maximum transform. *J Soc Indust Appl Math*, 1962, 10, 550-567 (a)

- Bellman, R., & Karush, W. On the maximum transform and semigroups of transformations *Bull Amer Math Soc*, 1962, 68, 516-518 (b)
- Bellman, R., & Karush, W. On the maximum transform *J Math Anal and Appl*, 1963, 6, 67-74
- Bellman, R., & Shapiro, H. N. The algebraic independence of arithmetic functions—I. Multiplicative functions *Duke Math J*, 1948, 15, 229-235
- Bharucha-Reid, A. T. *Elements of the theory of Markov processes and their applications* New York McGraw-Hill, 1960
- Bush, R. R., & Estes, W. K. *Studies in mathematical learning theory* Stanford Stanford Univer Press, 1959
- Bush, R. R., & Mosteller, F. *Stochastic models for learning* New York Wiley, 1955
- Dvoretzky, A. On stochastic approximation In J. Neyman (Ed.), *Proceedings of the third Berkeley symposium on mathematical statistics and probability*, Vol. 1, Berkeley Univer of Calif Press, 1956 Pp 39-56
- Erdos P. On the distribution function of additive functions *Ann Math*, 1946, 47, 1-20
- Feller, W. *An introduction to probability theory and its applications* New York Wiley, 1957
- Flood, M. A stochastic model for social interaction *Trans N Y Acad Sci*, Series II, 1954, 16, 202-205 (a)
- Flood, M. Game learning theory and some decision making experiments In R. M. Thrall, C. H. Coombs, & R. L. Davis (Eds.), *Decision Processes* New York Wiley, 1954 Pp 139-158 (b)
- Gnedenko, B. V. *The theory of probability* New York Chelsea, 1962
- Goldberg, S. *Introduction to difference equations* Englewood Cliffs, New Jersey Prentice Hall, 1958
- Hadamard, J. Two works on iteration and related questions *Bull Amer Math Soc*, 1944, 50, 67-75
- Harris T. E. *Branching Processes* Berlin Springer, 1963
- Hille, E., & Phillips, R. *Functional analysis and semi groups* New York Amer Math Soc, 1948
- Kanal, L. A functional equation analysis of two learning models *Psychometrika*, 1962, 27, 89-104 Reprinted in R. D. Luce, R. R. Bush, & E. Galanter (Eds.) *Readings in mathematical psychology, I* New York Wiley, 1963 Pp 360-375
- Karlin, S. Some random walks arising in learning models, I *Pacific J Math*, 1953, 3, 725-756 Reprinted in R. D. Luce, R. R. Bush, & E. Galanter (Eds.) *Readings in mathematical psychology, I* New York Wiley, 1963 Pp 381-403
- Karlin, S., Bradt, R., & Johnson, S. On sequential designs for maximizing the sum of n observations *Ann Math Stat*, 1956, 27, 1061-1074
- Kemeny, J., & Snell, L. *Finite Markov chains* Princeton Van Nostrand, 1960
- Kesten, H., & Furstenberg, H. Products of random matrices *Ann Math Stat*, 1960, 31, 457-469
- Koenigs, M. G. Nouvelles recherches sur les équations fonctionnelles *Ann Sci de l'école Normale Sup*, 1885, 2, 385-404
- Luce, R. D. A probabilistic theory of utility *Econometrica*, 1958, 26, 193-224
- Luce, R. D. *Individual choice behavior* New York Wiley, 1959, especially pp 29, 41-43, 47-51, 100 (a)
- Luce, R. D. On the possible psychophysical laws *Psychol Rev*, 1959, 66, 81-95 (b) Reprinted in R. D. Luce, R. R. Bush, & E. Galanter (Eds.) *Readings in mathematical psychology, I* New York Wiley, 1963 Pp 69-83

REFERENCES

- Luce, R D Some one-parameter families of commutative learning operators In R C Atkinson (Ed), *Studies in mathematical psychology* Stanford Stanford Univer Press, 1964 Pp 380-398
- Luce, R D, & Edwards, W The derivation of subjective scales from just noticeable differences *Psychol Rev*, 1958, 65, 222-237
- Luce, R D, & Galanter, E Discrimination In R D Luce, R R Bush, & E Galanter (Eds), *Handbook of mathematical psychology* New York Wiley, 1963 Pp 191-243
- Marcus, M *An introduction to information theory* The RAND Corporation, P-2609, 1962
- Montel, P *Leçons sur les récurrences et leurs applications* Paris Gauthier-Villars, 1957
- Polya, G, & Szego, G *Aufgaben und Lehrsätze aus der Analysis* New York Dover, 1945
- Prestrud, M Hierarchic algebra *Math Mag*, 1963, 36, 43-53
- Rausenberger, O *Lehrbuch der Theorie der periodischen funktionen* Leipzig Teubner, 1884
- Richter, H Zur Grundlegung der Wahrscheinlichkeitstheorie, I *Math Annalen*, 1952, 125, 129-139 (a)
- Richter, H Zur Grundlegung der Wahrscheinlichkeitstheorie, II *Math Annalen*, 1952, 125, 223-234 (b)
- Richter, H Zur Grundlegung der Wahrscheinlichkeitstheorie, III *Math Annalen*, 1952, 125, 335-343 (c)
- Richter, H Zur Grundlegung der Wahrscheinlichkeitstheorie, IV *Math Annalen*, 1953, 126, 362-374
- Richter, H Zur Grundlegung der Wahrscheinlichkeitstheorie, V *Math Annalen*, 1954, 128, 305-339
- Schroder, E Über eine eigentümliche Bestimmung einer Funktion durch formale Anforderungen *J reine angew Math*, 1881, 90, 189-220
- Shannon, C E, & Weaver, W *The mathematical theory of communication* Urbana: Univer of Illinois Press, 1949
- Titchmarsh, E C *Fourier integrals* Oxford Clarendon Press, 1937.
- Tou, J T *Optimum design of digital control systems* New York Academic Press, 1963
- Ulam, S, & Hyers, D Approximately convex functions *Bull Amer Math Soc*, 1952, 3, 821-828
- Weierstrass, K T W, & Schwarz, H A (Eds) *Formeln und Lehrsätze zum Gebrauche der elliptischen Funktionen* Berlin J Springer, 1893

Author Index

Page numbers in boldface indicate bibliography references

- Abel, N H, 494-495, 500
 Abelson, R M, 335, 394-395, 402
 Aczel, J, 272, 402, 491, 499-502, 510
 Adams, E W, 258, 268-269, 273, 281, 309, 402
 Allais, M, 328, 402
 Andersen, E S, 434, 484
 Anderson, N H, 393, 402
 Anderson, T W, 484
 Armstrong, W E, 279, 402
 Arrow, K J, 258, 305, 402, 404-405, 407, 409
 Aspinall, E, 43
 Atkinson, R C, 202, 211, 247, 369-370, 378, 393-396, 403, 405, 407, 409, 513
 Audley, R J, 369, 403
 Aumann, R J, 290, 403

 Baratta, P, 321-326, 408
 Bartley, S H, 101, 111, 154
 Bauch, H, 83, 93
 Baumgardt, E, 108, 154
 Beardslee, D C, 308, 404
 Beck, J, 47, 93
 Becker, G M, 314, 325, 331, 335, 360-362, 397, 400-401, 403
 Bekesy, G von, 5, 24, 29-30, 32-36, 38, 40, 43, 52, 69-70, 73-80, 90, 93-94, 104, 154
 Bellman, R, 490, 492-494, 496, 498-503, 505-510, 511-512
 Bentham, J, 258, 403
 Beranek, L L, 11, 94
 Bernoulli, D, 282, 403
 Bezold, W von, 139
 Bharucha-Reid, A T, 503-505, 512
 Biddulph, R, 47-49, 96
 Birdsall, T G, 57, 94
 Birkhoff, G, 264, 403
 Blackwell, D, 299, 403
 Bliss J C, 104, 154
 Block, H D, 331, 334-336, 338, 351-352, 354, 358, 384, 403
 Bouman, M A, 106, 154
 Bourne, L E, 215-223, 246, 247
 Bower, G H, 209-210, 212-214, 231, 233-237, 240, 242-243, 245-246, 247, 369, 403
 Bradley, R A, 335, 402-403
 Bradt, R, 510, 512
 Brindley, G S, 108, 158
 Brock, P, 490, 511
 Brown, W R J, 130, 148, 150-151, 154
 Brownlee, O H, 384, 386, 407
 Brucke, E, 139
 Burch, J M, 115-119, 159
 Burke, C J, 172, 392-393, 403, 405
 Burnham, R W, 134, 154
 Bush, R R, 151, 154-159, 167-170, 176, 183, 188, 193, 202-203, 371, 403, 408-409, 425, 427, 439, 455, 457, 474, 484-485, 506, 512-513
 Campbell, F W, 108, 158
 Cantor, G, 263, 403
 Cauchy, A L, 499
 Chipman J S, 264, 267, 340, 385-386, 404
 Chomsky, N, 432
 Chung, K L, 434, 451, 461-462, 468, 470, 482, 484
 Churchman, C. W, 404
 Ciucu, G, 484
 Cohen, B P, 457-458, 484
 Cohen, J., 330, 404
 Cohen, R D, 108, 158
 Cole M, 393, 405

- Coombs, C H, 272-274, 279, 307-308, 328-330, 351, 386-390, 404, 406-407, 512
- Cornsweet, T N 106, 155
- Cotton, J W, 392, 404
- Cox, D R, 483, 484
- Criswell, J H, 403
- Crowe, S J, 34
- Crozier, W J, 101, 106-111, 155-156
- Caroczy, Z, 510
- Dartnall H J, 108, 156
- Davidson, D, 275, 279, 290, 292-294, 308, 314-320, 325, 330, 341, 380-382, 384, 401, 404
- Davis, H, 23, 25, 35, 47, 49-50, 66, 69, 94, 96
- Davis, R L, 404, 406-407, 512
- Davson, H, 156, 158
- Debreu, G, 267-268, 271-272, 336-337, 350, 400-401, 404
- DeFinetti B, 294, 296, 298, 306, 405
- DeGroot M H, 314, 320, 331, 335, 360-362, 397, 400-401, 403, 405
- DeLange, Dzn H, 112, 156
- Dembo, T, 304, 405
- deParma, J J 103, 157
- Deutsch, M, 301, 405
- deVries H, 106, 156
- Dobrusin, R L, 483, 484
- Dolbear, F T Jr, 320, 401-402, 405
- Doob, J L, 414, 437, 460, 476-479, 483, 484
- Dreyfus, S, 494, 507-508, 511
- Dvoretzky, A 503, 512
- Edgeworth, F Y, 267, 405
- Edwards W, 249, 256, 258, 268, 293, 320, 324-325, 327-328, 375, 377, 380, 391, 393-395, 405-406, 495, 513
- Egan J P, 69, 94
- Ellis, A J, 95
- Emde, F, 31-32, 95
- Engler, J, 392, 405
- Erdos P, 501-502, 512
- Evans W K, 167, 170, 202, 211, 341, 369, 392-393, 403, 405, 425, 427, 454-455, 457, 474, 484, 506, 512
- Euler L 493
- Fagot, R F, 268-269, 273, 309, 402, 405
- Farnsworth, D, 140, 156
- Farnsworth, P R, 405
- Fechner, G T, 49, 80, 102, 120, 125, 129, 332, 334, 401, 495
- Feldman, A S, 43, 97
- Feldtkeller, R, 52, 57, 68, 83, 85, 94, 97
- Feller, W, 414, 435-436, 471, 484, 505, 512
- Ferry, E S, 111
- Field, H W, 154
- Fisher, I, 328, 405
- Fletcher, H, 25, 52, 66, 68, 70, 83, 94
- Flood, M 509, 512
- Flottorp, G, 52, 83, 97
- Ford, L R, Jr, 335, 405
- Fort, J G, 393, 395, 407
- Fouraker, L E, 305, 409
- Fourier, J B J, 7-8
- Franklin, W, 384, 386, 407
- Frey, A R, 21, 28, 95
- Friedman, M P, 393, 405
- Friend, M B, 509, 511
- Fuchs, W H T, 434, 484
- Furstenberg H, 505, 512
- Galambos R, 69, 94
- Galanter E, 47, 96, 151-152, 154-159, 193, 202-203, 327, 394-395, 406, 408-409, 485, 495, 512-513
- Gardner, R A, 392, 406
- Garner, W R, 56-57, 63, 90, 94
- Georgescu Roegen, N, 340, 406
- Gerlach, M W, 280-281, 406
- Ghurye, S, 403
- Gibson, K S, 135, 157
- Ginsberg, R, 213-214, 236-237, 246, 247
- Girshick, M A, 300, 403
- Glaze, J A, 315, 406
- Gnedenko, B V, 431, 484, 504, 512
- Goldberg, S, 492, 512
- Goldstein, D A, 394-395, 409
- Goldstein, M H, Jr, 91-92, 96
- Goodnow, J J, 394-395, 406
- Graham, C H, 115, 130, 156
- Granit, R, 108, 156
- Grant, D A, 392, 406

AUTHOR INDEX

- Green, D. M., 1, 57, 94
 Greenberg, M., 388, 404
 Greenwood, D. D., 77-78, 94
 Griffith, R. M., 323-326, 406
 Griswold, B. J., 390, 406
 Guilbaud, G., 340, 406
 Guild, S. R., 34
 Gulliksen, H., 335, 406
 Guthrie, H. M., 210

 Hadamard, J., 494, 512
 Hagins, W. A., 108, 158
 Hake, H. W., 69, 94, 392, 406
 Hall, V. E., 154
 Hall, J. L., 1
 Hansel, C. E. M., 330, 404
 Harlow, H. F., 233, 247
 Harlow, M. K., 247
 Harris, T. E., 464, 485, 494, 496, 503, 506, 511-512
 Hartline, H. K., 73, 95-96, 104, 156
 Hartridge, H., 140, 156
 Hausner, M., 290, 406
 Hawkins, J. E., Jr., 70-71, 95
 Hecht, S., 101, 105-108, 156
 Hellman, R. P., 57, 64, 81-82, 95, 97
 Hellyer, S., 392, 403
 Helmholtz, H. L. F. von, 38, 52, 95, 120-125, 129-130, 156
 Helson, H., 135, 156
 Hering, E., 114, 156
 Herstein, I. N., 285-288, 406
 Hille, E., 494, 498, 512
 Hoagland, H., 108, 155
 Hoefding, W., 403
 Holman, E. W., 161, 338
 Holway, A. H., 106-107, 110, 155-156
 Hooke, R., 15-16
 Hornseth, J. P., 392, 406
 Huggins, W. H., 69, 95
 Hurst, P. M., 273, 276, 307-309, 406
 Hurvich, L. M., 106, 110, 136-137, 139-142, 146, 156-157
 Hurwicz, L., 384, 386, 407
 Hyers, D., 501, 513
 Hyman, R., 392, 406

 Jahn, T. L., 106, 157
 Jahnke, E., 31-32, 95
 Jameson, D., 136-137, 139-143, 146, 156-157
 Jarvik, M. E., 392, 406
 Jevons, W. S., 267
 Johnson, S., 510, 512
 Jones, R. A., 42-43, 95
 Judd, D. B., 114-115, 119, 128, 134-135, 151, 157-158
 Jung, R., 154, 157
 Jureen, L., 259, 410

 Kalaba, R., 494, 505, 509, 511
 Kanal, L., 193, 203, 506, 512
 Kanazawa, K., 151, 157
 Kanno, Y., 87, 95
 Karlin, S., 404-405, 407, 409, 455-456, 462, 472-473, 485, 506, 510, 512
 Karush, W., 509, 511-512
 Katsuki, Y., 69, 87-90, 95
 Katz, L., 393, 395, 407
 Keller, L., 393, 405
 Kelly, D. H., 101, 112-113, 157
 Kelly, K. L., 135, 157
 Kemeny, J. G., 439, 444, 446, 448, 452-453, 455, 466, 485, 505, 512
 Kesten, H., 505, 512
 Kiang, N. Y.-S., 91-92, 96
 Kinsler, L. E., 21, 28, 95
 Koch, S., 95, 156, 247
 Koenigs, M. G., 496, 512
 Kolmogorov, A. N., 431, 449, 451, 484-485
 Komorita, S. S., 303, 404
 Konig, A., 122, 140, 157
 Koopman, B. O., 298, 406
 Kornhuber, H., 154, 157
 Kraft, C. H., 296-297, 406
 Krantz, D. H., 161, 187
 Kries, J. von., 129, 157
 Kurland, L., 509, 511
 Kyburg, H. E., Jr., 294, 404-406

 Lamb, H., 25, 95
 Lamperti, J., 187, 191, 195, 203, 471, 475, 484
 Lawrence, M., 34, 96

 Indow, T., 151, 157
 Irwin, I. W., 249, 252-254, 364, 406

- Suydam, M M , 393, 395, 407
 Szego, G , 502, 513

 Tanner, W R , Jr , 57, 94
 Tasaki, I , 35, 59, 66, 69, 96
 ten Doesschate, J , 106, 154
 Terry, M E , 335, 403
 Theodorescu, R , 484
 Thomson, L C , 106, 159
 Thrall, R M , 404, 406-407, 512
 Thurstone, L L , 310, 332, 335, 337, 349, 383-384, 389, 409
 Titchmarsh, E C , 503, 513
 Toda, M , 325, 409
 Tonndorf, J , 35, 96
 Torgerson W S , 134, 151, 159
 Tornquist, L , 335, 409
 Tou, J T , 509, 513
 Trabasso, T R , 214, 231, 233-236, 240-243, 245-246, 247
 Tucker, A W , 409
 Tukey, J W , 268-270, 277, 333, 407

 Uchiyama, H , 69, 95
 Ulam, S , 501, 513
 Uzawa, H , 259, 409

 Valavanis Vail, S , 340, 409
 Verrillo, R T , 57, 64, 97
 Volkmann, J , 46-47, 96
 von Neumann, J , 281, 284-287, 290-291, 300, 302, 306, 310-312, 314, 409

 Walras, L , 267
 Walsh, K , 319, 330, 408-409

 Warren, J M , 231, 240, 247
 Watanabe, T , 69, 95
 Weale, R A , 108, 146, 159
 Weaver, W , 502, 513
 Weber, E H , 126, 129
 Weiterstrass, K T W , 500-501, 513
 Wever, E G , 27, 34, 38, 47, 55, 67-68, 96
 Whalen, R E , 393, 402
 White, S D , 66, 96
 Wiener, F M , 45-46, 96
 Wiener, N , 280, 410
 Wiesen, R A , 327, 408, 410
 Willens, R , 386
 Willner, D , 404
 Winet, M , 273-274, 277, 293, 409
 Wing, G M , 505, 511
 Wold, H , 259, 409
 Wolf, E , 106-111, 155
 Wright, H N , 65, 97
 Wright, W D , 115, 144, 146, 159
 Wyckoff, L B , Jr , 172-176, 203
 Wyszecki, G , 114, 119, 128, 134-135, 151, 157

 Young, T , 15, 120

 Zerrahn Wolf, G , 106-108, 110, 155
 Zewi, M , 108, 156
 Zinnes, J L , 341, 388, 404, 409
 Zwicker, E , 47-49, 51-52, 65, 68, 70, 80, 83, 85, 94, 97
 Zwislocki, J , 24, 27, 29, 31-32, 40-43, 46, 57-58, 60-66, 81-82, 95, 97

Subject Index

- Abel Schroeder equation, 494-495
- Abelian function, 500
- Absolute threshold, *see* Threshold
- Absorbing point, 189-193
 - see also* specific topics
- Absorption properties of terminal state, 242
- Absorption spectra, 124
- Abstract space process, 417
 - see also* Stochastic process
- Acoustic resonator, 13
- Acoustic transfer function, 22-23
- Actuarial model, 402
- Adaptation, *see* specific topics
- Adaptation rate, 216, 218
- Adaptive processes, 504, 507, 509
- Additive utility model, *see* Utility model
- Additivity of categories, 238-240
 - of cues, 222-223, 230-231, 239
 - irrelevant, 246
 - of luminance, 114-115
 - of stimuli, 5, 490
 - of utility, 259, 267-272, 293, 309
- Adjustment, method of, 46-47, 145-148
- Algebraic model(s), 256
 - see also* specific models
- Allocation processes, 508-509
- All or-none assumption, 58, 113, 242-243, 246
- α mixtures, 284-285
- Alternation of response theory, 111
- Amplitude, 6
 - decay of, 9
 - modulation of, 51-52
- Approach avoidance response, 369
- Archimedean axiom, 270, 285-286, 290
- Arc sine law, 434
- Aspect, 170
 - see also* Cue
- Asymptotic behavior, 164-166, 168, 174, 186-187, 189, 377-378, 396, 456, 495
- Asymptotic behavior, in animal experiments, 164-166, 169
 - in learning experiments, 255-256
 - and perfect learning, 184, 186-190, 201, 219
 - in prediction experiments, 391-393
 - in preference experiments, 255-256
 - in single process linear model, 186-187
- stable, 187
- uniqueness of, 187
- Asymptotic choice probability, 360
 - 367-368, 390-397
- Asymptotic conditioning, 168
- Asymptotic distribution, of beta model, 194-199
 - for experimenter controlled linear operators, 187-189
 - means of, 188
 - uniqueness of, 187
- Attending response, 172-175, 369
 - see also* Observing responses
- Attitude scaling, 268, 272
- Auction game, 321-322
- Audibility, threshold of, 3, 56-68
- Auditory nerve, *see* Neural activity, etc
- Auditory stimulus, 5, 21-23
- Auditory system, pathological changes in, 81
- Autoregression, 480
- Axiom(s), Archimedean, 270, 290
 - cancellation, 270, 272
 - choice, 152-153, 336-337, 339, 342, 367, 369, 374, 396-397
 - closure, 285
 - conditioning, 170, 209-211, 216-218, 224-225, 234-235, 243-245
 - see also* Conditioning axiom
 - of conjoint measurement, 269
 - connectivity, 260, 276, 290
 - continuity, 260

- Axiom(s), for equally spaced alternatives, 277-278
 for expected utility, 284-287
 for higher ordered metrics, 276-277
 linear, 426
 nonsatiety, 260
 of path independence, 176-178
 perceptual, 170
 for qualitative probability, 294-296
 see also Subjective probability
 response, 171, 210-211, 216, 225, 234
 see also Response axiom(s)
 Savage's, 298-299
 solution of equations, 270
 stimulus, 209-210, 216, 225, 230, 234, 246
 see also Stimulus axiom
 for stimulus sampling model, 209-210
 for subjective probability, 291-299
 transitivity, 260 309-310, 319
 for utility, *see* Utility models
 weak ordering, 260
- Backward equation, 465, 470-471
 Base for a topology, 266
 Basic event, 415
 Basic probabilities, 415-417
 as conditional probabilities 415
 for linear model, 425
 for pattern model, 427-428
 for stimulus sampling model, 427-428
 Basilar membrane, compliance of, 29-30
 Bayesian estimation schemes, 327
 Behavioral inconsistency, 331
 Bernoulli sequence, 171, 235, 242, 244
 Bernoullian utility, 258
 Bessel function, 31
 Beta model, 193-201, 371-372
 asymptotic properties of, 194-199
 with equal presentation probabilities, 199
 experimenter controlled, 200-201
 single process, 193-201
 subject-experimenter controlled 371-372, 375
 symmetric operators for, 199-200
- Betting behavior, 322-323
 see also Gambling experiment
- Bezold Brucke effect, 139
 Bilateral adjacent triple, 389
 Bilateral monopoly, 305
 Bilateral pair, 388
 Bilateral split triple, 389
 Biological variability theory, 109
 Birth process, 471-473
 Bisymmetric relation, 291
 Blood pressure, 416
 Branching processes, 463, 477, 494
 see also Stochastic process
 Brightness contrast, 101-102
 Brightness discrimination, 106-107, 120-121, 125-126
 Brightness scale, 142
 Brownian motion, 436-437
 Bulk modulus of elasticity, 15
 Bush-Mosteller model, 167-169
 for noncontingent case, 439-440
 see also Linear operator model
- Cancellation axiom, 270, 272
 Cancer cells, growth of, 483-484
 Cantor function, 440
 Capacitance, 11
 Cardinal, *see* Utility and Scale
 Categories, 238-239
 see also Cues
 Category judgments, method of, 47
 Category selection, 235
 Cauchy's problem, 499
 Causality, law of, 493
 and functional equations 497-498
 Central limit theorem, 423-424, 430-432
 and Brownian motion, 436
 generality of, 430
 Certain outcomes, theories for, 257
 CFF, *see* Critical flicker frequency
 Chain(s), absorbing, 442
 of infinite order, 473-475
 ossicular, 40
 see also Markov chains
 Characteristic function, 297
 Choice, *see* Concept utilization, Responses and Utility

SUBJECT INDEX

- Choice axiom, 152-153, 336-337, 339, 342, 367, 369, 374
 experimental tests of, 396 fn, 396-397
- Choice theories, 188, 258-306
see also Choice axiom
- Chromaticity chart, 117
- Chromaticity space(s), 116-117
 CIE, 118-119
 linear transformation, 122-123
 logarithmic 3 space, 124
- CIE stimulus specification coordinates
 135
 and color difference equations, 119
 135
 and Riemannian metrics, 149-150
- Circular triads, 384, 387
- Closure axiom, 285
- Cochlea, 23
 cross sectional area of, 27-29
 input impedance of, 35-37, 40
 mathematical treatment of transmission in, 24
 maximum vibration amplitude in
 32-34, 37-38, 52-54, 69
 sound transmission in, 23-38
 wave motion in, 34, 68-69
- Cochlear microphonics, 66-67
- Cochlear partition, 24-25
 acoustic impedance of, 26, 29
 acoustic resistance of, 29-30
 compliance of, 29-30
 displacement of, 32, 66-67
 mass of, 29
 phase of, 34
 phase velocity of, 35
 pressure waves of, 32
 resistance of, 30
 vibration of, 32-34, 37-38, 52-54, 69
- Coefficient law and color adaptation
 129
- Coin tossing experiment, 414-424
- Color adaptation, coefficient law and,
 129
 neural aspects of, 106
- Color constancy, 135
- Color contrast and induction, 136-138, 141, 143
- Color difference, 132-133, 139, 151
 equations for, 121-122, 125, 132-133
 metric representation of, 150-154
- Color dimensions, 113, 135-136
- Color discrimination, 106, 130, 145, 147-148
 ellipses for 130-133
 experiments on, 145
 instrumental errors in, 147-148
 measurement of, 133-134
 neural aspects of, 106, 124, 130-135
 probabilistic model for, 148-149
 threshold of, 145
- Color distance, 149
- Color equivalence relations, 114-116
- Color gamut, 141, 142
- Color matching 114-120 130, 146-148
see also Color discrimination and Color mixture
- Color metric, 125, 131, 149-150
 statistical basis of, 148-149
- Color mixture, 114, 136-139
 curves for, 115-116 122-123
 data 115, 122
 error distributions of, 147-148
 experiment, 114-120
 treatment of data of, 122-123
 hue cancellation in, 137-138, 144-145
 linear transformation of, 112-122, 136-139
- Color notation, 133-135
- Color perception, 113-114, 116-117
- Color preference, 387
- Color scaling, 133-135, 151-154
- Color solid, 141
- Color space, 130
 CIE, 130-133
 geometry of, 124-125, 131, 149-150, 152
 material samples used, 133-135
 perceived, 120, 133-134, 140-141
 development of, 135-136
 usefulness of, 135
- Color spacing, uniform, 134-135
- Color sphere, 134
- Color vision, theory of, 120, 125, 135-144

- Colorimetric purity discrimination, 125, 145-146
- Commodity bundle(s), 259
 - additivity of, 267
 - ordering of, 260
 - preference relation over, 259-260
 - utility of, 261-272
- Common elements 167
 - see also* Cue
- Comparison, method of, 151
- Competitive components, 269
- Complete identification experiment, 164
- Complex sound, 68
- Compliance of basilar membrane, 29-30
 - of cochlear partition, 29-30
- Complementary components, 269
- Component, competitive, 269
 - complementary, 269
 - conditioning, 208
 - response, 208-209
 - stimulus 208
- Component model stimulus, 455
- Concept, 214-215
 - see also* Cue
- Concept utilization 207-246
 - as choice behavior, 207
 - as cue conditioning, 207
 - one element stimulus sampling model applied to 209-214
 - as paired associate learning 207, 209-214
 - response axiom for, 208-209
 - see also* Response axiom, applied to concept utilization
 - as selection and conditioning, 207, 233-245
 - as strategy selection, 207, 223-233
 - typical experiment in, 207-208
- Condition *see* specific topics
- Conditional outcome schedule, *see* Outcome
- Conditional response probability, *see* Response probability
- Conditioned cue, 170, 218-219
- Conditioning asymptotic 168
 - direct 178-179
 - generalized, 178-179
 - instrumental, 252
 - process of, 166-168
- Conditioning axiom, 208
 - of cue conditioning model, 216
 - in one element model, 209-210
 - in Restle model, 170
 - for stimulus sampling model, 209-210
 - in stimulus selection model, 234
 - modified, 243
 - in strategy selection model, 225
- Conformity, experiments on, 457-458
- Confusability, λ scale assumptions about, 153-154
- Confusable stimuli, 201
- Conjoint measurement, axioms of, 269
- Connected topological space, 271
- Connectivity, axiom of, 260, 276, 290
- Conservative system, 469
- Constant apparent brightness, index of, 144
- Constant outcomes, 298
- Constant utility model, 332-337
- Contiguity principle, 210
- Continuity, axiom of, 260
 - condition of, 286-287, 499
 - definition of, 266-267
 - equation of, 15-16
 - stochastic, 350
- Continuous space process, 417
 - see also* Stochastic process
- Continuous time Markov chain, 464-470
 - see also* Markov processes
- Contours of equal hue and luminance, 144-145
- Contrast effect, 72-73
- Contrast interaction, 142
- Control theory, 494, 506
- Convergence, in distribution 423
 - in measure, 424
- Convexity, 509
- Coombs' random utility model, 387-389
 - see also* Utility model
- Covariance, 449
- Criterion responses 144
- Critical band, auditory, 70-71, 83-86
 - visual, 52, 54-55, 70-71
- Critical event, 234-235, 237, 240-243
- Critical flicker frequency, 107-113
 - and brightness enhancement, 111

SUBJECT INDEX

- Critical flicker frequency, and intensity discrimination, 109-110
 effect of temperature on, 107-108, 110
 light-dark ratio in, 110-112
 and luminance, 111-112
 and neural physiology of, 111-112
 organic invariance of parameters for, 110
 parametric analysis of, 110
 Critical fusion frequency, *see* Critical flicker frequency
 Critical ratio, 52, 70-72
 Critical spectrum width, 52
 Cue(s), adaptation of, 170, 216, 218
 adapted, 219
 additivity of, 222-223, 230-231, 239
 common, 167-169
 conditioned, 170, 218-219
 conditioning of, 207, 214-218
 correct, 224
 discriminative, 167
 irrelevant, 169-170, 215-216, 219, 223-224, 242
 observed, 167
 partitioning of, 215-216
 redundant, 220-222
 relevant, 169-170, 215-216, 219, 220-222, 233
 suppressed, 170-171
 wrong, 224
see also Category, Concept, Stimulus element, Situational variable, and other specific topics
 Cue conditioning model, 215-220, 246
 axioms of, 216-218
 and delayed reinforcement, 222-223
 and individual variability, 223
 and partial reinforcement, 222-223
 response bias in, 219
 Cumulative distribution, 421
 Curves, *see* specific topics
 Dark adaptation, 106
 Death processes, 471-473
 Decision processes, multistage, 494
see also Decision theory, multistage
 Decision theory, 252-253, 256
 multistage, 494, 507, 509-510
 Decomposition assumption, 363-367, 374
 experimental tests of, 390 fn, 396-397
 Density of enervation, *see* Neural density
 Density function 421-423
 neural, 55, 68
see also Distribution
 Denumerable Markov chains 458-464, 467-471
 Detection, 163
 Diabolic subject, 383
 Dice game, 396-397
 Differential equations 490, 496-502
 Differential sensitivity 49
see also Discrimination, Just noticeable difference, and Threshold
 Dimensions, color, 113 135-136
 relevant 242
 stimulus, 207, 214-215, 220-222
 Direct conditioning operator, 178
 Discrete distribution, mean of, 422
 Discrete space-continuous time stochastic process, 420-421
see also Markov and Stochastic process
 Discrete space process, 417-418
see also Markov and Stochastic processes
 Discrimination, 163, 252, 254-255
 brightness, 106-107, 120-121, 125-126
 color, *see* Color discrimination
 intensity, 80, 106-107, 120-121, 125-126
 neural aspects of, 106, 130-135
 preference interlocking with, 252, 254
 purity, 146
 of stimuli, 254-255
 wavelength, 146
see also Differential sensitivity, Just noticeable difference, and Threshold
 Discrimination learning, 175, 240
 experiments in, 163-164, 199, 231, 233
see also Identification learning
 Discrimination operator, 168-169
 Discriminative stimulus 172, 254

- Displacement, assumption of, 126
of cochlear partition, 32
- Distribution, 421-423
discrete, 422
fundamental for vision, 125-129
Gaussian, 104, 109, 422-423, 430-431, 503
geometric, 445-446
infinitely divisible, 431
logistic, 335
luminance, 103-104
of neural excitation, 72
normal, 104, 109, 422-423, 430-431, 503
Poisson, 422-423, 430-431
of sensation, 78-80
spectral, 126, 129, 136-137
see also specific topics
- Drive, 252
- Dual process models, 166-175
Bush-Mosteller, 167-169
for identification learning, 166
Restle, 169-172
stimulus sampling, 175
Wyckoff, 172-175
- Dual standard sequence, 270
- Duality theory, 509
- Dynamic programming, 494, 506-509
- Ear, transmission characteristic of, 4, 37, 42-46, 67
- Ear canal, pressure transformations in, 44
- Eardrum, 40
acoustic resistance at, 43
effective area of, 40
impedance at, 41-42
postmortem impedance at, 43
- Effect, law of, 179
- Effect of reward, direct, 180-181, 201
generalized, 180-181, 201
- Elasticity, bulk modulus of, 15
- Electrical impedance, 12
- Electroacoustic analogies, theory of, 12-14
- Electroacoustic transducers, 5-6, 8
- Electromechanical analogies, theory of, 11-12
- Elements, *see* specific topics
- Elliptic functions, 500-501
- Equally spaced alternatives, axioms for, 277-278
- Equation, *see* specific topics
- Equilibrium, *see* Asymptotic behavior and specific topics
- Equipment and replacement theory, 506
- Equissection experiments, 47
- Equivalence relation, 114-116, 271
- Ergodic chain, 447
limit theorems for, 449-452
- Error(s), expected number of, 221-222, 227-231, 244-245
last, 235-237, 244-245
role in learning, 245
role in strategy selection, 224
total, 239-240
- Error trials, learning on, 243-245
- Esthetic judgments, 307-308
- Excitation, neural, 58-59, 90-91
function, 122-123
- Expected, *see* Errors, Utility, and other specific topics
- Experiment, *see* specific topics
- Experimenter controlled event model, 183
- Experimenter controlled linear learning model, 188-189, 371, 376
- Exponential functional equation, 498
- Fechnerian utility model, 334, 401
- Fechner's (psychophysical) law, 49, 120-121, 129, 134
- Feedback control processes, 507-508
- Ferry-Porter law, 111
- Fidelity of optical systems, 103
- Finite Markov chain, 440-458
ergodic behavior of, 447-452
transient behavior of, 442-447
- Finite difference system, 277-278
- Fixed point, 189-190
- Flicker contours, phylogenetic, 110
- Flicker frequency, 107-108, 144
see also Critical flicker frequency
- Forced-choice experiment, *k*-alternative, 163-164
- Forced oscillation, 8-10
- Formation of associative *S R* connection, 209
- Form invariance assumption, 126, 129

SUBJECT INDEX

- Forward equation, 465-466, 469-470
 Fourier analysis, 8, 112
 Fourier theorem, 7-8
 Fourier transforms, 104
 Foveal tritanopia, 140
 Fractionation, method of, 46-47
 Frequency, *ind* for, 47-54, 68
 modulation of, 47-48, 50-51
 sensitivity to, 69
 spectrum of, 57-58
 Function, Bessel, 31
 Cantor, 439-440
 characteristic, 297
 elliptic, 500
 excitation, 122-123
 gamma, 493-494
 Hankel, 31-32
 linear, 499-500
 linearizing, 494-496
 loudness, 80-93
 monaural, 83, 85
 luminance, 144-145
 multiplicative, 502
 order preserving 268-269
 outcome, 415
 of Brownian process, 437
 psychophysical, 49, 105-106, 109,
 120-121, 125, 129, 134
 simple harmonic, 6 16-17, 103-104
 sinewave response, 103-104
 spectral distribution, 126 129, 136-
 137
 transfer, 22-23, 113
 utility, *see* Utility function
 Functional equations, 489-510
 approximate, 501
 causality and, 497-498
 Erdos', 501-502
 exponential, 498
 recurrence 506
 Fundamental, *see* specific topics
 "Funneling" process, 104

 Gambling experiment, 327-330, 367-
 376, 380-384, 385-386 391-402
 Gamma function, 493-494
 Gaussian distribution, 104 109, 422-
 423, 430-431, 503
 Generalization decrements, 179-180
 Generalized, *see* specific topics

 Geometric distribution, 445-446
 Guthrie tradition, 210

 Hankel function, 31-32
 Hearing, physiological mechanisms of,
 3
 see also Auditory
 Helmholtz line element, 120-124
 Herstein Milnor system of utility, 286-
 290
 Heterochromatic brightness matching
 144
 Higher neural center, 87-88
 Higher ordered metric, 272-279
 Hooke's law, 15-16, 489
 Hue, relation to luminance, 142-143
 spectral discrimination of, 145

 Ideal utility point 279, 388
 Identification learning, 163-202
 and beta model, 193-201
 dual process models for, 166-175
 see also Dual process models
 satisfactory parsimonious theory of,
 201-202
 and single process model 201
 Immediate successor relation, 278
 Impedance, 10-14
 acoustic, 13-14 20
 acoustic input, 21
 of cochlea 35-37, 40
 of cochlear partition, 26 29
 at the ear, 14
 at the eardrum 14 19
 electrical, 12
 input 35-37, 40
 mechanical 10-11
 in the middle ear, 39-40
 specific, 12-13, 21, 35-37
 Implicit response 369
 see also Observing response and
 Orienting response
 Incentive, 292
 Inconsistency, behavioral 311-312
 Increment threshold 125-130
 Independence of added alternatives
 342
 condition of, 249-270
 of pair 176-177

- Independent increment, 420-421
 Independent increment processes, 436-440
 Brownian motion, 436-437
 Poisson process, 437-438
 stationary increments of, 421
 Independent processes, 428-440
 Independent stochastic process, 418-419
 see also Stochastic processes
 Indifference, 292, 310
 Indifference relation nontransitive, 279-281
 transitive, 260
 Inductance, 11
 Inequalities, study of, 501
 Infinite difference structures, 277
 Infinitely divisible distribution, 431
 Information theory, 413, 420, 481-482
 Inhibition area, 73-80
 in the ear, 80
 extent of, 77-80
 in the eye, 80
 in the skin, 73-75, 80
 Inhibitory interaction in eye of limulus 73
 Input impedance, *see* Impedance
 Instantaneous threshold 106
 Insurance, 282
 Integral equations, 490
 Intensity, 38-39
 discrimination of, 80, 106-107, 120-121, 125-126
 Interleaving of presentation sets, 377-378, 395-396
 Interpolation problem, 493-494
 Interval scale, 272, 276, 284, 307, 310, 314, 333
 Introspection, 273
 Invariant, relative, 494
 Invariant imbedding, 494, 505
 Inventory, 506
 Irrelevant, *see* Categories, Cues, and other specific topics
 Isometry relation, 291
 Isosensitivity curves, *see* ROC curves
 Iterated logarithm, law of, 435-436
 Iteration, 492-494, 497
 of a matrix, 504
 multiplicative characteristic of, 493
 stochastic, 496, 503
 JND, *see* Just noticeable difference
 Judgment probability, 362
 Jumping stand experiment, 163
 Just noticeable difference, assumptions
 about, 279-281
 and Fechner's problem, 495
 frequency, 47-54, 68
 intensity, 80, 106-107, 120-121, 125-126
 subjective size of, 80
 see also Discrimination, Semiorder, and Threshold
 Large numbers, law of, 464
 strong law of, 424, 432
 weak law of, 429, 431
 Lateral neural inhibition, 73-75
 Law, *see* specific topics
 Learning, 504, 509
 animal, 164-166, 186
 nongeneralization and, 187
 perfect, 168, 173-174, 184, 186-187, 201, 219
 see also Asymptotic behavior, Beta model, Conditioning, Linear operator model, Perfect learning, and related specific topics
 Learning curves, 165
 Learning process, 164, 504-506
 see also Conditioning process
 Learning rate, 209, 214, 219, 239, 372-373
 Learning theory, 367-368, 424-425
 see also Learning and specific topics
 Least preferred outcome, choice of, 356-358
 Less probable than relation, 294-296
 Level of aspiration, 304-305
 Lexicographic ordering relation, 261, 264
 Limit law, *see* Limit theorem
 Limit theorem, 428-432, 464, 502
 see also Central limit theorems
 Limulus eye, 73
 Linear function 499-500
 Linear operator model, 167-168, 217, 425-427, 455-457, 474-475
 basic probabilities for, 425
 experimenter-controlled, 188-189
 path independence of, 176-178

SUBJECT INDEX

- Linear operator model, and position habits, 189-193
 single process, 185-193
 with symmetric operators, 186-187
 Linear programming model, 317-319, 384, 401
 vs actuarial model, 319-320
 criticism of, 319
 methods for, 401
 and newspaper content preferences, 384
 probabilistic version of, 320
 Linearizing functions, 494-496
 Line element, 120-130
 in color space, 128-129
 test of theory, 150
 Logarithmic decrement, 9
 Logarithmic 3 space, 124
 Logistic distribution function, 335
 Longitudinal plane waves, 16
 Loss matrix, 375
 Loudness, 4
 and burst duration 90-93
 contribution of critical band to, 83-86
 of intermittent white noise, 92-93
 of masked tone, 81-86
 of a sinusoid, 84-86
 and stimulus duration, 90-93
 Loudness function, 80-93
 constants of, 88
 and critical band 83-86
 for 1000 cycle tone, 88
 Loudness matching 81
 Luminance, 135
 additivity of, 34, 114-115
 and response strength 140
 and saturation, 139
 and value, 134
 and wavelength discrimination, 139
 Luminosity, 125, 144-145

 Mach bands 101, 102-104
 Magnitude balancing (loudness), 81-82
 Magnitude estimation method of, 47, 151-152
 Marginal flicker recognition, 109
 Markov chains, 440-473, 405
 central limit theorem for, 450-451, 454
 continuous time 464-473
 denumerable, 458-464, 467-471
 finite, 441-458
 law of iterated logarithms for, 450-451, 454
 law of large numbers for, 450-451
 and nonstationary transition probabilities, 483
 Markov process 418-421, 426-428, 440-473, 504-505
 and dynamic programming 506-510
 random walk, 505
 see also Markov chains and Stochastic processes
 Martingale stochastic processes, 418-419, 421, 475-478
 convergence theorem for, 477-478
 system theorem for 476-477
 Masking, 69-72, 77-78, 81-86
 Mass, of cochlear partition, 29
 Matching color, 114-120, 130 146-148
 loudness, 81
 see also Color discrimination and Color mixture
 Matrix, fundamental 444-446
 iteration of, 504
 loss 375
 payoff, 164, 182, 393
 regret, 303, 375
 transition, 210-211, 504
 Maximin strategy, 302-303
 Maximum, see specific topic
 Maze experiments, 167, 172-173, 180, 186 230-231
 Mean first passage time, 448
 Mean learning curve, 396
 Measurement see specific topic
 Mechanical, see specific topic
 Mel scale 46-48
 and anatomical findings, 47, 52-54
 and physiological findings, 47, 52-54
 Metameric colors 119
 Metathetic continuum 80
 Method see specific topics
 Metric operation 290-291
 Metric scaling model 150-154
 Metrically transitive process 479
 Middle ear 40-41
 network analogue of 42
 sound transmission in, 39-44

- Minimal process, 470, 472
- Minimally discriminable differences, 121-125
 - see also* Discrimination, Just noticeable difference, and Threshold
- Minimax strategy, 302-303, 326, 402
- Minimax theorem, 302
- Mixture spaces, 284-290
- Model, *see* specific topics
- Moderate stochastic transitivity, 340, 344-346
 - see also* Stochastic transitivity
- Monaural loudness function, 83, 85
- Monte Carlo runs for strong stochastic transitivity, 386-387
- More probable than relation, 294-296
- Mosteller Nogee experiment, 310-314
 - criticism of, 314
- Motivation, 252
- Moving average, 480
- Multistage decision processes, 494, 507, 509-510
- Multiplication condition, *see* Product rule
- Multiplicative condition, 297, 341
 - and other properties and models 343-347
- Multiplicative function, 502
- Munsell color notation 133-134
- N*-bets assumption, 320
- Nervous system, response of, 58
 - see also* Neural activity and related topics
- Nets 272
- Neural activity, 87-88, 90
 - peripheral 90
 - spontaneous, 59, 72, 89-90
- Neural availability, 110
- Neural behavior, specific properties of, 101
- Neural density, 55, 68
- Neural effect of visual excitation, 109
- Neural excitation, 58-59, 90-91
- Neural firing 88-90
- Neural refractory period, 113
- Neural response, characteristics of, 87-88, 111
 - to pairs of short pulses, 60
- Neural responsiveness 136
- Neural unit, 5, 73, 104
 - see also* Response unit
- Newspaper content, preference in, 383-384
- Newton's second law, 17
- Noise, 5, 84
- Nonconfusable stimuli, 201-202
- Nonadapted *see* Categories and Cues
- Nongeneralization, assumption of, 184
- Nonlinear programming model for measurement of utility, 319-320
- Nonreversal shift experiment 223, 242
 - see also* Reversal shift experiment
- Nonreward effect of, 189, 193
- Nonsatiety axiom, 260
- Normal distribution, 104, 109, 422-423, 430-431, 503
- Observing response, 172-175, 369-370
 - see also* Orienting response
- Observing response model 172-175 369-370 376, 396
- Objective probabilities, 284, 291
 - estimation of, 322-325
 - and subjective probabilities, 323-326
 - see also* Response, Stimulus, and special topics
- One element stimulus sampling model, applied to concept utilization 212-214, 246
 - applied to gambling experiment, 360, 368-371
 - applied to paired associate learning 209-211, 246
 - applied to probability prediction experiment, 368-370, 395-396
 - asymptotic behavior, 369
 - axioms for, 209-210
 - experimental test of, 395-396
- Open conditions 269, 276
- Open set, 266
- Operations research, 506-507, 509
- Operators, identity, 169
 - symmetric, 181-183, 186-187, 189-200
 - see also* Beta model and Linear operator model
- Opponent process model, 135-144

SUBJECT INDEX

- Optical imagery, 103-104
- Optimal strategies, 300-301
- Optional stopping system (of play), 476
- Order preserving function, 268-269
- Ordered metric, 272, 274, 307-310
 - additivity of, 309
 - experimental tests of, 309-310
- Ordinal, *see* Scale and specific topics
- Oriented behavior, 108
- Orienting response(s), 172-175, 369-370
- Ossicular chain, 40
- Outcome(s), certain, 257, 331
 - conditional schedule, 257
 - consistency of, 180-181
 - constant, 298
 - elementary, 331
 - preference over, 178-179
 - set of, 164
 - uncertain, 257, 281-306, 359-377
 - see also* Payoff
- Outcome function, 415
 - of Brownian process, 437
- Oval window, effective area of, 40
- Overlearning 202
- p vs π , asymptotic values for, 390-399
- Paired associates experiment, 163-164, 207, 209-212, 246
- Paired comparisons, method of, 152
- Pareto optimal strategies, 300-301
- Pari mutual betting, 322-323
- Particle velocity, 19, 21
- Partitioning of cues, 215-216
 - of strategies, 225
- Path independence, 169, 175-178
- Pattern model, 427-428, 452-455
 - and level of risk, 329-330
- Payoff matrix, 164, 182, 393
- Penny-matching game, 433-435, 476-477
- Perceived brightness, 102-104
- Perceived color, 136, 138-139
- Perceived color gamut and color induction, 142-143
- Perception axiom, 170
- Perceptual process, 167
- Perfect learning, 168, 173-174, 184, 186-187, 201, 219
 - see also* Asymptotic behavior
- Period, 7
- Periodicity pitch, 46
- Peripheral receptors, photochemical events at, 101
- Perseveration model, 231-233
- Phase, 6, 34-35
 - sensitivity to, 51-52
 - velocity, 16, 35
- Phasor, 6-7
- Photochemical theory, 101, 105-108
- Photopigment concentration, 108
- Photoreceptor process, description of, 105-107
- Physiological interaction, 103, 141-142
- Pinball machine experiment, 390
- Pitch, 3, 46-55
- Place learning, 240
 - vs response learning, 230-231
- Place theory of hearing, 52-54, 84
- Plane wave, 19
- Poisson distribution, 422-423, 430-431
- Poisson process, 436-438
- Poker hands, 311
- Policy, 507
 - optimal, 507
- Position habits, 189-193, 198, 201, in discrimination learning, 255
 - with equal presentation probabilities, 199
 - predicted by linear model, 189-193
 - and symmetry in experimental situation, 199-200
- Potential surprise, 305-306
- Potential theory, 446-447, 451-452, 460
- Power law, 5, 82
- Predicted errors, *see* Errors
- Preference, 252-258
 - experiments on, 254-255, 396-397
 - and judgment of likelihood 363-364
 - for shades of gray, 386-387
- Preference relation 260-263
- Preference theory, classification of 255-258
 - origins of, 252-253
- Presolution stationarity, 235-236
- Pressure, auditory, 32, 44-45

- Principle of optimality, 507
- Prisoner's dilemma, 300-301, 303
- Probabilistic choice theories, 331-350, 359-377
 - and asymptotic behavior, 377-378
 - constant utility models for, 332-337
 - random utility models for, 337-339
 - response probabilities in, 331-332
 - transitivity in, 339-342
 - for uncertain outcomes, 359-377
 - decomposition assumption in, 362-367
 - expected utility models, 359-362
- Probabilistic ranking theories, 351-358
- Probability, basic, 415-417, 425, 427-428
 - of errors in strategy selection model, 225-226
 - objective, 284, 291, 323-325
 - response, 164, 168, 171-172, 332
 - subjective, 293, 321-327
 - see also* Subjective probability
 - theory of, 421-424, 502-506
 - transition, 210-211
 - see also* Markov and Stochastic processes and specific topics
- Probability matching hypothesis, 369, 393
- Probability measure, 293
- Probability prediction experiment, 390-396
- Probability preference, 327-328
- Process, *see* specific topics
- Product rule, 341-346, 379
- Prothetic continuum, 80, 82
- Psychiatry, 509
- Psychophysical experiment, 254
 - characteristics of, 144-150
 - response bias in, 255
 - see also* Discrimination, Magnitude estimation, Threshold, and other specific topics
- Psychophysical function, 105-106, 109
 - Fechner's, 49, 120-121, 129, 134
- Pulse sequence, 57-58
- Pure risk, 257, 330
- Quadruple condition, 275, 341, 343-346 350
- Qualitative probability, axioms for, 294-296
 - see also* Subjective probability
- Quasi ordering, 274
- Queues, 437-438, 483
- Random utility model, *see* Utility model
- Random variable(s), 421-424
 - uniformly bounded, 429
- Random vector, 337-338
- Random walk, 194-197, 440, 455, 460, 462-463, 505-506
- Ranking probabilities of, 351-359
- Ratio scale, 152, 335-336
- Rational behavior, 256
- "Rational" man, 253
- Reactance, 11
- Receiver operating characteristic curve, *see* ROC curve
- Receptor process, visual, 105
- Recognition experiment, 163-166, 184, 201
- Recognition learning model, 188
- Recurrence equations, 492
- Recurrence functional equation, 506
- Recurrence theorem, 435
- Redundancy, relevant, 220-222
- Redundant, *see* Categories, Cues, and other specific topics
- Reflexive relation, 274
- Refractory period, neural, 113
- Regret matrix, 303, 375
- Regularity condition, 342-344, 346
 - experimental test of, 400-401
- Reinforcement, effect of, 180-184, 186-187
 - in maze experiments, 180
 - partial, 169
 - schedules of, 425-427
 - in shuttle box, 180
 - theory of, 179
- Reissner's membrane, 24
- Relation, bisymmetric, 291
 - color equivalent, 114-116
 - equivalence, 114-116, 271
 - immediate successor, 278
 - indifference, nontransitive, 279-281
 - transitive, 260

SUBJECT INDEX

- Relation, isometric, 291
 less probable than, 294-296
 lexicographic ordering, 261, 264
 more probable than, 294-296
 nontransitive, 279-281
 preference, 260, 262-263
 psychophysical, 49, 105-106, 109,
 120-121, 129, 134
 reflexive, 274
 transitive, 260, 270, 276-277, 297
 violation of, 279-281, 309-310,
 319
 weak ordering, 260, 263, 270, 275,
 285, 298
 see also specific topics
 Relative expected loss minimization
 rule, 375-377
 RELM rule, *see* Relative expected loss
 minimization rule
 Relative invariant, 494
 Relevant, *see* Categories, Cues, Dimen-
 sion, Redundancy, and other
 specific topics
 Renewal theory, 438-439
 Replacement theory, 506
 Resistance (acoustic), 11
 of cochlear partition, 30
 Resolving power of auditory system, 49
 Response(s), implicit, 369
 observing, 172-175, 369-370
 orienting, 172-175, 369-370
 probability of, 168, 171-172 332
 conditional 164
 Response axiom(s), applied to concept
 utilization, 208-209
 of cue conditioning model, 216
 of one element model, 210
 of Restle's model 171
 for stimulus selection model, 234
 for strategy selection model, 225
 Response bias, 219, 255, 331
 Response learning 209
 vs place learning 230-231, 240
 Response set, 164
 Response unit, 73-77, 79-80
 see also Neural unit
 Restle model, 169-172
 Restoring force, 15-16
 Reversal shift experiment, 223, 242
 Reward and nonreward, effect of sym-
 metric, 181-182
 equality of effect of, 183-184, 200
 generalization of, 181, 187
 relative effect of, 180-181, 186
 see also Reinforcement
 Riemannian metric, 131, 133
 Risk, 257, 330
 ROC curves, 165, 188
 R-order dense set, 263

 Sample patch of Brownian process, 437
 Savage's axioms for utility and sub-
 jective probability, 298-299
 Scale, attitude, 268-272
 cardinal, 259
 interval, 272, 276, 284, 307, 310
 314, 333
 Mel, 46-48, 52-54
 ordinal, 259-264, 272, 284, 333
 ratio, 152, 335-336
 subjective, 332-333
 v, 152-153
 see also Color scaling Discrimina-
 tion Loudness and Magnitude
 estimation
 Selection conditioning model, 233-245
 Selective listening, 84
 Semigroups, 494, 498, 508
 Semiorde, 280
 Sensation fundamental, 121
 Sensation area, 73, 79
 distribution over, 79-80
 Sensation difference, 121-123, 130
 see also Color difference, Discrimina-
 tion and Threshold
 Sensation identity, 80-81
 see also Color mixture and Matching
 Sensation level, masking noise in 81-82
 Separable topological space, 266, 271
 Sequence, Bernoulli, 171, 235, 242, 244
 dual standard, 270
 standard 278
 Sequential analysis, 506
 Set, *see* specific topics
 Shift transformation, 479
 Shuttlebox experiment 180
 Similarity index, 167-169
 Simple harmonic function 6 17

- Sinewave response function, 103-104
- Single element assumption, 210
- Single process models, 176-201
- beta, 193-201
 - linear, 185-193
 - experimenter controlled operators for, 188-189
 - with symmetric operators, 186-187
 - nongeneralization, 184-185
- Situational variables, 170
- see also* Categories and Cues
- Skeleton chain, 465
- Skewness of bets, 329
- Skin sensitivity, 73-76
- Solution of equations axiom, 270
- Sone, 82
- Sound, diffraction of at listener's head
- 46
 - frequency of and pitch, 46
 - intensity of, 21
 - transmission of, in the cochlea, 24-38
 - in the ear, 21-46
 - in the middle ear, 22, 38-43
- Sound pressure, 13-14, 19, 21
- measurement of, 21-22
 - and particle velocity, 21
- Space, base for, 266
- color, *see* Color space and Chromaticity space
 - connected, 271
 - logarithmic 3, 124
 - mixture, 285-290
 - separable, 266-271
 - topological, 266-267, 287
 - see also* Topological space
- Spatial contrast, 101
- Spectral distribution functions, 126, 129, 136-137
- Spectral measure, 462
- Spectrum, frequency, 57
- Speech, approximation to, 420
- Spherical wave, 19
- Spontaneous neural activity, 59, 72, 89-90
- Stability, 501
- Stable law, 431
- Stapes footplate, area of, 40
- State(s), absorbing, 189-193, 442
 - equilibrium, in photoreceptor process, 105-106
- State(s), for finite Markov chains, 442
 - of nature, 282, 298
 - stable, 467
 - transient, 442
- State variables, 491
- Stationary processes, *see* Stochastic processes
- Stationary transition probabilities, 419
 - see also* Markov process
- Statistical decision theory, 252-253, 256
 - multistage, 494, 507, 509-510
- Steady state behavior, 10, 255-256, 495
 - equation of, 106-107
 - see also* Asymptotic behavior, Equilibrium state, and Perfect learning
- Stimuli, additivity of, 5, 490
 - confusable, 164-166
 - differentiation of, 209
 - dimensions of, 214-215, 220-222, 233
 - discriminative, 172, 254
 - distribution of in cochlea, 75-77
 - generalization of, 179
 - identity of, 80-81
 - mathematical description of, 5
 - presentation probability of, 164
 - at sensory cell, 22, 27
 - time pattern of, 57-58
 - see also* Category and Cue
- Stimulus axiom, of cue conditioning model, 216
 - of one element model, 209, 246
 - of stimulus sampling model, 209-210
 - of stimulus selection model, 234
 - for strategy selection model, 225
- Stimulus categories, 233-234
- Stimulus component model, 455
- Stimulus elements, 167
 - conditioned subset of, 167-168
- Stimulus power, additivity of within critical band, 5
- Stimulus presentation, 163-164, 166-167
- Stimulus sampling model, 175, 209-210, 368-370, 427-428, 452-457
 - axioms for, 209-210
 - basic probabilities for, 427-428
 - for paired associates learning, 209-213, 246

SUBJECT INDEX

- Stimulus selection model, 233-235
 Stimulus transformation, 22, 87
 Stochastic continuity, 350
 Stochastic iteration, 496, 503
 Stochastic theories of preference, 256
 Stochastic process(es), 175, 256, 413-484
 abstract space, 417
 classification of, 417-421
 continuous space, 417
 definition of, 414-417
 discrete space, 417
 independent, 418
 in information theory, 413
 Markov, 418
 Martingale, 418
 see also Martingale stochastic process
 nonbranching 171-172
 path dependent, 175
 path independent, 169, 176-178
 stationary, 418
 see also Markov processes
 Stochastic transitivity, definition of, 340
 experimental tests of, 380-390
 mild, 340
 moderate, 340, 344-346
 and multiplicative condition, 344-345
 strong, 340, 344-345, 364-365 384-387
 weak 340, 342, 344-345, 384, 386-387
 Strategy(ies), 223-224, 300
 classification of, 225
 equilibrium, 301-302
 maximum, 302-303
 minimax, 326, 402
 mixed, 302
 Pareto optimal, 300-301
 sampling probability of, 225-227
 selection of, 207, 224-227
 Strategy selection model, 224-233, 246
 additivity of cues in, 230-231
 and discrimination experiment, 233
 experimental tests of, 233
 Strict higher ordered metric, 276-277
 see also Higher ordered metric
 Strict utility model, *see* Utility model
 Strong conditioning model, 370, 376
 Strong higher ordered metric, 276-277
 see also Higher ordered metric
 Strong law of large numbers, 424, 432
 Strong stochastic transitivity, *see* Stochastic transitivity
 Strong utility model, *see* Utility model
 Subjective expected utility model, *see* Utility model
 Subject experimenter controlled Beta model 371-377
 Subjective probability, 252
 axiom system for, 291-299
 of compound event 326
 of $\frac{1}{2}$ 292-293, 314-315
 measurement of, 292, 314-315, 321-327
 Subjective scale 332
 see also Psychophysical relation and specific topics
 Subject's knowledge of outcome 378
 Subliminal differences, 280
 Suppression of irrelevant cues 170-171
 Sure thing principle, 298 300-302 319
 Surprise, potential, 305-306
 Survival of family names, 463-464
 Symmetric operators, 181-183
 in beta model 199-200
 in linear operator model, 186-187
 Systems engineering 101
 Temporal summation, 4, 65, 90-93
 Thomsen condition, 272
 Threshold, absolute photopic, 144
 auditory, 3, 56-68
 and critical band, 68 70-72
 and density of innervation, 68
 elevation of 69-72
 increment, 125-130
 instantaneous, 106
 of preference 317
 of three paired system 140-141
 see also Discrimination Just noticeable difference, and Semioorder
 Threshold model, applied to gambling experiment, 372-373, 376
 Thurstone model, 349, 383-384 389
 Time-dependent responses, 113
 Time history of system 491, 504
 T maze experiment 180, 240
 Tone burst, 90-93

- Topological space, 266–267, 287
 - base for, 266
 - connected, 271
 - separable 266, 271
- Transfer function, 22–23, 103–104, 113
- Transform, maximum, 508–509
- Transient response, 10
- Transient state, 441–442
- Transition matrix of probabilities, 210–211, 504
- Transitive process, metrically, 479
- Transitive relation, 260, 270, 276–277, 297
 - violation of, 279–281 309–310 319
- Transitivity, stochastic, *see* Stochastic transitivity
- Transmission characteristics of ear, 4, 37, 42–46, 67
- Triads method of, 307
- Trial independence, 210, 332
- Trials to last error, 235–237, 244–245
- Triangle condition 340 342–344, 350
- Tritanopia 140
- Two person game method, 325–326

- Uncertain outcomes 257, 281–306, 359–377
- Unfolding technique, 279, 307–308, 329, 387–388
- Uniformly bounded variable, 429, 436
- Unilateral pair, 388
- Unilateral triple, 388
- Uniqueness, *see* specific topic
- Utility, Bernoullian, 258
 - cardinal, 259
 - defined, 258
 - expected, 281–284
 - of gambling, 330
 - maximization of, 258, 282, 295
 - ordinal, 259
- Utility curves, *see* Utility function
- Utility difference(s), 272–279
 - measurement of, 308–310, 315–319
- Utility function(s), 258–306
 - additivity of, 267–272
 - continuous, 264–265, 267
 - discontinuous, 264–265
 - equally spaced, 315
 - existence of, 261–264, 267, 271, 277–278, 280 286, 289, 299
- Utility function(s), measurement of, 310–321
 - ordinal 259–264
 - and R order dense set, 263–264
 - uniqueness of, 262–263, 271, 278, 286, 289
- Utility of gambling model, 330
- Utility intervals, *see* Utility difference
- Utility model(s), additive, 267–272
 - asymptotic, 374–375
 - constant, 332–337
 - expected, 281–284, 292, 327–330
 - see also* Herstein Milnor, Savage, and von Neumann Morgenstern, utility models
- Herstein Milnor, 286–290
 - probabilistic expected, 359–367
- random, definition of, 338
 - necessary and sufficient conditions for, 352–353
 - relations to other concepts, 344, 346–349, 356–358
- random expected definition of, 361
 - experimental tests of, 397–400
 - relation to other concepts, 361–362
- Savage, 298–299
 - strict, definition of, 335
 - experimental tests of, 397–400
 - relations to other concepts, 335–336, 344, 350, 354–356, 379
 - strict expected, definition of, 360
 - experimental test of, 397–400
 - relations to other concepts, 361–362, 373–374
 - strong definition of, 334
 - relations to other concepts, 334–335, 339, 344, 350
 - strong expected, definition of, 360
 - von Neumann Morgenstern, 285–286, 290
 - weak, definition of, 333
 - experimental tests of, 380
 - relations to other concepts, 334, 339, 344, 349–350
 - weak expected, definition of, 360
- Utility theory, *see* Utility model
- Utility of variability model, 374–375

SUBJECT INDEX

- v scale, 152-153
- Variable, random, 421-424
 - uniformly bounded, 429, 436
- Variance preference, 328-330
- Velocity of wave propagation, 16-19
- Vibration maximum in cochlea, 32-34, 52-54, 69
- Visual acuity, 103, 106
- Visual flicker perception, 101
 - see also* Critical flicker frequency
- Visual photochemistry, 106-108, 124
- Visual stimulus, quantum aspects of, 108
- Visual system, characteristics of, 104
- Von Neumann Morgenstern system of utility, 285-286, 290
- Voter's preference, 268, 279
- Waiting lines, 437-438
- Waves, in cochlea, 32, 34
 - plane, 19
 - propagation of, 14-20
- Waves, reflection of, 20-21
 - spherical, 19
- Wave equation, 16-17
- Wavelength, 17
 - discrimination of, 122 125, 144-145
 - see also* Color discrimination
- WDW unit coordinates of color equivalence, 115-118
- Weak conditioning model, 370, 376
- Weak higher ordered metric, 274-276
- Weak law of large numbers, 429, 431
- Weak ordering relation 260 263, 270 275, 285, 298
 - see also* Preference relation
- Weak stochastic transitivity *see* Stochastic transitivity
- Weak utility model, *see* Utility model
- Weber's law, 126, 129
- Webs theory of, 272
- Wyckoff model, 172-175
- Zero sum game, two person 302
- Zone of inconsistency, 312